BUSINESS MATHEMATICS

A COMPLIATION

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Student Edition

To

The Brothers of St. Gabriel in Thailand and Assumption Business Administration College

"Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."

Bertrand Russell.

PREFACE

This text is meant mainly for the students of Assumption Business Administration College (ABAC), often at the freshman level. It is a compilation of business topics from various books. The emphasis is on the business transactions and business models. It serves as a basis for decision making and for higher studies.

Throughout the book, practical problems are used to illustrate the application of the formulas and tables. This is particularly helpful to students who have a limited knowledge of algebra, and self-study would be much enhanced by exercises and given answers to the problems presented at the end of each topic or chapter.

The currencies used may be in baht or U.S. dollars and the interest rates used may be a little obsolete. But the approach to the problems would be the same regardless of these differences.

This is a one-semester course in Business Mathematics. The instructor is free to organize the sequence of the subject matters in the most suitable way. Omission or addition of certain topics may be done. The text is meant as a guideline for both the instructor and the students. The course can be developed into a two-semester course later on with more contents.

I wish to express my appreciation to the administrators of Assumption Business Administration College in general, and its staff members and students in particular who have encouraged me to undertake this piece of work and through the available sources rendering it to reach its completion.

I am in debted to Suda Kittikorncharoen and Chaiyong Truakul for their constant push towards its publication.

I am grateful to Songsamorn Chomchinda who had helped me to type the rough and final draft. Thanks are extended also to the following for their kind help in the production of the book: Narin Chomchinda, Rapeepan Suk-asa, and Decha Sethapun.

Bancha Saenghiran

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CHAPTER I

SIMPLE INTEREST

Interest *(I) What is an interest? It is the income from invested capital; or in a narrower sense, it is the fee paid for the use of money.*

Principal *(P) is the sum of money borrowed in an interest transaction.*

Time or term of the loan *(t) or (n) is the period during which the borrower has the use of all or part of the borrowed money.*

Interest rate *(r or i) is the price of a simple interest loan.*

Simple Interest: *means that the loans have interest computed entirely on the original principal.*

1.1 Simple Interest:

- It is defined as the product of principal, rate, and time.

$$I = Prt \qquad \text{______} (1)$$

Note that 'r' and 't' must be consistently stated. That is, if the rate is an annual rate, the time must be stated in years; if the rate is a monthly rate, the time must be stated in months.

- When an interest rate is stated as a percent, it should be converted to a fraction or decimal when used in the simple interest formula.

Example 1:

A couple buys a home and gets a loan for ฿ 300,000. The annual interest rate is 15%. The term of the loan is 15 years
a) Find the interest for the first month.

I = interest to be found out
P = principal = ฿ 300,000
$r = 15\% = 0.15/\text{year}$
$t = 1 \text{ month} = \frac{1}{12} \text{ year}$

$$I = Prt$$
$$= 300{,}000 \times \frac{15}{100} \times \frac{1}{12}$$
$$= ฿\ 3{,}750$$ Ans

b) If the monthly payment is ฿ 4,000 find the amount of house purchased with the first payment.

Since payment on interest for the first month is = ฿ 3,750
And the monthly payment is = ฿ 4,000
∴ For the first payment, the couple buys only
= 4,000 – 3,750 = ฿250 worth of the house. Ans

Example 2:

The interest paid on a loan of ฿ 10,000 for 4 months was ฿ 250. What was the interest rate?

Solution:

P = ฿ 10,000
t = 4 months
$= \frac{4}{12} = \frac{1}{3}$ year
I = ฿ 250
r = to be found out.

or $I = Prt$

$$r = \frac{I}{Pt}$$
$$= \frac{250}{10{,}000 \times \frac{1}{3}}$$
$$= \frac{250 \times 3}{10{,}000}$$
$$= 0.075$$
$$= 7\tfrac{1}{2}\%$$ Ans

Example 3:

A man gets ₿ 1,275 every 6 months from an investment which pays 4¼ % interest. How much money does he have invested?

Solution:

$$I = ₿\ 1{,}275$$
$$t = 6 \text{ months} = \frac{1}{2} \text{ year}$$
$$r = 4\tfrac{1}{4}\% = 0.0425$$

or

$$I = Prt$$
$$P = \frac{I}{rt} = \frac{1275}{0.0425 \times \frac{1}{2}} = ₿\ 60{,}000 \qquad \underline{\text{Ans}}$$

Example 4:

How long will it take ₿ 100,000 to earn ₿ 1,000 interest at 6%?

Solution:

$$P = ₿\ 100{,}000$$
$$I = ₿\ 1{,}000$$
$$r = 6\% = \frac{6}{100}$$
$$t = ?$$

$$I = Prt$$
$$t = \frac{I}{Pr} = \frac{1000}{100{,}000 \times \frac{6}{100}} = \frac{1}{6} \text{ year or 2 months}$$

<u>Ans</u>

1.2 Amount : (or) Future Value (F)

It is the sum of the pricipal and the interest. It is symbolized by F

$$F = P + I$$
$$= P + Prt$$
$$= P(1 + rt) \qquad \text{.................... (2)}$$

Example 1:

Method 1:

A man borrows ₱ 7,000 for 6 months at 10%. What amount must be repaid?

Solution:

$$P = ₱7{,}000 \qquad F = P(1 + rt)$$

$$t = 6 \text{ months} \qquad = 7000\left(1 + \frac{1}{10} \times \frac{1}{2}\right)$$

$$= \frac{1}{2} \text{ year} \qquad = 7000\left(1 + \frac{1}{20}\right)$$

$$r = 10\% \qquad = 7000(1.05)$$

$$= ₱7{,}350 \qquad \underline{\text{Ans}}$$

Method 2:

This problem can be solved by getting the simple interest and adding it to the principal.

$$I = Prt$$

$$P = ₱7{,}000 \qquad I = 7000 \times \frac{1}{10} \times \frac{1}{2}$$

$$r = 10\% \qquad = ₱350$$

$$= \frac{1}{10} \qquad \therefore F = P + I$$

$$= 6 \text{ months} \qquad = 7000 + 350$$

$$= \frac{1}{2} \text{ year} \qquad = ₱7{,}350 \qquad \underline{\text{Ans}}$$

Note: The amount of money at the starting of the period is the present value and the amount obtained at the end of the period (principal + interest) is the future value of the amount.

Example 2:

A debt of ฿ 20,000 due in one year with interest at 15% may be considered as an obligation of ฿ 23,000 due in one year.

If a time is not stated, we shall assume that the designated sum is the present or cash value. Thus; if a problem states that the price of a car is ฿ 400,000 this means the cash price. If a sum is due in the future *'without interest'* or if a rate is not stated, we shall assume that the given value is the maturity value of the obligation.

Example 3:

'฿ 60,000 due in 6 months' means. that 6 months hence the debtor must pay ฿ 60,000.

In the business world, this obligation would be worth less than ฿ 60,000 before the end of 6 months.

EXERCISE: 1 - 1

1. Find the simple interest on $750 for 2 months at 7% and the amount.

2. Find the simple interest on $1225 for 3 months at 3½% and the amount.

4. A man borrows $10,000 to buy a home. The interest rate is 6% and the monthly payment is $64.43. How much of the first payment goes to interest and how much to principal ?

5. A $15,000 home loan is to be repaid with monthly payments of $100. If the interest rate is 6½%, how much of the first payment goes to interest and how much to principal ?

6. A savings and loan association advertises a rate of $4\frac{1}{2}\%$ compounded semiannually. Deposits made by the 10th of the month earn dividends from the 1st of that month. Deposits made after the 10th earn dividends from the 1st of the following month. Dividends are credited on June 30 and December 31. A depositor had \$2045 in his account on June 30, 1973. He deposited \$1120 on September 7, 1973. How much will he have in his account on December 31, 1973 ?

7. A man has an account in a savings and loan association that pays 5% compounded semiannually. Dividend dates are June 30 and December 31. Money deposited by the 10th of a month earns dividends from the 1st of that month. Deposits after the 10th earn dividends from the first of the following month. This man had a balance of \$1600 on December 31, 1972. He deposited \$400 on February 8, 1973, and \$600 on April 25, 1973. Find the final entry in his passbook after interest is credited on June 30, 1973.

8. A man borrowed \$150 for two months and paid \$9.00 interest What was the annual rate ?

9. A mechanic borrowed \$125 from a licensed loan company and at the end of one month paid off the loan with \$128.75. What annual rate of interest did he pay ?

10. If a person loans \$6000 at 5%, how long will it take him to get \$75.00 interest ?

11. How long will it take \$8400 to earn \$24.50 interest at 3.5% ?

12. How long a time will be required for \$625 to earn \$25 interest at 4.8% ?

13. How long will it take \$1800 to earn \$63.75 interest at $4\frac{1}{4}\%$?

14. What is the amount if \$3.6 million is borrowed for 1 month at $2\frac{7}{8}\%$?

15. What is the amount if $14.4 million is borrowed for 2 months at 3¼% ?

16. A man borrows $95. Six months later he repays the loan, principal and interest, with a payment of $100. What interest rate did he pay ?

17. A loan shark made a loan of $50 to be repaid with $55 at the end of one month. What was the annual interest rate ?

18. A waitress, who was temporarily pressed for funds, pawned her watch and diamond ring for $55. At the end of 1 month she redeemed them by paying $59.40. What was the annual rate of interest ?

19. A loan shark was charging $10 interest for a $50 loan for a month. What was the annual interest rate ?

20. A teacher borrowed $200 from the credit union to which she belonged. Every month for 8 months she paid $25 on the principal and interest at the rate of 1% a month on the unpaid balance at the beginning of the month. The first interest payment would be 1% of $200, the second interest payment would be 1%, of $175, and so on until the loan was repaid. What was the total interest paid on this loan ?

21. A person borrowed $800 from a credit union which charges ¾% per month on the outstanding balance of a loan. Every month for 8 months he paid $100 on the principal plus interest on the balance at the beginning of the month. Find the total interest.

22. The value of fixed income investments is very sensitive to changes in interest rates. In 1963, when money was relatively easy or cheap, Cincinnati Gas and Electric $100 par value, 4% preferred stock was quoted at 94. Early in 1972, when interest rates were higher, the stock could be purchased for $61. What rates of return or yield rates were received at these prices ?

23. Philadelphia Electric \$100 par value, 3.8% preferred stock was quoted at 54½ in the January 31, 1972, issue of *The Wall Street Journal.* What was the yield rate ? Find the current yield on this stock.

1.3 Exact and Ordinary Interest.

When the time is in days and the rate is an annual rate, it is necessary to convert the days to a fractional part of a year when substituting in the simple interest formulas. Interest computed using a divisor of 360 is called *ordinary interest.* When the divisor is 365 or 366, the result is known as *exact interest.*

For a given rate of interest, a denominator of 360 results in a borrower paying more interest in dollars than would be the case if 365 or 366 were used. On individual loans the difference may not be large. But when many borrowers each pay a little more, the total difference is a substantial sum. This increased revenue makes the 360-day year popular with lenders.

Example

Get the ordinary and exact interest on a 60-day loan of \$300 if the rate is 8%.

Solution:

Substituting P = 300 and r = .08 in (1), we have

$$\text{Ordinary interest} = 300 \times .08 \times \frac{60}{360} = \$4.00$$

$$\text{Exact interest} = 300 \times .08 \times \frac{60}{365} = \$3.95$$

Note that ordinary interest is greater than exact interest. Also it is easier to compute when the work must be done without a calculator.

1.4 Exact and Approximate Time.

There are two ways to compute the number of days between calendar dates. The more common method is the *exact* method which includes all days except the first. A simple way to determine the exact number of days is to use Table 1 in the back of this book, which gives the serial numbers of the days in the year. Another method is to add the number of days in each month during the term of the loan, not counting the first day but counting the last one. The *approximate* method is based on the assumption that all of the full months contain 30 days. To this number is added the exact number of days that remain in the term of the loan.

Example:

Find the exact and approximate time between March 5 and September 28.

Solution:

From Table 1 we find that September 28 is the 271st day in the year and March 5 is the 64th day. Therefore the exact time = 271-64 = 207 days. If a table of calendar days is not available, we can set up a table as follows:

March	26	(31 - 5)
April	30	
May	31	
June	30	
July	31	
August	31	
Sept.	28	
Total	207	days

To get the approximate time we count the number of months from March 5 to September 5. This gives us 6 x 30 = 180 days. To this we add the 23 days from September 5 to September 28 to get a total of 203 days.

1.5 Commercial Practice.

Since we have exact and ordinary interest and exact and approximate time, there are four ways to compute simple interest:

1. Ordinary interest and exact time (Bankers' Rule)
2. Exact interest and exact time
3. Ordinary interest and approximate time
4. Exact interest and approximate time

This brings out the fact that in computing simple interest, as in all problems in the mathematics of finance, both parties to the transaction should understand what method is to be used. In this book, when the time is in days, we use ordinary interest and exact time unless another method is specified. This is known as *Bankers' Rule* and is the common commercial practice.

When an obligation has a stated time to run, it is necessary to determine the due date. If the time is stated in days, the due date is the exact number of days after the loan begins. If the time is stated in months, the due date is the same as the date on which the term of the loan begins unless the date of the loan is larger than the last date of the month in which the loan matures. When this happens, we take as the maturity date the last date of the month.

Example 1:

DATE OF LOAN	TERM OF LOAN	MATURITY DATE
June 15, 1972	60 days	August 14, 1972
June 15, 1972	2 months	August 15, 1972
Dec. 10, 1972	4 months	April 10, 1973
Dec. 10, 1972	120 days	April 9, 1973
Dec. 10, 1971	120 days	April 8, 1972
		(1972 is a leap year)
Dec. 28, 29, 30 or 31, 1972	2 months	Feb. 28, 1973
Dec. 29, 30 or 31, 1971	2 months	Feb. 29, 1972

Example 2:

On November 15, 1973, a man borrowed $500 at 5%. The debt is repaid on February 20, 1974. Find the simple interest using the four methods.

Solution:

We first get the exact and approximate time.

EXACT TIME		APPROXIMATE TIME	
From Table 1:			
November 15 is the 319th day		November 15 to February 15 is three months	
February 20 is the 51st day			
In 1973 there are 366-319 =	46 days	Three months × 30 days =	90 days
In 1974 there are	51 days	February 15 to 20 =	5 days
Total time	97 days	Total time	95 days

Ordinary interest and exact time (Bankers' Rule) $\quad I = 500 \times .05 \times \frac{97}{360} = \$6.74$

Exact interest and exact time $\quad I = 500 \times .05 \times \frac{97}{365} = \$6.64$

Ordinary interest and approximate time $\quad I = 500 \times .05 \times \frac{95}{360} = \$6.60$

Exact interest and approximate time $\quad I = 500 \times .05 \times \frac{95}{365} = \$6.51$

To encourage prompt payment of bills, many merchants allow discounts for payments in advance of the final due date. Terms of 2/10, n/30 mean that if payment is made within 10 days from the date of the invoice 2% of the amount of the invoice can be deducted. If the bill is paid after 10 days but on or before the 30th day, the net amount of the invoice must be paid. A buyer who takes advantage of cash discounts in effect lends money to the seller and receives as interest the cash discount. Interest rates earned in this way usually are so high that it is good business practice to take advantage of savings through cash discounts.

Example 3:

A merchant receives an invoice for $2000 with terms 2/10, n/30. If he pays on the 10th day, he earns what rate of interest ?

Solution:

The cash discount is $40 making the principal $1960.

$$I = Prt$$

$$1960 \times r \times \frac{20}{360} = 40$$

$$r = \frac{40}{1960} \times \frac{360}{20} = .367 = 36.7\%$$

Banks sometimes require business borrowers to keep a *compensating balance* on deposit with the bank as one of the conditions for getting a loan. If a borrower gets a $1000 loan with a 20% compensating banlace, he will be required to maintain $200 (20% of $1000) in his demand account. If he normally maintains less than $200, part of the loan proceeds must be used to meet the compensating balance requirement. While he pays interest at 8% on the full $1000, he will have the use of less than $1000. The effective rate of interest will be greater than the stated 8%. The difference will depend on the size of the compensating balance, and the balance the borrower would ordinarily keep anyway.

Example 4:

A borrower gets a $100,000 loan at 8% at a bank which requires the lender to keep a compensating balance of 20%. If all of this balance is a net increase in the demand account balace which the borrower would normally maintain, what is the equivalent rate paid by the borrower ?

Solution:

On an annual basis he is paying $8000 interest for the use of $80,000. This is equivalent to a rate of 10%.

Example 5:

If the borrower in Example 4 normally keeps a demand deposit balance of $5000, the required balance would result in what equivalent rate ?

Solution:

He is still paying $8000 interest on an annual basis. He now takes $15,000 out of the loan to get the required $20,000 compensating balance. In effect he is paying $8000 for the use of $85,000. On an annual basis the rate is 8000/85,000 = .094 = 9.4%.

Example 6:

Builders of homes, apartment houses, and commercial building often get construction loans. Construction loan funds are not supplied at one time but rather are advanced gradually as construction progresses. The loan may be repaid in a lump sum shortly after completion of the project.

The builder of an apartment building obtained an $800,000 construction loan at an annual rate of 9%. The money was advanced as follows.

March	1, 1973	$300,000
June	1, 1973	$200,000
October	1, 1973	$200,000
December	1, 1973	$100,000

The building was completed in February of 1974 and the loan repaid on March 1, 1974. Find the amount using ordinary interest and approximate time.

Solution:

Interest on each part of the loan is

$$300{,}000 \times .09 \times 1 = \$27{,}000$$

$$200{,}000 \times .09 \times \frac{9}{12} = 13{,}500$$

$$200{,}000 \times .09 \times \frac{5}{12} = 7{,}500$$

$$100{,}000 \times .09 \times \frac{3}{12} = \underline{2{,}250}$$

$$\text{Total interest} \quad \$50{,}250$$

$$\text{Amount of loan} = 800{,}000 + 50{,}250 = \$850{,}250$$

EXERCISE 1 - 2

1. Find the ordinary and exact interest on \$750 for 100 days at 3%

2. Find the ordinary and exact interest on \$6080.50 for 60 days at 3¼ %.

3. Find the ordinary and exact interest on \$1200 for 45 days at 4½ %.

4. If P = \$9800 and r = $2\frac{1}{8}\%$, get the ordinary and exact interest for a 120 - day loan.

5. Use exact time and find the ordinary and exact interest on \$300 at 5% from May 5, 1974, to September 12, 1974.

6. Use exact time and find the ordinary and exact interest on \$500 from November 30, 1974, to March 15, 1975, using a rate of 7½%.

Use Bankers' Rule in Problems 7 to 12

7. What amount must be repaid on November 21, 1973, if \$7000 is borrowed at 7% on November 1, 1973?

8. A man borrows \$7760 on December 15, 1974. He repays the debt on March 3, 1975 with interest at 6½ %. Find the amount repaid.

9. A debt of \$500 is due on June 15, 1973. After that date the borrower is required to pay 5% interest. If the debt is settled on January 10, 1974, what must be repaid?

10. On May 4, 1973, a man borrows \$1850 which he promises to repay in 120 days with interest at 3%. If he does not pay on time his contract requires him to pay 8% on the unpaid amount for the time after the due date. Determine how much he must pay to settle the debt on December 15, 1973.

11. A man borrows \$3050 on December 15, 1971, at 3¼ %. What amount must he repay on April 8, 1972? Note that for leap years, the number of the day after February 28 is one more than the number in Table 1.

12. A man borrows \$5000 on November 11, 1971, at 5½ %. What amount must he repay on March 10, 1972?

13. On December 31 a man has \$3000 in his account in a savings and loan association. His money will earn dividends at 4½ % if it is left on deposit until the next interest date which is June 30. Money withdrawn before June 30 does not get any interest. On May 1 the man needs \$1000. Instead of drawing the money out of his account, he makes a passbook loan using his passbook as security. He plans to repay this loan from his account on June 30 when he gets his interest. If the savings and loan association charges 6% for a passbook loan, this plan will save the man how much money as of June 30? The savings and loan association uses Bankers' Rule in getting interest on passbook loans.

14. Get the answer to Problem 13 if the loan is for \$1500 and is made on March 31.

15. A merchant receives an invoice for \$2000 with terms 2/10, *n*/60. If he pays on the 10th, he will earn what rate of interest?

(14.7%)

16. An invoice for \$5000 has terms 3/10, *n*/45. What rate of interest is earned if payment is made on the 10th?

1.6 Present Value at Simple Interest.

To get the amount of a principal invested at simple interest, we use the formula, $S = P(1 + rt)$. If the amount is known and we want to get the principal, we solve the formula for P.

$$P = \frac{S}{1 + rt} \quad \text{.................................}(3)$$

Example:

If money is worth 5% what is the present value of \$105 due in one year?

Solution:

Substituting $S = 105$, $r = .05$, and $t = 1$ in (3)

$$P = \frac{105}{1 + .05 \times 1} = \$100.00$$

This means that \$100 invested now at 5% should amount to \$105 in a year. Substituting in the amount formula verifies this.

$$S = 100(1 + .05 \times 1) = \$105.00$$

Getting the present value of a sum due in the future in called *discounting.* When the simple interest formula is used to get the present value, the difference between the amount and present value is called *simple discount.* Note that the simple discount on the future amount is the same as the simple interest on the principal or present value.

EXERCISE 1 - 3

1. What is the present value of \$1500 due in 9 month if money is worth 4%?

2. At an interest rate of 4½ % what is the present value of \$4300 due in 3 months?

3. At 6% interest what is the present value of \$600 due in 6 months?

4. What is the present value of \$600 due in 6 months at:
   (a) 2%, (b) 3%, (c) 4%?

5. What is the present value of \$100 due in 1 year at:
   (a) 4%, (b) 5%, (c) 6%?

6. At 5% find the present value of \$2000 due in: (a) 3 months; (b) 6 months; (c) 9 months.

7. A man can get a building lot for \$3000 cash or \$3100 in one year. He has the cash but can invest it at 4%. Which is more advantageous to him and by how much now?

8. If a person can earn 4½% on his money, is it better to pay \$1990 cash for an item or to pay \$2090 in a year? Give the cash equivalent of the savings resulting from adopting the better plan?

9. A man may discharge an obligation by paying either \$200 now or \$208 in 6 months. If money is worth 4% to the man, what is the cash equivalent of choosing the better plan?

10. A man can settle a debt by paying either \$1475 now or \$1500 in 3 months. If the man can earn 4% on his money, which plan is more advantageous and by how much now?

CHAPTER 2

COMPOUND INTEREST

When interest is periodically added to the principal and this new sum is used as the principal for the following time period and this procedure is repeated for a certain number of periods, the final amount is called compound amount.

P	$F_1 = P_2$	$F_2 = P_3$	$F_3 = P_4$	$F_{n-1} = P_n$	Fn
0	1	2	3	n-1	n

2.1 Derivation of the Compound Formula.

Let P = Principal amount

F_1 = Amount at the end of period one

F_2 = Amount at the end of period two

F_3 = Amount at the end of period three

F_n = Amount at the end of period 'n'

$F = P(1 + rn)$

$F_1 = P(1 + r)$. This F_1 becomes the principal amount of period two

$\therefore F_2 = P(1 + rn)$

$= P_2(1 + r) = P(1 + r)(1 + r) = P(1 + r)^2$. Thus F_2 becomes the principal amount of period three

$F_3 = P(1 + rn)$

$= P_3(1 + r) = P(1 + r)^2 (1 + r) = P(1 + r)^3$. Thus F_{n-1} is the principal amount of period n.

$F_n = P(1 + rn)$

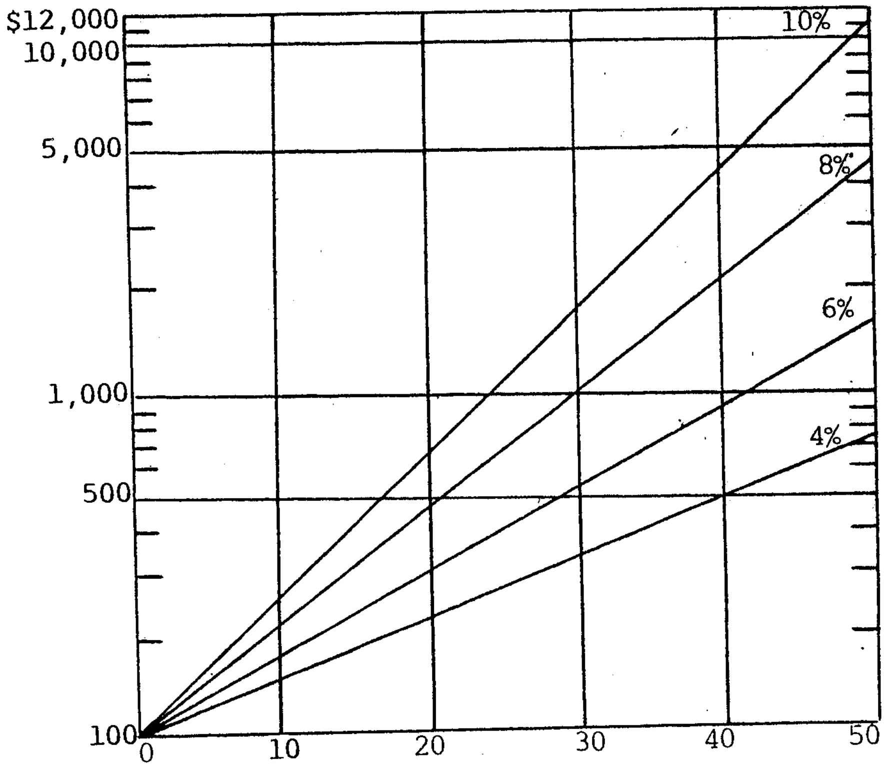
$= P_n(1 + r) = P(1 + r)^{n-1} (1 + r)$

$$F_n = P(1 + r)^n \qquad (3)$$

In calculations of compound interest the interest for an interest period is calculated of the principal plus the total amount of interest accumulated in previous periods. Thus compound interest means 'interest on top of interest

Applications of compound interest go far beyond keeping track of bank accounts. It is used for business and government planning, gauging the health of the economy, making judgments about the growth of the economy of a nation relative to other national economics and for many applications.

Growth of $100 at Compound Interest Ratio Scale



Therefore, compound interest is simple interest applied over and over to a sum which is increased by the simple interest each time it is earned.

Example 1:

An investment of £20,000 at 10% three years find:
a) the simple interest earned
b) total interest if the interest rate is 10% compounded annually.

Solution:

a) Simple Interest:

$$P = ₵\ 20{,}000 \qquad r = 10\% = \frac{1}{10} \qquad n = 3 \text{ years}$$

$$I = Prt = 2{,}000 \times \frac{1}{10} \times 3 = ₵\ 6{,}000 \quad \underline{\text{Ans}}$$

b) Compound Interest:

$$P = ₵\ 20{,}000 \qquad r = \frac{1}{10} \qquad n = 3 \text{ years}$$

$$F = P(1 + r)^n = 20{,}000\left(1 + \frac{1}{10}\right)^3 = 20{,}000(1.10)^3 = ₵\ 26{,}620$$

$$I = F - P = 26{,}620 = 20{,}000 = ₵\ 6{,}620 \quad \underline{\text{Ans}}$$

In many business transactions the interest is compounded annually, semiannually, quarterly, monthly, daily, or at some other time interval. The time between successive interest computations is called the *Conversion or Interest Period.*

Let i = interest rate per conversion period
n = total number of conversion periods

In most business transactions the practice is to quote an annual interest rate and the frequency of conversion. From this information the rate per period is determined.

Example 2:

6% compounded semiannually means that 3% interest will be earned every six months.

– The quoted annual rate (6%) is called the *nominal rate.*

$$F = P(1+i)^n$$

F = the amount at compound interest
P = the principal
i = the rate per conversion period
n = the number of conversion periods

The factor $(1+i)^n$ is called the *accumulation factor or Amount of 1 or single-payment compound-amount factor.*

Example 3:

Find the compound amounts of ₱250,000 invested at 8% for 5 years.

Solution:

$$P = ₱250{,}000 \qquad F = P(1+r)^t$$

$$r = 8\% \qquad = 250{,}000\left(1 + \frac{8}{100}\right)^5$$

$$= \frac{8}{100} \qquad = 250{,}000(1.469328)$$

$$t = 5 \text{ years} \qquad = ₱367{,}332 \qquad \underline{\text{Ans}}$$

$$F = \text{to be found out}$$

Example 4:

A principal of ₿10,000 is deposited at 6% for 10 years. What will be the compound amount and the compound interest if the interest is compounded annually, semiannually, quarterly, monthly?

Solution:

$$P = ₿10{,}000$$
$$r = 6\%$$
$$t = 10 \text{ years}$$

Frequency of Compounding	Rate per period i	Number of Conversion	Amount of 1. $(1 + i)^n$	Compound amount F.	Compound interest
1.Annually	0.06	10	1.790848	17908.48	7908.48
2. Semiannually	0.03	20	1.806111	18061.11	8061.11
3.Quarterly	0.015	40	1.814018	18140.18	8140.18
4.Monthly	0.005	120	1.819397	18193.97	8193.93

Remark:

As the frequency of conversion is increased, interest is added to principal more often, so the depositor has a larger amount to his credit.

Example 5:

A depositor planned to leave ₿40,000 in a savings and loan association paying 4½% compounded semiannually for a period of 5 years. However, at the end of 2½ years he had to withdraw ₿20,000. What will he have in his account at the end of the original 5 year period?

Solution:

At the end of 2½ years, the amount in the account is

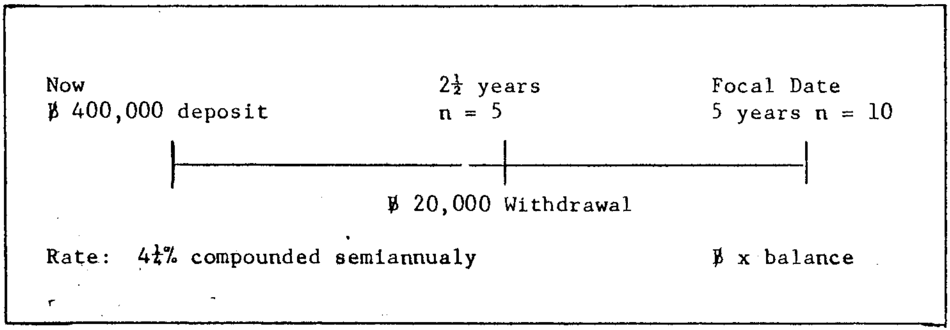
$$F = P(1 + i)^n$$

F	=	to be found out	F	=	$40{,}000\ (1 + 0.0225)^5$
P	=	₱ 40,000		=	40,000 (1.117678)
i	=	$2\frac{1}{4}\%$		=	₱ 44,707.12
n	=	5 periods			

After withdrawing ₱20,000 he has a remainder of ₱24,707.12. During the remainder of the 5 years this will grow to

$$\begin{aligned} F &= P(1 + i)^n \\ &= 24707.12\ (1 + 0.0225) \\ &= 24707.12\ (1.117678) \\ &= ₱\ 27{,}614.60 \end{aligned}$$

<u>Ans</u>



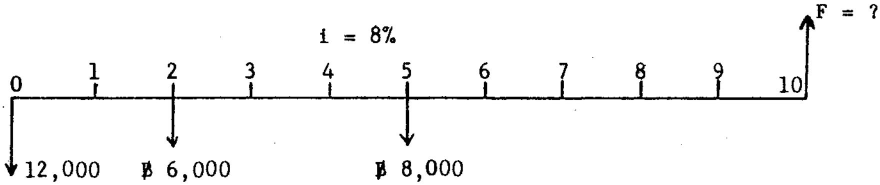
2.2 Cash-Flow Diagrams or a time diagram:/ Present and future value

A cash-flow diagram is simply a graphical representation of cash flows drawn on a time scale. The diagram should represent the statement of the problem and should include what is given and what is to be found. That is, after the cash-flow diagram has been drawn, an outside observer should be able to work the problem by looking at the cash-flow diagram. Time zero is considered to be the present and time 1 the end of time period 1. The direction of the arrows on the cash-flow diagram is important to problem solution. Therefore, we use a vertical arrow pointing up to indicate a positive cash flow.

Conversely, an arrow pointing down indicates a negative cash flow.

Example 1:

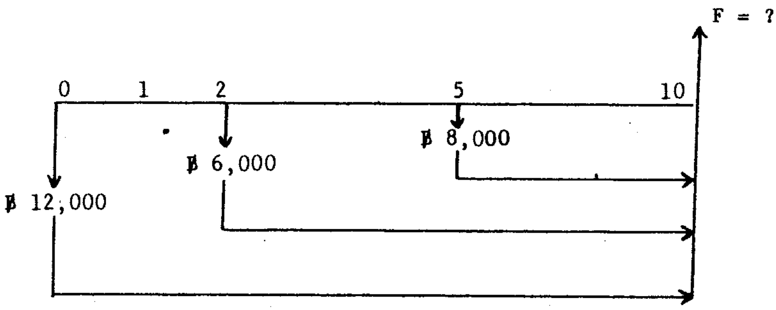
If a woman deposits ฿12,000 now, ฿6,000 two years from now and B8,000 five years from now, how much will she have in her account ten years from now if the interest rate is 8%?



Solution:

This is a problem for a future value. The first step is to draw the cash-flow diagram

Method 1:



Find the future worth of each amount and then sum them up.

AMOUNT ฿ 12,000:

$i = 8\% = 0.08$

$n = 10$ years

$$\therefore F = P(1 + i)^n$$
$$= 12000(1 + .08)^{10}$$
$$= 12000(2.158925)$$
$$= 25907.10$$

AMOUNT ฿ 6,000:

$i = 8\%$ $\quad\therefore F = 6000(1.08)^8$

$n = 8$ years $\quad = 6000(1.850930)$

(starting year 2) $\quad = 11105.58$

AMOUNT ฿ 8,000

$i = 8\%$ $\quad\therefore F = 8000(1.08)^5$

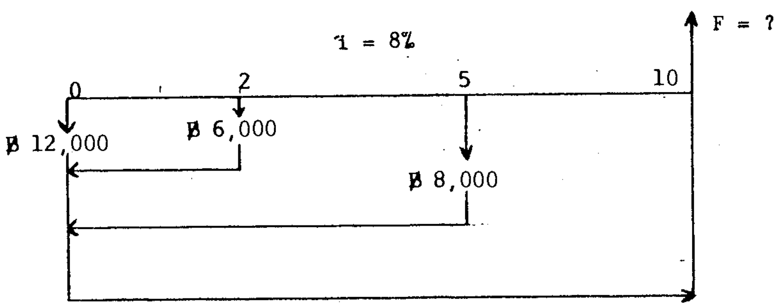
$n = 5$ years $\quad = 8000(1.469328)$

(starting year 5) $\quad = 11754.62$

$\therefore$ Total F = 25907.10 + 11105.58 + 11754.62

= ฿ 48767.30 Ans

Method 2:



The problem could also be solved by finding the present worth in year zero of the ฿6,000 and ฿8,000 deposits and then finding the future worth (year 10) of the total.

Present worth of ฿ 6,000 is: $P = F(1 + i)^{-n}$

$= 6000(1.08)^{-2}$

$= 6000(0.857339)$

$=$ ฿ 5,144.03

Present worth of ฿ 8,000 is: $P = F(1 + i)^{-n}$

$= 8000(1.08)^{-5}$

$= 8000(0.680583)$

$= 5444.66$

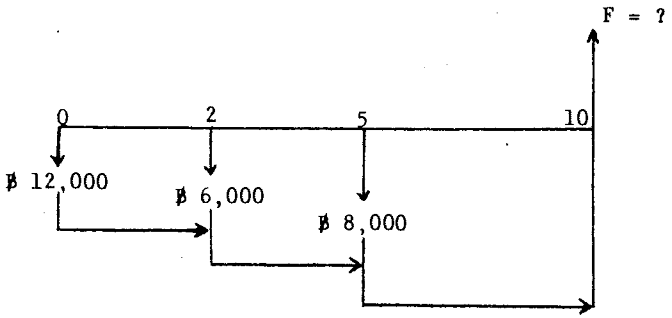
$\therefore$ Total present worth (at year 0) is

$= ₱\ 12{,}000 + 5{,}444.66 + 5{,}144.03 = 22{,}588.69$

$\therefore$ Total future worth (at year 10) is

$$F = P(1+i)^n$$
$$= 22588.69(1.08)^{10} = 22588.69(2.158925)$$
$$= ₱\ 48{,}767.29 \qquad \underline{\text{Ans}}$$

Method 3:



First find the future worth of ₱12,000 at year 2, then sum it up with ₱6,000. With this amount, find its future worth at year 5, and add it with ₱8,000. The last step is to find the future worth of this amount at year 10.

Future worth of ₱ 12,000 at year 2.

$$F = P(1+i)^n = ₱\ 12{,}000(1.08)^2 = 12000(1.166400)$$
$$= ₱\ 13{,}996.80$$

At year 2 the total amount is ₱ 6,000 + 13996.80

$$= ₱\ 19{,}996.80$$

Future worth of ₱ 19,996.80 at year 5

$$F = P(1+i)^n = 19996.80(1.08)^3$$
$$= 19996.80(1.259712)$$
$$= 25190.21$$

At year 5 the total amount is ₱ 8,000 + 25190.21

$$= ₱\ 33{,}190.21$$

Future worth of ฿ 33,190.21 at year 10

$$F = P(1+i)^n = 33190.21(1.08)^5$$
$$= 33190.21(1.469328)$$
$$= ฿\ 48{,}767.30 \qquad \underline{\text{Ans}}$$

Remark:

There are a number of ways the problem could be solved. All answers should be the same, except for round-off error.

2.3 Use of Interest Tables.

To avoid the cumbersome task of writing out the formulas each time one of the factors is used, a standard notation has been adopted which represents the various factors. This standard notation, which also includes the interest rate and the number of periods, will always be of the general form (x/y, i%, n).

Where as: x represents what you 'want to find'
y represents what is 'given'
i represents the interest rate in percent
n represents the number of periods involved.

For Example:

(F/P, 6% 20) means find future value F when principal amount P is accumulated for 20 periods and the interest rate is 6%.

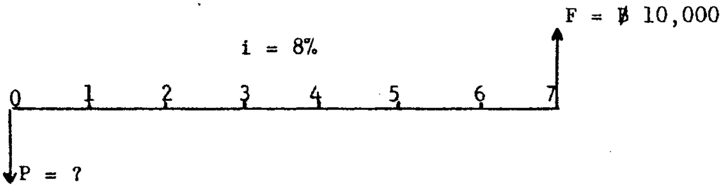
Computations using Standard Notation

To find	Given	Factor	Formula
P	F	(P/F, i%, n)	P = F(P/F, i%, n)
F	P	(F/P, i%, n)	F = P(F/P, i%, n)

Example 1:

How much money would you be willing to spend now in order to avoid spending ₱10,000 seven years from now if the interest rate is 8%?

Solution:



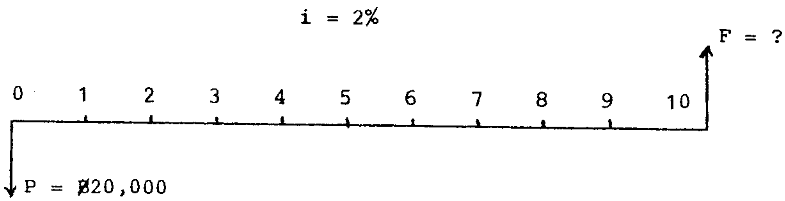
F = ₱ 10,000
P = ?
i = 8%
n = 7 years

P = F(P/F, i%, n)
= 10,000(P/F, 8%, 7)
= 10,000(0.583490)
= ₱ 5,834.90 Ans

Example 2:

Find the compound amount of ₱20,000 invested for 10 years at 8% compounded quarterly.

Solution:



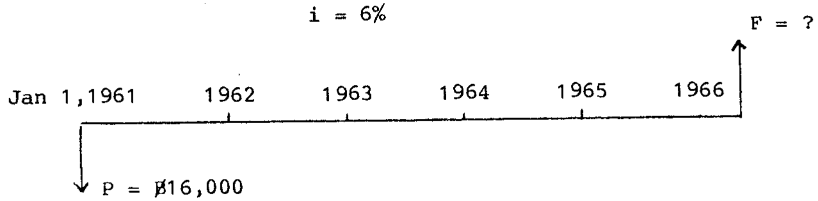
P = ₱20,000
F = ?
i = $\frac{8}{4}$% = 2% /period
n = 10 × 4 = 40 periods

F = P(F/P, i%, n)
= 20,000(F/P, 2%, 40)
= 20,000(2.208040)
= ₱44,160.80 Ans

Example 3:

On January 1, 1961 we deposit ฿16,000 in a saving bank which pays 6% per annum, and the interest is computed and added to the principal yearly on the 1^{st} of January. How much will the ฿16,000 have grown by the 1^{st} of January, 1966?

Solution:



P	=	฿16,000	F	=	P(F/P, i%, n)
F	=	?		=	16,000(F/P, 6%, 5)
i	=	6%		=	16,000(1.338226)
n	=	5 years		=	฿21,411.62 <u>Ans</u>

EXERCISE 2 - 1

1. Find the compound amount and the compound interest if \$1000 is invested for 10 years at 2% ?

2. Find the compound amount and the compound interest if \$24,500 is borrowed for 3 years at 5% converted monthly.

3. The day a boy was born, his father invested \$200 at 3½% compounded semiannually. Find the value of the fund on the boy's 18th birthday.

4. On a girl's 8th birthday her parents placed \$250 in her name in an investment paying 3½% compounded semiannually. How much will she have to her credit on her 21st birthday?

5. A town increased in population 2% a year during the period 1950 to 1960. If the population was 18,000 in 1960, what is the estimated population to the nearest hundred for 1970, assuming that the rate of growth remains the same?

6. The sales of a business have been increasing at the rate of 3% a year. If the sales in 1968 are $250,000, what are the estimated sales to the nearest thousand dollars for 1973?

7. Find the amount of $6000 for 8 years at 4% nominal, compounded: (a) annually; (b) semiannually; (c) quarterly; (d) monthly.

8. Find the amount of $10,000 for 10 years at 6% compounded: (a) annually; (b) semiannually; (c) quarterly; (d) monthly.

9. What amount of money will be required to repay a loan of $6000 on December 31, 1974, if the loan was made on December 31, 1968, at a rate of 5% compounded semiannually?

10. On June 1, 1965, a man incurred a debt of $3000 which was to be repaid on demand of the lender with interest at 3% converted semiannually. If the lender demanded payment on December 1, 1973, how much would he receive?

11. On June 30, 1974, a man put $15,000 in a deferred savings account paying 6% converted quarterly. Find the amount in the account when it matures on June 30, 1980.

12. As part of her retirement program, a woman put $12,000 in a deferred savings account paying 5% converted quarterly. The investment was made on her 57th birthday on August 15, 1973. What will be the maturity value of this account if it matures on her 62nd birthday?

13. On June 30, 1966, a man deposited $850 in a building and loan paying 4½% compounded semiannually. How much will he get if he draws out his money on June 30, 1973?

14. What amount of money will be required to repay a loan of $1835.50 on July 1, 1973, if the loan was made on October 1, 1969, at an interest rate of 5% compounded quarterly?

15. A person put $2500 in a savings and loan association paying 4% converted quarterly. He planned to leave the money there 6 years. and then use it for a trip. At the end of 2 years he had to withdraw $500. What was the amount in his account at the end of the original 6-year period?

16. What would be the answer to Problem 15 if the $500 withdrawal was made at the end of 4 years?

17. On July 1, 1969, a man put $5000 in a savings and loan association paying 4% converted semiannually. On January 2, 1971, he withdrew $2500 from his account. What was the balance in his account on January 2, 1974?

18. Interest dates for a bank are May 1 and November 1. Interest on savings accounts is 4% converted semiannually. A depositor opens an account on May 1, 1970, with a deposit of $1500. He withdraws $500 on November 1, 1971, and deposits $1000 on May 1, 1973. What is the balance in his account on May 1, 1974?

19. During the period 1965 to 1970 the annual earnings per share of Standard Oil Company of California increased about 5% a year. If the earnings in 1970 were $5.36, the same rate of increase would result in what predicted per share earnings in 1975?

20. During the period 1967 to 1971 the annual earnings per share of The Colorox Company increased about 6% a year. If the earnings were $1.68 in 1971, the same rate of increase would result in what predicted per share earnings in 1975?

21. On June 30, 1967, Charles Moser borrowed $3000 at 4% converted semiannually. How much would he have to repay on September 28, 1974. Use Bankers' Rule for simple interest computation.

22. Find the amount of \$750 for 4½ years at 5% effective.

23. If \$6000 in borrowed for 5 years and 4 months at 4½% converted semiannually, what amount would be required to repay the debt?

24. What is the amount of \$40,000 for 6 years and 3 months at 4½% converted semiannually?

25. To what sum of money does \$2000 accumulate in 3 years and 5 months at 6% compounded semiannually?

26. On June 1, 1970, a debt of \$4000 was incurred at a rate of 6% What amount will be required to settle the debt on September 15, 1974? Use Bankers' Rule for simple interest computation.

27. A savings and loan association advertises 'instant interest'. Funds received by the 10th of the month earn interest from the 1st. Interest is paid at the rate of 4% converted quarterly. Interest dates are March 31, June 30, September 30, and December 31. Interest is paid to the date funds are with-drawn. A man deposits \$2000 on January 7, 1972. If he closes out his account on Janaury 30, 1974, how much will he get? Allow simple interest for 1 month.

28. Another depositor put \$5000 on April 8, 1970, in the savings and loan association in Problem 27. He closed out his account on November 30, 1973. What was the balance at that time? Allow simple interest for 2 months.

29. Four thousand dollars was invested for 12 years. During the first 5 years the interest rate was 5% converted semiannually. The rate then dropped to 4½% converted semiannually for the remainder of the time. What was the final amount?

30. A principal of \$6500 earns 4% effective for 3 years and then 3½% compounded semiannually for 4 more years. What is the amount at the end of the 7 years?

31. William H. Maguire bequeathed \$400,000 to a university for the construction of a science building. The university got 4% on this investment for 9 years. The rate then dropped to 3½%. If the building was built 25 years after the gift was received, how much was in the fund at that time?

32. An investment of \$3000 earns 3% for 2 years, then 2½% compounded semiannually for 4 more years, then 4% compounded semiannually for 2 years. Get the amount at the end of the 8 years.

2.4 Continuous Compounding.

Interest is costomarily expressed on an annual basis.

If r is the annual interest rate
t is the number of years and
m is the frequency per year of compounding

The compound interest formula becomes:

$$F = P\left(1 + \frac{r}{m}\right)^{mt}$$

Certain assets earn on a continuous basis throughout the year. For example, capital equipment that is used in the manufacture of daily output contributes a continuous flow of earnings to the firm. The rate of return of this equipment is somewhat distorted if it is assumed that the return occurs only once each year. The actual rate of return should be calculated by incorporating the continuous flow of earnings into the formula.

The compound amount formula can be modified to incorporate continuous compounding. We must determine the effect of 'm' approaching infinity on the compound amount formula.

In calculus, it is shown that as m becomes infinite, the expression $(1 + \frac{1}{m})^m$ approaches a limit. This limit, designated by the symbol, e, has been calculated to thousands of decimal places. Thus

$$e = \lim (1 + \frac{1}{m})^m = 2.7182818285$$

Therefore, the formula for the amount for continuous compounding reduces to:

$$\boxed{F = P e^{rt}}$$

Example 1:

Assume the ₱200,000 is invested for 10 years at an annual interest rate of 6 percent. Compare the amount if the principal is compounded continuously and if it is compounded annually.

SOLUTION:

For continuous compounding:

$$\begin{aligned} F = Pe^{rt} &= 200{,}000e^{(0.06)10} \\ &= 200{,}000e^{0.6} \\ &= 200{,}000(1.822) \\ &= ₱364{,}400 \end{aligned}$$

For annual compounding:

$$\begin{aligned} F &= P(1 + r)^t \\ F &= 200{,}000(1.06)^{10} \\ &= 200{,}000(1.790848) \\ &= ₱358{,}169.60 \end{aligned}$$

∴ The difference over the 10-year period between continuous compounding and annual compounding is ₱6,230.40 <u>Ans</u>

Example 2:

If ₿2,000 is invested at an annual rate of 5 percent compounded continuously, find the compound amount at the end of (a) 1 year and (b) 5 years.

Solution:

a) Here $P = ₿2{,}000$, $r = 0.05$, $t = 1$

$$
\begin{aligned}
F &= Pe^{rt} \\
&= 2000e^{(0.05)(1)} \\
&\approx 2000(1.0513) \\
&= ₿2102.60 \qquad \underline{\text{Ans}}
\end{aligned}
$$

b) $P = ₿2{,}000$, $r = 0.05$, $t = 5$ years

$$
\begin{aligned}
F &= Pe^{rt} \\
&= 2000e^{(0.05)(5)} \\
&\approx 2000(1.2840) \\
&= ₿2568 \qquad \underline{\text{Ans}}
\end{aligned}
$$

Example 3:

A trust fund is being set up by a single payment so that at the end of 20 years there will be ₿50,000 in the fund. If interest is compounded continuously at an annual rate of 7 percent, how much money should be paid into the fund initially?

Solution:

Here $F = ₿500{,}000$, $r = 7$ percent, $t = 20$ years.

$$
\begin{aligned}
F &= Pe^{rt} \\
\text{or } P &= Fe^{-rt} \\
&= 500{,}000e^{-(0.07)(20)} \\
&\approx 500{,}000(0.24660) \\
&= ₿123300
\end{aligned}
$$

The present value is ₿123300 $\underline{\text{Ans}}$

EXERCISE: 2 - 2

In Problems 1 and 2, find the compound amount and compound interest if $4000 is invested for six years and interest is compounded continuously at the given annual rate.

1. 5 ½ %.
2. 9 %.

In Problems 3 and 4, find the present value of $2500 due eight years from now if interest is compounded continuously at the given annual rate.

3. $6\frac{3}{4}$ %.
4. 8 %.

5. If $100 is deposited in a savings account that earns interest at an annual rate of 5½ percent compounded continuously, what is the value of the account at the end of two years?

6. If $1000 is invested at an annual rate of 6 percent compounded continuously, find the compound amount at the end of eight years.

7. The board of directors of a corporation agrees to redeem some of its callable preferred stock in five years. At that time $1,000,000 will be required. If the corporation can invest money at an annual interest rate of 6 percent compounded continuously, how much should it presently invest so that the future value is sufficient to redeem the shares?

8. A trust fund is being set up by a single payment so that at the end of 30 years there will be $50,000 in the fund. If interest is compounded continuously at an annual rate of 5 percent, how much money should be paid into the fund initially?

9. If interest is compounded continuously at an annual rate of .05, how many years would it take for a principal P to triple? Give your answer to the nearest year.

10. If interest is compounded continuously, at what annual rate will a principal of P double in ten years? Give your answer to the nearest percent.

2.5 Effective Interest Rates:

The effective rate is the rate converted annually that will produce the same amount of interest per year as the nominal rate converted n times per year. We can determine the effective rate by putting different rates and frequencies of conversion on a comparable basis.

If the nominal rate is 6% converted annually, the effective rate will also be 6%. But if the nominal rate is 6% converted semiannually, the amount of ₱1 at the end of one year will be $(1.03)^2$ = ₱1.0609. This is simply the accumulation factor for a rate per period of 3% and two periods.

The interest on ₱1 for one year is then = ₱1.0609 – ₱1
= ₱0.0609

This is equivalent to an annual rate of = 6.09%

Thus 6.09% converted annually would result in the same amount of interest as 6% converted semiannually. The computations used to get the effective rate of 6.09% in this case can be summarized as follows:

$$
\begin{aligned}
1.0609 &= (1.03)^2 \\
\text{effective rate} &= 1.0609 - 1 \\
&= 0.0609 \\
&= 6.09\%
\end{aligned}
$$

Putting the above relationship in the form of a general equation we have

$$1 + \text{effective rate} = (1 + i)^n$$

$$\boxed{\textit{effective rate} = (1 + i)^n - 1}$$

where:

i = the rate per conversion period $= \frac{r}{n}$
n = frequency of conversion.

For the effective rate which corresponds to an annual rate of 'r' compounded continuously is

$$\boxed{\text{effective rate} = e^r - 1}$$

Note:

1. The nominal rate is the quoted or stated rate. When a nominal rate is quoted, the frequency of compounding should be stated.
2. The rate per period (symbol i) equals the nominal rate divided by the number of conversion period per year.
3. The effective rate is the rate actually earned in a year.

Example 1:

Find the effective rate of interest equivalent to 8% converted semiannually.

Solution:

$$i = \frac{8\%}{2} = 4\% \qquad\qquad r = (1.04)^2 - 1$$

$$n = 2 \qquad\qquad = 1.0816 - 1$$

$$= 0.0816$$

$$= 8.16\%$$

Thus 8.16% compounded annually will produce the same amount of interest as 8% compounded semiannually. Ans

Example 2:

To what amount will £12,000 accumulate in 15 years if invested at an effective rate of 5 percent?

Solution:

Since an effective rate is the actual rate conpounded annually, we have

$$F = 12{,}000(1.05)^{15}$$
$$\approx 12{,}000(2.078928)$$
$$\approx \$24{,}947.14 \qquad \underline{\text{Ans}}$$

Example 3:

Find the effective rate which corresponds to an annual rate of 5 percent compounded continuously.

Solution:

$$\text{The effective rate is } = e^{r} - 1$$
$$= e^{0.05} - 1$$
$$\approx 1.0513 - 1$$
$$= 0.0513$$
$$\text{or } = 5.13\% \qquad \underline{\text{Ans}}$$

EXERCISE 2 - 3

1. What is the effective rate of interest equivalent to 10% converted (a) semiannually; (b) quarterly?

2. What is the effective rate of interest equivalent to 7% converted. (a) semiannually; (b) quarterly?

3. Which gives the better annual return on an investment, $6\frac{1}{8}$ % converted annually or 6% converted quarterly? Show the figures on which you base your answer.

4. Which rate of interest gives the better annual yield, 4% compounded monthly or 4½% compounded semiannually? Show the figures on which you base your answer.

5. A savings and loan association in California advertised that they are paying 5.25% converted daily. According to the ad, this is equivalent to an effective rate of 5.39%. An investor in Ohio has his money in an account paying 5% converted quarterly. If the Ohio investor transfers $10,000 to the California association, he will get about how much additional interest a year?

6. A company has $50,000 to invest. What would be the difference in interest at the end of a year between earning 5% converted semiannually and 5% converted quarterly?

7. Complete the following table showing the effect of frequency of compounding on effective interest rates. Use the 3% values as a check on your work. Answers not available in this text have been obtained from more extensive interest tables.

Nominal	Effective Rate if Compounded:		
Annual Rate	Semiannually	Quarterly	Monthly
3%	3.02%	3.03%	3.04%
6			
9	9.20		
12			
15	15.56	15.87	
18	18.81	19.25	
21	22.10	22.71	
24	25.44		
27	28.82	29.86	
30	32.25	33.55	

8. A $6,000 certificate of deposit is purchased for $6,000 and is held 7 years. If the certificate earns 8 percent compounded quarterly, what is it worth at the end of that period?

9. How many years will it take for a principal of P to triple at the effective rate of r?

10. A major credit-card company has a finance charge of $1\frac{1}{2}$ percent per month on the outstanding indebtedness. (a) What is the nominal rate compounded monthly? (b) What is the effective rate?

11. How long would it take for a principal of P to double if money is worth 12 percent compounded monthly? Give your answer to the nearest month.

12. To what sum will $2,000 amount in 8 years if invested at a 6 percent effective rate for the first 4 years and 6 percent compounded semiannually thereafter?

13. How long will it take for $500 to amount to $700 if it is invested at 8 percent compounded quarterly?

14. An investor has a choice of investing a sum of money at 8 percent compounded annually, or at 7.8 percent compounded semi-annually. Which is the better of the two rates?

In Problems 15 - 18, find the effective rate of interest which corresponds to the given annual rate compounded continuously.

15. 4%.

16. 7%.

17. 10%.

18. 9%.

19. What annual rate compounded continuously is equivalent to an effective rate of 5 percent?

20. What annual rate r compounded continuously is equivalent to a nominal rate of 6 percent compounded semiannually?
Hint: First show that $r = 2 \ln 1.03$.

CHAPTER 3

ANNUITIES

ANNUITIES:

An annuity is a sequence of periodic payments, usually equal, made at equal intervals of time. For example, payments of rents, life insurance premiums, and installment payments.

When the payments are due or made at the ends of the payment intervals, the annuity is called *an ordinary annuity.* When the payments are due or made at the beginnings of the payment intervals, the annuity is called *an annuity due.*

Example:

Mr. A buys a television set for a cash payment of ฿12,000 followed by 12 monthly installments to ฿1,000 each, the first due 1 month after the date of sale. (Analysis: The monthly installments constitute an ordinary annuity whose term starts on the day of sale and continues for 1 year. The payment interval is 1 month).

In this chapter we shall work with the ordinary annuity which has the periodic payments made at the end of each period.

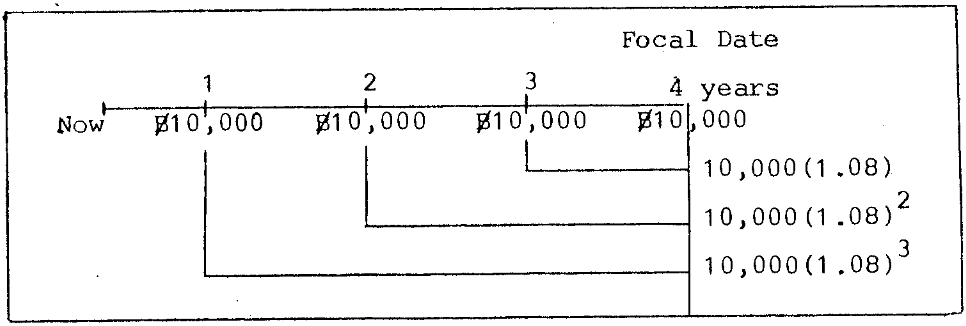
3.1 Amount of an Ordinary Annuity:

The final value of 'annuity' is the sum of all the periodic payments and the compound interest on them accumulated to the end of the term. In the case of an ordinary annuity, this will be the value of the annuity on the date of the last payment.

Example 1:

Starting 1 year from now, a man deposits ฿10,000 a year in an account paying 8% compounded annually. What amount does he have to his credit just after making the 4th deposit?

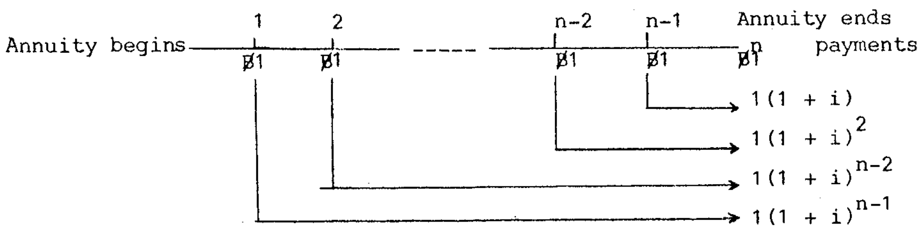
Solution:



Starting with the last payment and accumulating all of them of the focal date, we have

4^{th} payment	=	₿10,000
3^{rd} payment $10,000(1.08)^1$	=	₿10,800
2^{nd} payment $10,000(1.08)^2$	=	₿11,664
1^{st} payment $10,000(1.08)^3$	=	₿12,597.12
∴ Amount of the annuity	=	₿45,061.12 Ans

If there are many payments, the above method for getting the amount would require too much work. We now derive a general formula for the amount of an annuity of n payments of ₿1 each and a rate of i per period.



Symbol $S_{\overline{n|}}$ (read '*s sub n*' or '*s angle n*')

$S_{\overline{n|}i}$ is used to represent the amount of n payments of ₿1 each when interest rate per period is i.
(This amount can be termed as the Future Value)

$$\therefore\ s_{\overline{n}|i} = 1 + 1(1+i)^1 + 1(1+i)^2 + \ldots\ldots + 1(1+i)^{n-2} + 1(1+i)^{n-1}$$

By using the formula to find the sum of the terms in geometric progression

$$S = \frac{a(r^n - 1)}{r - 1} \; : \begin{cases} a = \text{the first term} = 1 \\ r = \text{Common ratio} = (1 + i) \end{cases}$$

$$\therefore\ s_{\overline{n}|i} = \frac{1\left[(1+i)^n - 1\right]}{(1+i) - 1} = \frac{(1+i)^n - 1}{i}$$

This factor is called 'Amount of an Annuity of 1 per period' If the periodic payment is ฿R per period instead of ฿1, we indicate the total amount of the payments by the symbol S_n.

$$\therefore\ S_n = R\, S_{\overline{n}|i} = \frac{R(1 + i)^n - 1}{i}$$

S_n = Amount of an ordinary annuity of n payments

R = periodic payment or rent

$S_{\overline{n}|}$ or $S_{\overline{n}|i}$ = Amount of 1 per period for n periods at the rate of i per period.

Example 2:

A father of a family will deposit ฿500 for his son at the end of each quarter in a saving fund that earns interest at the rate of 6%, compounded quarterly. How much will he have in the fund at the end of 10 years.

a) he has nothing in the account to-day
b) he has ฿20,000 in the fund at present?

Solution:

a) He has nothing in account to-day

$$S_n = R\, S_{\overline{n}|i}$$

$S_n = ?$ $\qquad \therefore\ S_n = 500\, S_{\overline{40}|\frac{3}{2}\%}$

$R = ₿500$ $\quad = 500(54.267894)$

$n = 4 \times 10 = 40$ periods $\quad = 27133.95$

$= ₿27{,}133.95$ Ans

b) He has ₿20,000 in the account to-day

$$\begin{aligned} \text{Total } S_n &= R\,S_{\overline{n|}i} + P(1+i)^n \\ &= 27133.95 + 20{,}000(1.015)^{40} \\ &= 27133.95 + 20{,}000(1.814018) \\ &= 27133.95 + 36280.36 \\ &= ₿63{,}414.31 \end{aligned}$$

Ans

Example 3:

Starting 1 year from now, a man deposits ₿500 a year in an account paying 4% interest compounded annually. What amount does he have to his credit just after making the 4th deposit?

Solution:

$$S_n = R\,S_{\overline{n|}i}$$

$S_n = ?$

$R = ₿500$

$n = 4$

$i = 4\%$

$$\begin{aligned} S_n &= 500\,S_{\overline{4|}4\%} \\ &= 500(4.246464) \\ &= ₿2{,}123.23 \end{aligned}$$

Ans

Example 4:

Find the amount of an ordinary annuity of ₿500 every 3 months for 12 years at 6% compounded quarterly.

Solution:

$$S_n = R\,S_{\overline{n|}i}$$

$S_n = ?$

$R = ₿500$

$n = 48$

$i = \frac{6}{4} = 1\frac{1}{2}\%$

$$\begin{aligned} S_n &= 500\,S_{\overline{48|}\frac{3}{2}\%} \\ &= 500(69.565219) \\ &= ₿34{,}782.61 \end{aligned}$$

Ans

EXERCISE: 3 - 1

1. Find the amount of an annuity of \$5000 per year for 10 years at: (a) 3%; (b) 4%. Interest is compounded annually.

2. Find the amount of an annuity of \$1200 at the end of each 6 months for 5 years if money is worth: (a) 4%; (b) 5%; (c) 6%. All rates are converted semiannually.

3. A man puts \$100 every 3 months in a savings account that pays 5% compounded quarterly. If he makes his first deposit on June 1, 1964, how much will he have in his account just after he makes his deposit on December 1, 1973?

4. Every 3 months a family puts \$50 in a savings account that pays 5% compounded quarterly. If they make the first deposit on August 1, 1964, how much will be in their account just after they make their deposit on February 1, 1974?

5. What is the amount of an annuity of \$100 at the end of each month for 6 years if money is worth 4% compounded monthly?

6. Find the amount of an annuity of \$35 at the end of each month for 12 years if money is worth 5% converted monthly.

7. To provide for his son's education, a man deposits \$150 a year at the end of each year for 18 years. If the money draws 3% interest, how much does the fund contain just after the 18th deposit is made? If no more deposits are made, but if the amount in the fund is allowed to accumulate at the same interest rate, how much will it contain in 3 more years?

8. Two hundred dollars at the end of each year for 6 years is equivalent in value to what single payment at the end of 6 years if the interest rate is 6% effective?

9. A child 12 years old received an inheritance of \$400 a year. This was to be invested and allowed to accumulate until the child reached the age of 21. If the money was invested at 4% effective and if the first payment was made on the child's 12th birthday

and the last payment on his 21st birthday, what amount did he receive when he reached the age of 21?

10. A man deposits \$125 at the end of each 3 months for 5 years in a fund that pays 5% converted quarterly. How much will he have to his credit just after the last deposit is made?

11. A man deposits \$300 at the end of each year for 3 years in an investment paying 3% converted annually. He then allows his account to accumulate for 2 more years without making any more deposits. What is the amount in his account at the end of the 5 years?

12. On March 1, 1968, a man deposited \$150 in an investment that pays 3% converted semiannually. He continues to make \$150 deposits every 6 months until September 1, 1974, when he makes his final deposit. If he lets the money continue to draw interest, how much will he have in his account on March 1, 1976?

13. A family has been paying \$75.50 a month on their home. The interest rate on the mortgage is 5% compounded monthly. Because of sickness they miss the payments due on May 1, June 1, July 1, and August 1. On September 1 they want to make a single payment which will reduce their debt to what it would have been had they made all payments on time. What single payment on this date will be equivalent to the 5 payments from May to September inclusive?

14. A dealer purchased merchandise and agreed to pay \$200 on August 1, September 1, October 1, November 1, and December 1. He was to be charged 6% converted monthly on any payments not made on time. He made the August payment, but let all the others go to December 1. At that time what payment did he have to make to settle all his obligations?

15. A person has an income of \$250 every 3 months from preferred stocks. He deposits this in a savings and loan association paying 5% converted quarterly with interest dates on March 31, June 30, September 30, and December 31. If he makes his first deposit on June 30, 1965, and his last deposit on December 31, 1969, how

much will be in his account just after he makes the last deposit?

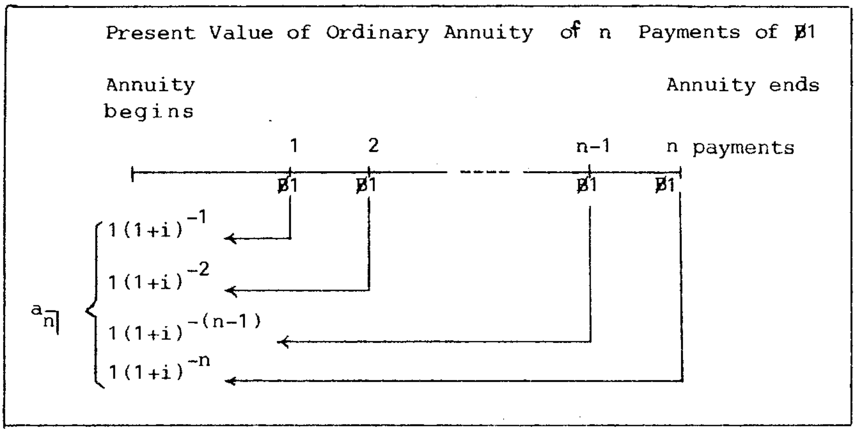
16. In 1949, a man put $500 in common stocks. He continues to make the same deposit every year. If the stocks increase in market value at about 4% a year, what will be the value of the man's portfolio just after he makes his purchase in 1966?

17. To provide for a son's education, a man opens an account with a deposit of $300 in a savings and loan association which pays 4½% converted semiannually. The first deposit is make on the boy's 6th birthday on June 23, 1960. (This deposit will earn interest from July 1.) Additional deposits of $100 each are made every 6 months. How much will be in the account on the interest date immediately following the boy's 18th birthday?

3.2 Present value of an Ordinary Annuity:

The present value of an annuity is the sum of the present values of all the payments of the annuity.

To get the present value we shall assume an annuity of n payments of ₱1 each and a rate of i per period. We then discount each payment to the beginning of the annuity. The sum of these discounted values is designated by the symbol $a_{\overline{n}|}$ or $a_{\overline{n}|i}$



$$\therefore \quad a_{\overline{n}|} = (1+i)^{-1} + (1+i)^{-2} \ldots\ldots + (1+i)^{-(n-1)} + (1+i)^{-n}$$

Using the formula to find the sum of the the terms in geometric

Progression: $S = \dfrac{a(r^n - 1)}{r - 1}$ where : $\begin{cases} a = (1+i)^{-1} \\ b = (1+i)^{-1} \end{cases}$

$$\therefore \quad a_{\overline{n}|i} = \frac{(1+i)^{-1}\left[\left((1+i)^{-1}\right)^n - 1\right]}{(1+i)^{-1} - 1}$$

Multiplying numerator and denominator by $(1 + i)$

$$\therefore \quad a_{\overline{n}|i} = \frac{(1+i)^{-n} - 1}{1 - (1+i)} = \frac{(1+i)^{-n} - 1}{1 - 1 - i}$$

$$= \frac{(1+i)^{-n} - 1}{-i} = \frac{1 - (1+i)^{-n}}{i}$$

If the periodic payment is ฿R per period instead of ฿1, we indicate the present value of the annuity by the symbol A_n

$$\therefore \quad A_n = R\, a_{\overline{n}|i} = \frac{R[1 - (1+i)^{-n}]}{i}$$

A_n = present value of an ordinary annuity of n payments

R = periodic payment

$a_{\overline{n}|}$ or $a_{\overline{n}|i}$ = Present Worth of 1 per period for n periods at the rate i per period.

Example 1:

A man wants to provide a ฿6,000 scholarship every year for 10 years. The first scholarship is to be awarded one year from now. If the school can get a 4% return on their investment, how much money should the man give now?

Solution:

$$A_n = R\, a_{\overline{n}|i}$$

A_n = ?

R = ฿6,000

n = 10

$A_n = 6000\, a_{\overline{10}|4\%}$

$= 6000(8.110896)$

$=$ ฿48,665.38 Ans

Example 2:

Mr. A agrees to pay Mr. B ฿20,000 at the end of each year for 5 years. If money is worth 8%.

a) what is the cash equivalent of this debt?

b) If Mr. A does not make any payments until the end of 5 years, how much should he pay at that time if this single payment is to be equivalent to the original payments using an interest rate of 8%?

Solution:

$$A_n = R\, a_{\overline{n}|i}$$

$A_n = ?$

$R = ฿20,000$

$n = 5$ years

$i = 8\%$

$$A_n = 20000\, a_{\overline{5}|8\%}$$
$$= 20000(3.992710)$$
$$= ฿79,854.2 \quad \underline{\text{Ans}}$$

(b) If a payment is made after it is due.

$$S_n = R\, s_{\overline{n}|i}$$

$S_n = ?$

$R = ฿20,000$

$n = 5$

$i = 8\%$

$$S_n = 20000\, s_{\overline{5}|8\%}$$
$$= 20000(5.866600)$$
$$= ฿117,332 \quad \underline{\text{Ans}}$$

Check:

Thus ฿79,854.2 now is equivalent in value to the five ฿20,000 payments at the end of each year if money is worth 8%. Likewise ฿117,332 in five years is equivalent in value to the original obligations.

$$79854.2(1.08)^5 = 79854.2(1.469328)$$
$$= ฿117,332 \quad \underline{\text{Ans}}$$

EXERCISE: 3 - 2

1. Find the present value of an ordinary annuity of \$5000 per year for 10 years at: (a) 3%; (b) 4%. All rates are converted annually. Carry the present value for the 3% rate forward 10 years. How does the accumulated amount compare with the amount of an annuity of \$5000 a year.

2. Find the present value of an ordinary annuity of \$750 at the end of each year for 8 years at: (a) 2%; (b) 4%; (c) 6%; (d) 8%.

3. Find the present value of an ordinary annuity of \$1200 at the end of each 6 months for 5 years if money is worth: (a) 4%; (b) 5%; (c) 6%. All rates are converted semiannually.

4. If money is worth 4½% converted semiannually, what is the present value of \$145.50 due at the end of each 6 months for 2 years?

5. A television set is bought for \$50 cash and \$18 a month for 12 months. What is the equivalent cash price if the interest rate is 24% converted monthly?

6. A refrigerator can be purchased for \$57.47 down and \$20 a month for 24 months. What is the equivalent cash price if the rate is 30% converted monthly?

7. Find the cash value of a car that can be bought for \$400 down and \$80 a month for 24 months if money is worth 18% converted monthly.

8. An heir receives an inheritance of \$500 every half year for 20 years, the first payment to be made in 6 months. If money is worth 4½% converted semiannnually, what is the cash value of this inheritance?

9. A contract for the purchase of a home calls for the payment of \$81.90 a month for 20 years. At the beginning of the 6th year (just after the 60th payment is made) the contract is sold to a buyer at a price which will yield 5% converted monthly. What

does the buyer pay?

10. A man wants to provide a $3000 research fellowship at the end of each year for the next 4 years. If the school can invest money at 4%, how much should the man give them now to set up a fund for the 4 scholarships?

11. A home was purchased for $3000 down and $100 a month for 15 years. If the monthly payments are based on 6% converted monthly, what was the cash price of the house?

12. An apartment was purchased for $6000 down and $500 at the end of each 6 months for 8 years. If the payments are based on 5% converted semiannually, what was the cash price of the apartment?

13. If a person can get 4% converted semiannually on his invested money, is it better for him to pay $11,500 cash or $3000 down and $1000 every 6 months for 5 years for a store building?

3.3 Extension of Tables:

In some problems the number of payment is greater than can be found directly in the tables. Such problems can be solved by dividing the annuity into parts and then accumulating or discounting the amount or present value of each part of annuity to the desired point in time.

Example 1:

Find the amount of an annuity of $2,000 at the end of each month for 30 years at 6% converted monthly.

Solution:

n = 360 payments. Divide the annuity into 2 annuities of 180 payments each.

The amount of the first 180 payments is

$$S_n = R\, s_{\overline{n}|i}$$

$$S_{180} = 2000\, s_{\overline{180}|\frac{1}{2}\%}$$

$$= 2000(290.818712)$$

$$= ₱581,637.42$$

This amount of the first 180 paymetns at the end of the term would be obtained at compound interest.

$$F = P(1+i)^n$$

$$= 581637.42(1.005)^{180}$$

$$= 581637.42(2.4540936)$$

$$= ₱1,427,392.67$$

The amount of the second 180 payments will be the same amount as the first 180 payments $= ₱581,637.42$

Total amount is $= 1,427,392.67 + 581,637.42$

$= ₱2,009,030.09$ Ans

0 1 2 180 360 Focal Date

——— first 180 periods ——→ $581637.42(1.005)^{180}$ ——→ 1,427,392.67

Amt. of annuity for second 180 periods ——→ 5,81637.42

Total Amt. = ₱2,009,030.09 Ans

Example 2:

Find the present value of an annuity of ₱100 at the end of each month for 30 years at 6% converted monthly.

Solution:

Divide the annuity into 2 annuities of 180 payments each. The present value of the first 180 payments is

$$A_n = R\, a_{\overline{n}|i}$$

$$= 100\, a_{\overline{180}|\frac{1}{2}\%} = 100(118.503515)$$

$$₱11{,}850.35$$

The present value of the second 180 payments at a point of time one period before 181 at payment is also = ₱11,850.35. Toget the value of the second 180 payments at the beginning of the term is by means of compound interest.

$$P = R(1+i)^{-n}$$

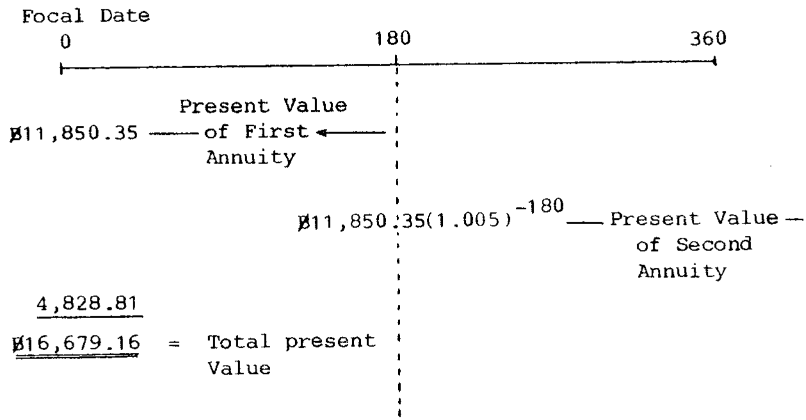
$$= 11850.35(1.005)^{-180}$$

$$= 11850.35(0.407824)$$

$$= ₱4{,}828.81$$

The total present value is $=$ ₱11,850.35 + 4,828.81

$=$ ₱16,679.16 <u>Ans</u>



EXERCISE: 3 - 3

1. Find the amount of an annuity of \$40 at the end of each year for 80 years if money is worth 3%.

2. What is the amount of an annuity of $60 at the end of each month for 30 years at a rate of 5% converted monthly?

3. Find the present value of the annuity in Problem 1.

4. Find the present value of the annuity in Problem 2.

5. A home can be purchased for $5000 down and $200 a month for 20 years. If the monthly payments are based on 5% converted monthly, find the total cash price of the house.

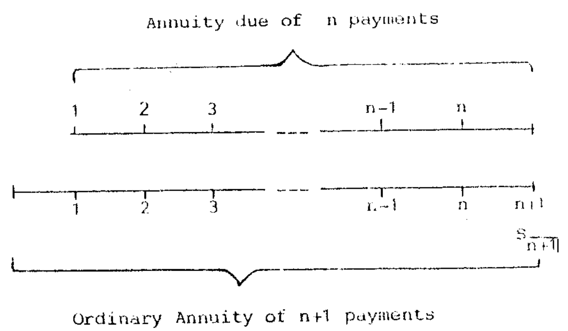
6. An apartment can be purchased for $10,000 down and $1000 every 3 months for 18 years. If the payments are based on 5% converted quarterly, find the total cash price.

3 - 4 Annuity Due:

An *annuity due* is one in which the payments are made at the beginning of the payment interval, the first payment being due at once. Insurance premiums, and property rentals are examples of annuities due.

To get the amount and present value formulas for the annuity due, we modify the formulas already derived for the ordinary annuity so that the ordinary annuity tables can be used with simple modifications.

Annuity Due of n Payments.



If we obtain the amount of an ordinary annuity of n+1 payments, we would have the amount of the corresponding annuity due except that we have included a final payment which is not actually made at the end of the last period of the annuity due. Therefore all we have to do is subtract one payment to permit us to use the ordinary annuity tables to get the amount of an annuity due.

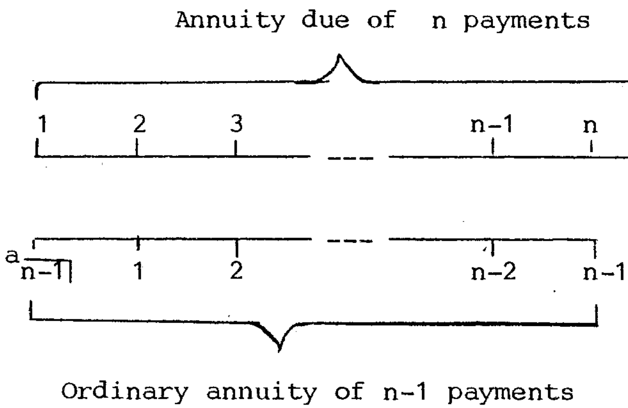
The formula is:

$$S_n = R\,S_{\overline{n+1}|} - R$$

or

$$S_n = R(S_{\overline{n+1}|} - 1)$$

3.5 Present Value of an Annuity Due:



If we obtain the present value of an ordinary annuity of n-1 payments, we would have the present value of the corresponding annuity due except that we would not be including a payment which is made at the beginning of the term of the annuity due. Therefore all we have to do is add this first payment. The formula for the present value of an annuity due is:

$$A_n = R\,a_{\overline{n-1}|} + R$$

$$A_n = R(a_{\overline{n-1}|} + 1)$$

Example 1:

A man invest ₱4,000 at the beginning of each year for 10 years. If he gets interest at 3½% effective, how much does he have to his credit at the end of 10 years?

Solution:

$$S_n = R(s_{\overline{n+1}|} - 1)$$

$R = ₱4,000$

$n+1 = 11$

$i = 3\frac{1}{2}\ \%$

$$S_{10} = 4000(s_{\overline{11}|} - 1)$$
$$= 4000(13.141992 - 1)$$
$$= 4000(12.141992)$$
$$= ₱48,567.97 \qquad \underline{\text{Ans}}$$

Example 2:

A student wants to have ₱12,000 four years from now. How much must he invest at the beginning of each year starting now if he gets 4% compounded annually on his savings?

Solution:

$$S_n = R(s_{\overline{n+1}|\,i} - 1)$$

$R = ?$

$S_4 = ₱12,000$

$i = 4\%$

$n+1 = 5$

$$\text{or} \quad R = \frac{S_n}{(s_{\overline{n+1}|\,i} - 1)} \quad \begin{matrix} n+1 & i \\ 5 & 4\% \end{matrix}$$
$$= \frac{12000}{5.416323 - 1}$$
$$= \frac{12000}{4.416323}$$
$$= ₱2,717.19 \qquad \underline{\text{Ans}}$$

Example 3:

The premium on a life insurance policy is ₱1,200 a quarter payable in advance. Find the cash equivalent of a year's payments if the insurance company charges 6% converted quarterly for the privilege of paying this way instead of all at once for the year.

Solution:

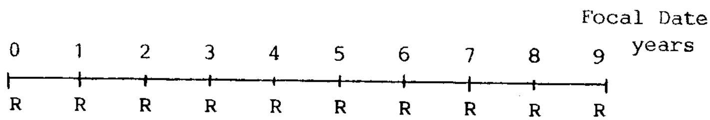
$$R = 1{,}200$$
$$n-1 = 3$$
$$i = 1\tfrac{1}{2}\ \%$$

$$A_n = R(a_{\overline{n-1}|i} + 1)$$
$$= 1200(a_{\overline{3}|1\frac{1}{2}\%} + 1)$$
$$= 1200(2.912200 + 1)$$
$$= 1200(3.9122)$$
$$= ₱4{,}694.64 \quad \underline{\text{Ans}}$$

Example 4:

The beneficiary of a life insurance policy may take ₱200,000 cash or 10 equal annual payments, the first to be made immediately. What is the annual payment if money is worth 6%?

Solution:



$$A_n = ₱200{,}000$$
$$R = ?$$
$$n-1 = 9$$

$$A_n = R(a_{\overline{n-1}|} + 1$$
$$R = \frac{200000}{a_{\overline{9}|6\%} + 1}$$
$$= \frac{200000}{6.801692 + 1}$$
$$= \frac{200000}{7.801692}$$
$$= ₱25{,}635.46 \quad \underline{\text{Ans}}$$

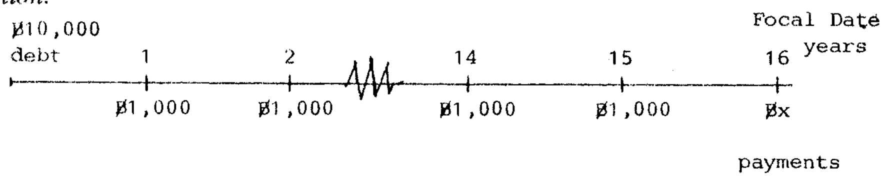
Outstanding Balance.

A common financial problem in determining the outstanding balance on a loan after several payments have been made.

Example 1:

A ₱10,000 loan is to be amortized with payments of ₱1,000 at the end *of each year and a final concluding payment. Find the number of full payments and the size of the concluding payment if the rate is 6% converted annually.*

Solution:



$$A_n = R\, a_{\overline{n}|}$$

A_n = Ƀ10,000

R = Ƀ1000

$$\therefore \quad a_{\overline{n}|} = \frac{A_n}{R} = \frac{10{,}000}{1000} = 10$$

Then look for this factor in the 'Present Worth of 1 per period' column in Table, we find that there will be 15 full payments. We get the concluding payment by taking the Ƀ10,000 forward as a single sum for 16 years and subtracting the amount of an annuity due of 15 payments.

$$10{,}000(1.06)^{16} = 10{,}000(2.5403517) = Ƀ25{,}403.52$$

$$1000(s_{\overline{15+1}|} - 1) = 1000(24.672528) = Ƀ24{,}672.53$$

$$\therefore \text{ Concluding payment } = Ƀ\ 730.99 \qquad \underline{\text{Ans}}$$

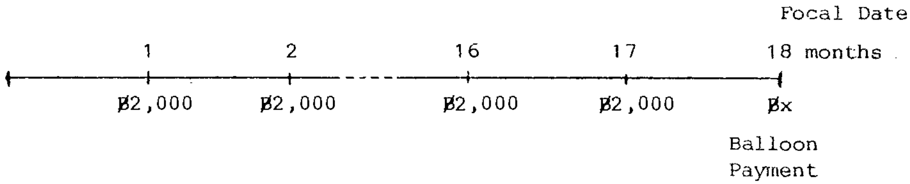
Example 2:

A Ƀ48,000 refrigerator is sold for Ƀ8,000 down, 17 payments of Ƀ2,000 a month, and an 18th Balloon payment. *If the interest rate is 24% converted monthly, find the size of the balloon payment.*

Remark:

An item is sold on time for a series of payments that the borrower can meet. At the end there is a much larger payment. This final amount to be paid is called *balloon payment.*

Solution:



Rate: 24% Converted monthly

Price of the refrigerator is	=	₱48,000
Down payment is	=	₱ 8,000
∴ Loan is	=	₱40,000

At the end of 18 months, the total amount to be paid is

$$F = ?$$
$$P = ₱40{,}000$$
$$i = \frac{24}{12}\% = 2\%$$
$$n = 18$$

$$F = P(1+i)^n$$
$$= 40000(1.02)^{18}$$
$$= 40000(1.428246)$$
$$= ₱57{,}129.84$$

By 17 payments of ₱2,000 a month, we have already paid

$$S_n = R(s_{\overline{n+1}|\, i} - 1)$$
$$= 2000(s_{\overline{17+1}|\, 2\%} - 1)$$
$$= 2000(20.412312)$$

$$= 2000(21.412312 - 1)$$
$$= ₱40{,}82462$$

∴ The remaining amount or balloon payment to be paid is

$$= ₱57{,}129.84 - 40{,}824.62$$
$$= ₱16{,}305.22 \qquad \underline{\text{Ans}}$$

EXERCISE: 3 - 4

1. If money is worth 4% converted semiannually, what is the present value of an annuity due of $500 every 6 months for 5 years?

2. At 5% converted monthly, what is the present value of an annuity due of $45 every month for 12 years?

3. What is the amount of the annuity due in Problem 1?

4. What is the amount of the annuity due in Problem 2?

5. On July 6, 1967, a man deposited $250 in a savings and loan association paying 4½% converted semiannually. Interest dates are June 30 and December 31, and deposits made by the 10th of a month earn interest for the entire month. This man continues to make $250 deposits every 6 months up to and including January 4, 1973, when he makes his last deposit. How much will be in his account after interest is credited on June 30, 1973?

6. On June 1, 1967, a man deposits $200 in a savings and loan association that pays 4½% converted semiannually. He continues to make $200 deposits every 6 months. If he makes his last deposit on December 1, 1973, how much will be in his account on June 1, 1974? Interest dates are May 31 and November 30. Deposits made by the 10th of the month earn interest for the entire month.

7. A couple expects to need $4000 in June, 1974. In June, 1970, they made the first of 4 equal annual deposits in an investment paying 4%. What is the size of each deposit needed to accumulate the desired amount?

8. A debt of $7500 is due in 10 years. The debtor agrees to settle this debt with 10 equal annual payments, the first payment to be made now. Find the size of the payment if money is worth 4%.

9. The premium on a life insurance policy is $15 a month payable in advance. A policy holder may pay a year's premium in advance. Find the annual premium if it is based on 6% converted monthly.

10. A life insurance policy has premiums at the beginning of each month of \$25. Find the equivalent annual premium at the beginning of each year if it is based on 9% converted monthly.

11. The annual premium on a life insurance policy is \$250 payable in advance. However, payments may be made quarterly in advance in which case the insurance company charges interest at 6% converted quarterly. Find the size of the quarterly payment.

12. The annual premium on a life insurance policy is \$185 payable in advance. What would be the monthly premium based on 6% converted monthly?

13. A man has \$5000 in a building and loan association which pays 4½% compounded semiannually. If he withdraws \$500 at the beginning of each half year, how many such withdrawals can he make? What will be the size of the concluding withdrawal?

14. In Problem 13 how many \$250 witdrawals could be made and what would be the size of the concluding witdrawal?

15. Instead of taking \$5000 from an inheritance, a person decides to take 60 monthly payments with the first payment to be made immediately. If interest is allowed at 3% converted monthly, what will be the size of each payment.

16. Instead of taking \$20,000 cash from an inheritance, the heir elects to receive quarterly payments for 10 years with the first payment to be made right away. If the money is invested at 4% converted quarterly, what will be the size of each payment?

17. A man wants to take out enough life insurance to provide 120 monthly payments certain of \$150 each to his family. If the first payment is to be made upon proof of the man's death, he should take out what size policy? Round the answer to the nearest \$1000. The insurance company pays 3% converted monthly on money left with them.

18. A man is planning an insurance program which will pay 180 monthly payments certain of \$200 each to his widow. The

insurance company will pay 3% converted monthly on money left with them. If the first payment is to be made upon proof of the man's death, he should take out what size policy? Round answer to nearest $1000.

19. A store can be rented for $300 a month payable in advance. If the renter pays 3 months in advance, the owner will allow interest at 5% converted monthly. What is the size of the payment at the beginning of each 3 months that is equivalent to $300 at the beginning of each month?

20. A store can be rented for $350 per month payable in advance. The landlord will accept a single cash payment at the beginning of a year for a year's lease if interest is computed at 5% converted monthly. What would be the cost of a year's lease?

21. A man knows that if he dies his wife will not get any social security benefits until she reaches age 60. He has several insurance policies. One for $10,000 is to be used to provide a monthly income for her between the time he dies and she reaches age 60. She is 53 at the time he dies and she elects to get 84 equal monthly payments from this policy with the first payment to be made immediately. What will be the size of each payment if 4% converted monthly is paid on the proceeds of this policy?

22. The widow in Problem 21 decides to wait until age 62 to start receiving social security payments so they will be larger. She elects to get 108 equal monthly payments from the $10,000 policy with the first payment to be made immediately. Find the size of the payments.

23. On June 1, 1967, a man opened a savings account for his daughter with a deposit of $25 in a bank paying 3% converted semiannually. If he continues to make semiannual deposits of $25 until December 1, 1974, when he plans to make the last deposit, how much will be in the account on June 1, 1975?

24. A man invests $300 every year, making his first deposit on August 1, 1962, and his last deposit on August 1, 1972. How much will be in his account on August 1, 1973, if he gets interest at 4%

compounded annually?

25. A man aged 25 is considering two types of life insurance policies. The first is an ordinary life with an annual premium of \$200 for a \$10,000 policy. The second is a twenty-year endowment with an annual premium of \$510. He decides to take out the ordinary life and deposit the difference in premiums each year in an investment paying 4% compounded annually. If he dies just before making his 17th deposit, how much will be in his savings account? Allow interest for the year after the 16th deposit.

26. A man aged 30 is considering two \$15,000 insurance policies. The first is an ordinary life with an annual premium of \$277. The second is a 20-pay life with an annual premium of \$410. He decides to take out the ordinary life and put the difference in premiums in an investment paying 3½%. How much will his family get from this investment if he dies after interest is earned on the 15th deposit but before he makes the 16th deposit?

27. A factory owner has a fire insurance policy for which the annual premium is \$300 payable in advance. If interest is 6% converted annually, what would be the equivalent single premium which would provide insurance for 5 years?

28. A home owner has been paying his insurance at the beginning of each year. The annual premium is \$60. At 8% converted annually, what would be the equivalent premium for a three-year policy?

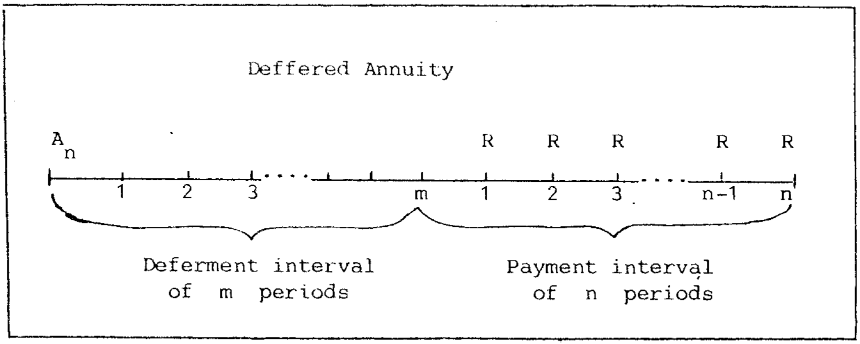
29. A \$2200 used car is sold for \$200 down, 23 payments of \$80 a month and a 24th balloon payment. If the interest rate is 18% converted monthly, what is the size of the balloon payment?

3.6 Deferred Annuity:

A *deferred annuity* is one in which the first payment is not made at the beginning or end of the first period but at some later date. The interval of deferment ends one period before the first payment.

When the first payment is made at the end of 10 periods, the annuity is said to be deferred 9 periods. Similarly, an annuity which is deferred for 12 periods will have the first payment at the end of 13 periods.

The sketch shows a deferred annuity of n payments which is deferred for m periods.



To get a formula for the present value of deferred annuity.

Method 1:

Assume that a payment is also made at the end of each period during the interval of deferment (m periods). Then we would have an ordinary annuity of (m + n) payments and the present value would be $R\ a_{\overline{m+n}|}$. But this value includes the assumed payments during the interval of deferment. These assumed payments also form an ordinary annuity and their present value is $R\ a_{\overline{m}|}$

Therefore, to get the present value of deferred annuity is to subtract the present value of the m assumed payments from the present value of the (m + n) payments.

$$\therefore A_n = Ra_{\overline{m+n}|} - Ra_{\overline{m}|}$$

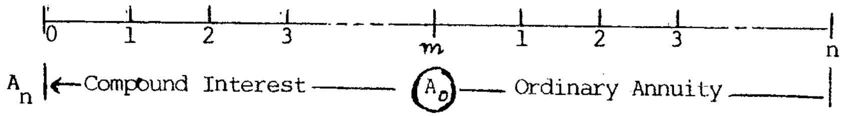
$$A_n = R(a_{\overline{m+n}|} - a_{\overline{m}|}) \quad \text{where;}$$

A_n = present value of a deferred annuity

R = periodic payment

$a_{\overline{m+n}|}$ and $a_{\overline{m}|}$ = Present Worth of 1 per period.

Method 2:



Another method to find the present value of a deferred annuity is to use the present value of an ordinary annuity to get the value of the payments one period before the first payment is made and then discount this equivalent single sum for the m periods in the interval of deferment.

1st step: To get the value of the payments one period before the payment is made

$$A_0 = Ra_{\overline{n}|}$$

2nd step: To discount this equivalent single sum for the m periods in the interval of deferment

$$\therefore A_n = Ra_{\overline{n}|}(1+i)^{-m}$$

Example 1:

Find the present value of a deferred annuity of ฿10,000 a year for 10 years deferred 5 years. Money is worth 3%.

Solution:

Deferment interval (m) = 5 years R = ฿ 10,000
Number of payments (n) is = 10 i = 3%

$$\begin{aligned} A_n &= R(a_{\overline{m+n}|\,i} - a_{\overline{m}|\,i}) \\ &= 10{,}000(a_{\overline{5+10}|\,3\%} - a_{\overline{5}|\,3\%}) \\ &= 10{,}000(11.93794 - 4.57971) \\ &= 10{,}000(7.35823) \\ &= ฿\ 73{,}582.3 \qquad \underline{\text{Ans}} \end{aligned}$$

Example 2:

Find the present value of an annuity of ฿1,000 every three months for

5 years if the first payment is made in 3 years. Money is worth 4% converted quarterly.

Solution:

$n = 20$

$i = \frac{4}{4}$

$m = 11$

$R = ฿\ 1{,}000$

$$A_n = R(a_{\overline{m+n}|i} - a_{\overline{m}|i})$$

$$A_{20} = 1000(a_{\overline{11+20}|1\%} - a_{\overline{11}|1\%})$$

$$= 1000(26.5423 - 10.3676)$$

$$= 1000 \times 16.1747$$

$$= ฿\ 16{,}174.7 \quad \underline{\text{Ans}}$$

Example 3:

A woman inherits ฿400,000. Instead of taking the cash, she invests the money at 3% converted semiannually with the understanding that she will receive 20 equal semiannual payments with the first payment to be made in 5 years. Find the size of the payments.

Solution:

The interval of deferment (m) = 9 periods

The present value of the annuity (A_n) = ฿ 400,000

$i = \frac{3}{2}\% = 1\frac{1}{2}\%$

$$A_n = R(a_{\overline{m+n}|i\%} - a_{\overline{m}|i\%})$$

$$R = \frac{A_n}{a_{\overline{m+n}|i} - a_{\overline{m}|i}}$$

$$= \frac{400{,}000}{a_{\overline{9+20}|1\frac{1}{2}\%} - a_{\overline{9}|1\frac{1}{2}\%}}$$

$$= \frac{400{,}000}{23.3760756 - 8.3605173}$$

$$= \frac{400{,}000}{15.015583}$$

$$= ฿26{,}639.04 \quad \underline{\text{Ans}}$$

EXERCISE: 3 - 5

1. Find the present value of a deferred annuity of ฿10,000 a year for 6 years deferred 5 years, if money is worth 4%.

2. Find the present value of a deferred annuity of ฿5,000 every 6 months for 4 years deferred 5 years and 6 months, if the rate is 4% converted semiannually.

3. Find the value on September 1, 1972, of a series of ฿1,600 monthly payments, the first of which will be made on September 1, 1976, and the last on December 1, 1978, if money is worth 6% compounded monthly.

4. Find the value on June 1, 1973, of a series of payments of ฿8,500 every 6 months if the first of these payments is to be made on December 1, 1976, and the last on June 1, 1983. Use an interest rate of 4½% converted semiannually.

5. What sum put aside on a boy's 12th birthday will provide 4 annual payments of ฿40,000 for college expenses if the first payment is to be made on the boy's 18th birthday? The fund will earn interest at 4% compounded annually.

6. What sum of money should be set aside today to provide an income of ฿3,000 a month for a period of 5 years if the first payment is to be made 4 years hence and money is worth 6% compounded monthly?

7. A child aged 12 wins ฿160,000 on a quiz program. It is placed in a trust fund earning 4% converted annually. If she takes the money in 4 equal annual payments with the first payment to be made 6 years later, what will be the size of each payment?

8. Under the terms of a will, a child on his 18th birthday will recieve ฿240,000. He will get ฿40,000 of this in cash. The remainder will be set aside to provide a monthly income from his 21st to his 25th birthday inclusive ($n = 49$). If the money is invested at 4% compounded monthly, what will be the size of the monthly payments?

9. On his wife's 59th birthday, her husband makes provision for her to receive ₱3,000 a month for 5 years with the first payment to be made on her 65th birthday. If the investment earns 4% converted monthly, how much money must be set aside?

10. On his 57th birthday a man wants to set aside enough money to provide an income of ₱3,000 a month for 10 years with the first payment to be made on his 60th birthday. If he gets 3% compounded monthly on his money, how much will this pension plan cost on his 57th birthday?

11. On June 1, 1968, a man deposits ₱60,000 in an investment paying 5% compounded annually. On June 1, 1974, he makes the first 4 equal annual withdrawals from his account. Find the size of the withdrawals so that the account will be closed with the last withdrawal.

12. If the account in Problem 11 had earned 4% interest, what would be the size of the annual withdrawal?

13. A philanthropist gave a college ₱800,000 on September 1, 1959, with the provision that the money is to be used to provide annual scholarships of ₱80,000 with the first scholarship to be awarded on the September 1 following his death. The money is invested at 4%. If the donor dies on June 18, 1974, how many full scholarships can be awarded?

14. A widow decides to take the ₱200,000 proceeds from an insurance policy in payments of ₱4,000 a month with the first payment to be made in 5 years. How many full payments will she get if the insurance company allows 3% converted monthly on the money left with them?

15. On June 1, 1969, a minor received an inheritance of ₱130,000. This was placed in a trust fund earning 3% compounded semiannually. On December 1, 1974, the child will reach age 18 and be paid the first of 8 equal semiannual payments from the fund. How much will each payment be?

16. A school receives ₱800,000 in 1968. This is to be used to provide

annual scholarships of $60,000 with the first scholarship to be awarded in 1974. If the money is invested at 4%, how many full scholarships can be awarded? How much will be left in the fund one year after the last full scholarship to apply to a partial scholarship?

CHAPTER 4

INEQUALITIES AND LINEAR PROGRAMMING FORMULATION AND GRAPHIC SOLUTIONS.

4.1 The nature of Inequalities:

Equations are used to represent a condition where two quantities are equal whereas inequalities express the condition that two quantities are not equal.

Inequality Symbols.

$<$ less than

$>$ greater than

$\leqslant$ less than or equal to (no more than)

$\geqslant$ greater than or equal to (no less than)

both conditions cannot hold simultaneously. One and only one is the case.

a) Strict inequalities: Those which items being compared can never equal one another.

Example:

$3<5$	3 is less than 5
$2>-5$	2 is greater than -5
$-10<-2$	-10 is less than -2
$0<2<5$	2 is greater than 0 and less than 5
$0<x<10$	x is greater than 0 and less than 10

– The direction of the inequality symbol is referred to as the *sense* of the inequality.

– The comparisons among three items ($0<2<5$ and $0<x<10$) are called *Double Inequalities.*

b) Mixed Inequalities: Situations in which items being compared may or may not be equal. The symbols used are
$\leqslant$ and $\geqslant$
$\leqslant$ means less than or equal to.
$\geqslant$ means greater than or equal to.

Example:

1) $x \leqslant 25$ implies that the variable x can assume values which are less than or equal to 25.

2) The double inequality as $0 \leqslant x \leqslant 25$ states that x may assume values which are greater than or equal to 0 and less than or equal to 25.

4.2 Inequality - Preserving Operations

1. We may add (subtract) the same quantity to (from) both sides.

Example:

$3 < 4$ add 5 to both sides we get $3 + 5 < 4 + 5$
$10 > 5$ subtract 2 from both sides we get $10 - 2 > 5 - 2$

2. We may multiply (divide) both sides by the same positive number.

Example:

$4 < 6$ multiply both sides by 3 we get $4 \times 3 < 6 \times 3$ But if we multiply both sides of an inequality by -3 the sense of inequality has to be reversed.
$4 < 6$ multiply both sides by -3 we get

$$4 \times (-3) > 6 \times (-3) \text{ or } -12 > -18$$

3. If both sides of the inequality are always positive, we may raise both sides to the same positive power.

Example:

$5 > 3$ raise both sides to the same positive power we get
$5^2 > 3^2$
$25 > 9$

4. If both sides of the inequality are always positive, we may take the logarithm of both sides.

Example:

$100 < 1000$ Take logarithm of both sides we get
$\log 100 < \log 1000$
$\log 10^2 < \log 10^3$
$2 \log 10 < 3 \log 10$
$2 < 3$

5. If two inequalities are of the same type ($\leqslant$, $<$, $\geqslant$, or): we may add (but not subtract) the inequalities and obtain an inequalities of the same type.

Example:

$$\begin{array}{ll} \text{we may add} & 3 < 5 \\ & \underline{2 < 8} \ + \\ \text{to obtain} & \underline{\underline{5 < 13}} \end{array}$$

Def: a) Conditional inequalities are inequalities involving a variable and which hold for some values of the variables but not others.

Example:

$x > 10$ is a conditional inequality, it holds only for all values of x greater than 10.

b) Absolute inequalities are inequalities involving a variable and which hold for all values of the variables.

Example:

$x^2 \geqslant o$ is an absolute inequality; it holds for all real numbers.

c) $3 < 4$ does not involve a vairable. It is an inequality that always holds.

Intervals:

Let x be any variable and a and b be constants.

$a < x < b$ is called the open interval from a to b.
$a \leqslant x \leqslant b$ is called the closed interval from a to b.
$a < x \leqslant b$ is called the open-closed interval from a to b.
$a \leqslant x < b$ is called the closed-open interval from a to b.

The points a and b are called the end points of the interval.

4.3 Solution of Inequalities:

Solving an inequality means finding the region in which it holds, in other words, finding the set of real numbers for which it is true.

All inequalities can be written in standard form with zero to the right of the symbol.

Example:

$x \geqslant 10$ in standard form it becomes

$$x - 10 \geqslant 0$$

To solve for Conditional Inequalities, follow the following steps:

1. *Solve as if the inequality were an equality.*
2. *Put it in Standard Form.*
3. *Solve for all x values such that $f(x) = 0$.*
4. *For one point in each segment, evaluate $f(x)$.*

If the inequality holds for the point chosen, it holds for the entire segment.

I. Example 1:

$$5x + 7 \leqslant 2x + 1$$

$$5x - 2x + 7 - 1 \leqslant 0$$

$$3x + 6 \leqslant 0$$

$$x \leqslant -\frac{6}{3}$$

$$x \leqslant -2$$

Ans

Example 2:

$$2x + 3 < 4x + 9$$

$$2x - 4x + 3 - 9 < 0$$

$$-2x - 6 < 0$$

$$-2x < 6$$

$$\therefore \quad x > \frac{6}{-2}$$

$$x > -3$$

Ans

Example 3:

$$4x - 3 \leqslant 2x + 3$$
$$4x - 2x - 3 - 3 \leqslant 0$$
$$2x \leqslant 6$$
$$x \leqslant 3 \qquad \underline{\text{Ans}}$$

Example 4:

$$3x + 2 < 6x - 7$$
$$3x - 6x < -7 - 2$$
$$-3x < -9$$
$$x > 3 \qquad \underline{\text{Ans}}$$

Example 5:

$$3x - 1 \geqslant 4x + 2$$
$$3x - 4x \geqslant 2 + 1$$
$$-x \geqslant 3$$
$$x \leqslant -3 \qquad \underline{\text{Ans}}$$

Example 6:

$$x - 3 > 1 + 3x$$
$$x - 3x > 1 + 3$$
$$-2x > 4$$
$$x < -2 \qquad \underline{\text{Ans}}$$

Example 7:

$$2x - 3 \leqslant 5x - 9$$
$$2x - 5x \leqslant -9 + 3$$
$$-3x \leqslant -6$$
$$x \geqslant 2 \qquad \underline{\text{Ans}}$$

Example 8.

$$\frac{1}{2}x + \frac{x-2}{3} < 2x - \frac{1}{12}$$
$$6x + 4(x - 2) < 24x - 1$$
$$6x + 4x - 8 < 24x - 1$$
$$6x + 4x - 24x < -1 + 8$$
$$-14x < 7$$
$$x > -\frac{1}{2} \qquad \underline{\text{Ans}}$$

Example 9:

Solve each inequality; i. e. rewrite the inequality so that x is alone between the inequality signs.

a) $3 \leqslant x - 4 \leqslant 8$

$7 \leqslant x \leqslant 12$ $\quad$ <u>Ans</u>

b) $-1 \leqslant x + 3 \leqslant 2$

$-4 \leqslant x \leqslant -1$ $\quad$ <u>Ans</u>

c) $-9 \leqslant 3x \leqslant 12$

$-3 \leqslant x \leqslant 4$ $\quad$ <u>Ans</u>

d) $-6 \leqslant -2x \leqslant 4$

$3 \geqslant x \geqslant -2$ $\quad$ <u>Ans</u>

e) $3 < 2x - 5 < 7$

$8 < 2x < 12$

$4 < x < 6$ Ans

f) $-7 \leqslant 2x + 3 < 5$

$-10 \leqslant 2x \leqslant 2$

$-5 \leqslant x \leqslant 1$ Ans

Example 10:

Solve for the x values where the following Conditional inequalities hold.

a) $x^2 + 4x + 4 \geqslant 0$

$(x + 2)^2 \geqslant 0$ for all x values

b) $x^3 - 2x^2 - x + 2 > 0$

$(x + 1)(x - 1)(x - 2) > 0$

$$\left.\begin{array}{l} \text{for } -1 > x > +1 \\ \text{and for } \quad x > 2 \end{array}\right\} \text{Ans}$$

c) $x^2 - 1 < 0$

$(x + 1)(x - 1) < 0$

$x > -1$ and $x < 1$

$-1 < x < 1$ Ans

d) $x^2 < 0$

never holds for x = a real number

Ans

II. *A company has a selling price for a particular product of ₿5 per unit. The variable cost of production is ₿3 per unit, and they have fixed costs of ₿1,000. They would like to know the values of sales that will give them some positive profit.*

Solution:

Let x be units sold.

$$\begin{aligned}
\text{Revenue} &= \text{price x quantity}\\
&= 5x\\
\text{Variable cost} &= 3x\\
\text{Fixed cost} &= ₱1{,}000\\
\therefore \text{total cost} &= 3x + 1000\\
\text{Profit} &= 5x - (3x + 1000)\\
\therefore \text{Positive Profit} = f(x) &= 5x - (3x + 1000) > 0\\
&\quad 5x - 3x - 1000 > 0\\
&\quad 2x > 1000\\
&\quad x > 500\\
\therefore \text{The inequality holds for} &\quad x > 500
\end{aligned}$$

Ans

III. ***The firm has low variable costs initially, and that as space become limited, costs begin to rise. Suppose that variable cost per unit is given by:***

$$c(x) = 1 + 0.001x$$

They want to find x such that:-

$$f(x) = 5x - (1 + 0.001x)\,x - 1000 > 0$$

Solution:

$$\begin{aligned}
f(x) &= 5x - (1 + 0.001x)\,x - 1000 \geqslant 0\\
&\quad 5x - x - 0.001x^2 - 1000 \geqslant 0\\
&\quad -0.001x^2 + 4x - 1000 \geqslant 0\\
&\quad 0.001x^2 - 4x + 1000 \leqslant 0\\
&\quad x^2 - 4000x + 1000000 \leqslant 0
\end{aligned}$$

Using quadratic formula

$$(x - 268)(x - 3732) < 0$$

The inequality hold for $268 < x < 3732$ Ans

4.4 Application of Inequalities:

1. *For a manufacturer of thermostats, the combined cost for labor and material is ₱80 per thermostat. Fixed costs (costs incurred in a given time*

period, regardless of output) are ₱1,200,000. If the selling price of a thermostat is ₱140, how many must be sold for the company to earn a profit?

Solution:

Let x be the number of thermostats that must be sold

Then their cost is	= 80x
Total cost is	= 80x + 1,200,000
The total revenue from the sale of x thermostats will be	= 140x
But Profit	= Total revenue - total cost and we want profit > 0. Thus

Total revenue - Total cost > 0

140 - (80x + 1,200,000) > 0

140x - 80x - 1,200,000

60x > 1,200,000

x > 20,000

Therefore, at least 20,001 thermostats must be sold for the company to earn a profit. Ans

4.5 Systems of Linear Inequalities:

To plot a linear inequality in two unknowns:

1. Draw the line representing the relationship as though it were an equality.
2. The inequality holds for all points on one side of the line and for no points on the other side.
3. Evaluate the inequality for one point (one on the line) to determine which side is which.

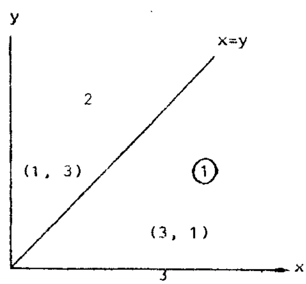
Example 1:

$x \quad x \geqslant y$

Step 1. Draw the line $x = y$

Step 2. The inequality holds for all points either in region (1) or in region (2)

Step 3. Then pick out one point in region (1) and evaluate the inequality.



- If we take point (3, 1), then evaluate it in $x \geqslant y$. We get $3 > 1$ which is correct.

But if we take the point in region 2 (1, 3) and evaluate in $x \geqslant y$, we get $1 > 3$ which is incorrect.

$\therefore$ If $x = y$, the points will be on the line and if $x > y$ the points will be in the shaded region or region 1 as shown.

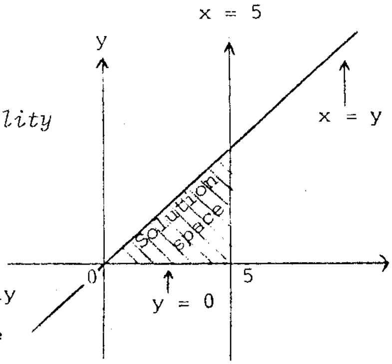
Example 2:

To plot more than one inequality

$y \geqslant 0$

$x \leqslant 5$

$x \geqslant y$



The darkly shaded region shows the only area where $x \geqslant y$, $y = 0$, and $x \leqslant 5$ are all simultaneously satisfied.

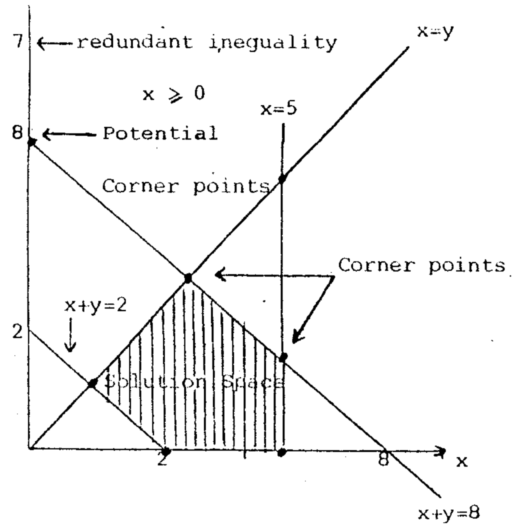
The region where all are simultaneously satisfied is called the *solution space* or *technical feasibility polygon*.

Example 3:

Find the solution space of the following inequalities:

$x \geqslant y$	$y \leqslant 0$
$x \leqslant 5$	$x+y \leqslant 8$
$x+y \geqslant 2$	$x \geqslant 0$

The shaded region is the required solution space.



Some facts about a solution space formed by linear inequalities

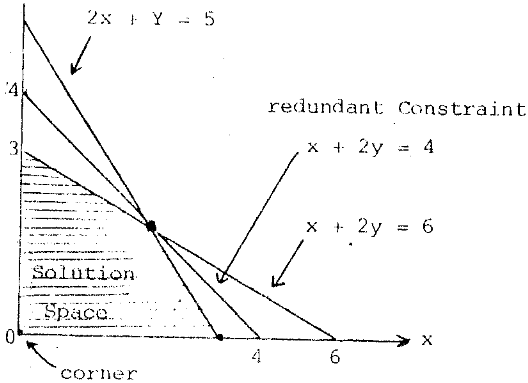
1. Corner points = are the corners of straight lines that form the solution space.
2. Potential corner points: they are the intersection of two inequalities, but the point of intersection is not in the solution space.
3. Redundant inequality : is an equality that does not affect the solution space. In this example, we have $x \geqslant 0$ as redundant inequality.
4. Inbusiness problems, the inequalities are termed as *constraints* and they can be separated into two categories, namely.
   a) Nonnegativity constraints:
   guarantee that each decision variable will not be negative.
   $$x \geqslant 0$$
   $$y \geqslant 0$$
   b) Structural Constraints reflect such things as resource limitations and other restrictions explicitly identified in the statement of the problem.
   $$x + y \leqslant 8$$
   $$x + y \geqslant 2............\text{etc.}$$

5. All the corner points can be obtained by solving two of the five inequalities as equations.

Example 4:

Draw a solution space for the following systems of inequalities:

$$\begin{aligned} 2x + y &\leqslant 5 \\ x + 2y &\leqslant 6 \\ x + y &\leqslant 4 \\ x &\geqslant 0 \\ y &\geqslant 0 \end{aligned}$$



EXERCISE: 4 - 1a

Make a graph showing the solution space.

1. $x \geqslant 0$, $y \geqslant 0$, $y \geqslant -1$, $x \leqslant 4$.

2. $x \geqslant 0$, $y \geqslant 0$, $x \leqslant 3$, $x \leqslant y$.

3. $x \geqslant 0$, $y \geqslant 0$, $2x + y \geqslant 6$, $x + 4y \leqslant 8$.

4. $x \leqslant 3$, $y \leqslant 1$, $x + 3y \geqslant 3$.

5. $x \geqslant 0$, $y \geqslant 0$, $7x + 4y \leqslant 28$, $x + 2y \leqslant 8$.

Find solutions, if any exist.

6. $x \geqslant 0$, $y \geqslant 0$, $2x + 3y \leqslant 12$, $x - 2y \geqslant 2$.

7. $x \geqslant 0$, $y \geqslant 0$, $4x + y \leqslant 12$, $x + 5y \geqslant 8$.

8. $$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 4x + y &\geq 12 \\ x + 5y &\leq 8. \end{aligned}$$

9. $$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 4x + y &\geq 12 \\ x + 5y &\geq 8. \end{aligned}$$

10. $$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 4x + y &\leq 12 \\ x + 5y &\leq 8. \end{aligned}$$

Specify in two ways the side of the line which contains the points satisfying the following inequalities.

11. $x - y < 5.$

12. $x + y > 3.$

13. $2x - 3y < -5.$

14. $-2x + 5y > 2.$

15. $4x - 3y > -5.$

Solve for x.

16. $2 - 3x \leq 6.$

17. $2x + 7 \geq 4.$

18. $7 - 4x \leq 0.$

19. $3 \geq 5 - 2x.$

20. $0 \leq 2x + 3.$

Write the solution, if any exists, for each of the following systems.

21. $$\begin{aligned} x &< 4 \\ x &> 6. \end{aligned}$$

22. $$\begin{aligned} x &> -2 \\ x &> 2. \end{aligned}$$

23. $$\begin{aligned} x &> 2 \\ x &< 6. \end{aligned}$$

24. $$\begin{aligned} x - 3y &\leq 5 \\ 2x - 6y &\geq 12. \end{aligned}$$

25. $$\begin{aligned} x - 3y &\leq 5 \\ 2x - 6y &\geq 8. \end{aligned}$$

26. $$\begin{aligned} x - 3y &\leq 5 \\ 2x - 6y &\leq 15. \end{aligned}$$

27. $$\begin{aligned} x + 2y &\geq 40 \\ 4x + 3y &\geq 120. \end{aligned}$$

28. $$\begin{aligned} x + 2y &\geq 40 \\ 4x + 3y &\leq 120. \end{aligned}$$

29. $$\begin{aligned} x + 2y &\geq 40 \\ 6x + 7y &\leq 190. \end{aligned}$$

Make a graph showing the solution space for each of the following, assuming in each system that the nonnegativity constraints $x > 0$ and $y > 0$ prevail.

30. $x - 2y \leq 0$
$x + y \leq 2.$

31. $x + y \geq 2.$
$x + 2y \leq 4.$

32. $x - 2y \leq 0$
$x + 2y \leq 4.$

33. $x - 2y \geq 0$
$x + 2y \geq 4$
$x \geq 3.$

34. $x + y \geq 2$
$x - 2y \leq 0$
$x + 2y \leq 4.$

35. $x + y \geq 2$
$x - 2y \geq 0$
$x \leq 3$
$x + 2y \leq 4.$

36. A mixture is to made containing x units of food A and Y units of food B. Food A weighs three ounces per unit; food B, five ounces per unit. Food A contains 0.5 ounces of nutrient per unit; food B, one ounce of nutrient per unit. The mixture is to have at least eight ounces of nutrient, and the total weight is not to exceed 45 ounces. Find the general solution, showing all permissible combinations of the two foods.

37. Type A and type B items require the same amount of storage space per unit. Type A costs \$3 per unit, B costs \$8 per unit. Sufficient storage space is available for 500 units (total), and \$2400 is available to spend on the items. Write the general solution, showing permissible combinations of items which may be purchased and stored without exceeding total space and money restrictions.

38. It takes two hours to make a unit of A and three hours to make a unit of B. The number of units of B must not be more than twice the number of units of A. If up to 36 hours are available to make A and B, find the permissible combinations of numbers of units of A and B.

39. A costs \$2 per pound and B costs \$5 per pound. If we mix x pounds of A with y pounds of B, the mixture contains x + y pounds. This proportion of A in the mixture therefore is the

rotio $x/(x + y)$. Mixtures are to be made at a total cost not exceeding $286. In these mintures, the proportion of A is to be not less than 0.2 (20 percent) and not greater than 0.8 (80 percent). Write the general solution showing permissible combinations of A and B.

40. The table shows, for example, that Department III has 190 hours available and that one unit of A requires 6 hours of Department III time. Write the general solution showing the combinations of A and B which can be made in the time available.

Department	Available Hours	Hours Required to Make One Unit of	
		Product A	Product B
I	120	4	3
II	40	1	2
III	190	6	7

41. A mixture of foods A and B is to weigh not more than 5 pounds and must contain at least 24 ounces of nutrient. Food A contains six ounces of nutrient per pound and B contains 4 ounces of nutrient per pound. Write the general solution showing the permissible mixtures of the two foods.

42. A type A batch of a product contains one unit of substance P and three units of substance Q. A type B batch contains three units of P and four units of Q. If 70 units of P and 110 units of Q are available, write the solution showing permissible combinations of the two types of batches.

EXERCISE: 4 - 1b

Algebraically solve for the values of x that satify the following inequalities.

1. $3x - 2 \leqslant 4x + 8$

2. $x \geqslant x + 5$.

3. $-4x + 10 \geqslant -10 + x$.

4. $15x + 6 \geqslant 10x - 24$

5. $12 \geqslant x + 16 \geqslant -20$.

6. $50 \leqslant 4x - 6 \leqslant 25$

7. Write the general expression for the values of x which are in the solution set of the inequality $2x + 8y \leq 80$. What are the values for x when $y = 5$? Determine the general expression for values of y which are in the solution set. What are the values of y when $x = -40$?

Graphically determine the half-space which satisfies the inequality.

8. $-4x + 2y \leqslant 20$

9. $5x + 3y \leqslant 30$.

10. $5x - 4y \leqslant -24$.

11. $3x_1 + 2x_2 \leqslant 36$

$x_1 + 4x_2 \leqslant 16$

12. $3x_1 + 4x_2 \leqslant 24$

$4x_1 + 3x_2 \leqslant 24$

13. $4x_1 - 2x_2 \geqslant 4$

$x_1 \leqslant 5$

$x_2 \geqslant 2$

14. $x_1 + x_2 \leqslant 8$

$3x_1 - x_2 \leqslant 9$

$x_1 + 2x_2 = 5$

$x_1 \geqslant 0$

$x_2 \geqslant 0$

15. A firm produces two products, each of which must be processed through three departments. The time requirements in each department are shown. Formulate the three inequalities associated with production during the coming week. Graphically determine the combinations of the two products which can be produced within the three departments

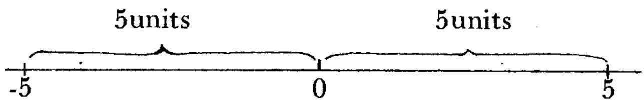
	Hours Required per Unit		
	Department 1	Department 2	Department 3
Product A	2	3	1.5
Product B	4	2	3
Hours Available per Week.	40	36	30

4 - 6 Absolute Value:

On the real number line, the distance between a number x and 0 is called the ***absolute value*** and is denoted by $|x|$

For example, $|5| = 5$
and $|-5| = 5$

because both 5 and -5 are five units from 0.



$$|5| = |-5| = 5$$

Absolute value can be defined as follows:

The ***absolute value*** of a real number x, written $|x|$, is

$$|x| = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Applying the definition, we have

$$\left.\begin{aligned} |3| &= 3; \\ |-8| &= -(-8) = 8; \\ \left|\tfrac{1}{2}\right| &= \tfrac{1}{2}; \\ -|2| &= -2; \\ \text{and} \quad -|-2| &= -2. \end{aligned}\right\}$$

— Notice that $|x|$ is always positive or zero; that is, $|x| \geqslant 0$.

Example 1:

a) Solve $|x - 3| = 2$
This equation states that $x - 3$ is a number two units from 0, Thus, either
$$x - 3 = 2 \quad \text{or} \quad x - 3 = -2$$
Solving these gives $x = 5$ or $x = 1$ Ans

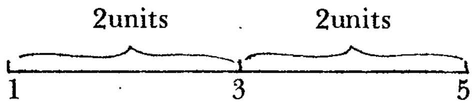
b) Solve $|7 - 3x| = 5$
The equation is true if $7 - 3x = 5$ or if $7 - 3x = -5$.
Solving these gives $x = \frac{2}{3}$ and $x = 4$ Ans

c) Solve $|x - 4| = -3$
The absolute value of a number is never negative.
Thus the solution set is $\emptyset$ Ans

- Another example would be
$$|9 - 5| = |4| = 4$$
$$|5 - 9| = |-4| = 4$$
The numbers 5 and 9 are 4 units apart.

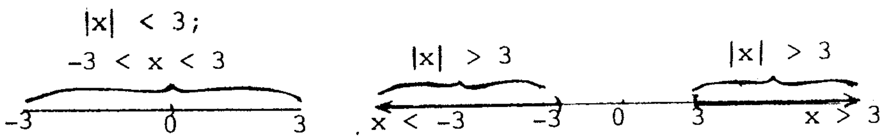
In general, we may interpret $|a - b|$ or $|b - a|$ as the distance between a and b.

For example, the equation $|x - 3| = 2$ states that the distance between x and 3 is 2 units. Thus x canbe 1 or 5 as shown below.



Absolute value and Inequalities:

If $|x| < 3$, then x is less than 3 units from 0. Thus x must lie between -3 and 3. That is, $-3 < x < 3$. On the other hand, if $|x| > 3$, then x must be greater than 3 units from 0. Thus there are two cases: either $x > 3$ or $x < -3$. See the figures below for better understanding.



We can extend these ideas. If $|x| \leqslant 3$, then $-3 \leqslant x \leqslant 3$. And if $|x| \geqslant 3$, then $x \geqslant 3$ or $x \leqslant -3$. In general, the solution of $|x| < d$ or $|x| \leqslant d$, where d is a positive number, consists of one interval, namely $-d < x < d$ or $-d \leqslant x \leqslant d$. However, when $|x| > d$ or $|x| \geqslant d$. There are two intervals in the solution, namely $x < -d$ and $x > d$, or $x \leqslant -d$ and $x \geqslant d$.

Example 1:

a) Solve $|x - 2| < 4$

Solution:

The number $x - 2$ must be less than 4 units from 0.

This means that $-4 < x - 2 < 4$

Then $-4 + 2 < x < 4 + 2$

$-2 < x < 6$

Thus the solution is $-2 < x < 6$. This means that all numbers between -2 and 6 satisfy the original inequality. <u>Ans</u>

b) Solve $|3-2x| \leqslant 5$

Solution:

$$|3-2x| \leqslant 5 \text{ means that}$$
$$-5 \leqslant 3 - 2x \leqslant 5$$
$$-5 - 3 \leqslant -2x \leqslant 5 - 3$$
$$-8 \leqslant -2x \leqslant 2$$

The sense of the inequality is reversed when it is divided by -2.

$$4 \geqslant x \geqslant -1$$

or $\quad -1 \leqslant x \leqslant 4$ <u>Ans</u>

Example 2:

a) Solve $|x+5| \geqslant 7$

Solution:

$|x+5| \geqslant 7$ means that $x + 5$ must be <u>at least</u> 7 units from 0.

Thus either $x + 5 \leqslant -7$ or $x + 5 \geqslant 7$. It also means that either $x \leqslant -12$ or $x \geqslant 2$ <u>Ans</u>

b) solve $|3x-4| > 1$

Solution:

$|3x - 4| > 1$ means that either $3x - 4 < -1$

or $\quad 3x - 4 > 1$

Thus, either $3x - 4 + 4 < -1 + 4$

$3x < 3$

$x < 1$ Ans

or $3x - 4 + 4 > 1 + 4$

$3x > 5$

$x > \frac{5}{3}$ Ans

Example 3:

Using absolute value notation, express the following statements:

a) x is less than 3 units from 5. $= |x - 5| < 3$
b) x differs from 6 by at least 7. $= |x - 6| \geqslant 7$
c) $x < 3$ and $x > -3$ simultaneously $= |x| < 3$
d) x is more than 1 unit from −2 $= |x - (-2)| > 1$, $|x + 2| > 1$
e) x is strictly within σ units of μ $= |x - \mu| < \sigma$

Three basic Properties of absolute value:

They are:-

1. $|ab| = |a| \cdot |b|$
2. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
3. $|a - b| = |b - a|$

Example:

a) $|(-7) \cdot 3| = |-7| \cdot |3| = 21$

$|(-7)(-3)| = |-7| \cdot |-3| = 21$

b) $|4 - 2| = |2 - 4| = 2$

c) $|7 - x| = |x - 7|$

d) $\left|\frac{-7}{3}\right| = \frac{|-7|}{|3|} = \frac{7}{3}$

$\left|\frac{-7}{-3}\right| = \frac{|-7|}{|-3|} = \frac{7}{3}$

e) $\left|\frac{x-3}{-5}\right| = \frac{|x-3|}{|-5|} = \frac{|x-3|}{5}$

EXERCISE: 4 - 2

In Problems 1 - 10, write an equivalent form without the absolute value symbol.

1. $|-13|$.
2. $|2^{-1}|$.
3. $|8-2|$.
4. $|(-4-6)/2|$.
5. $\left|3\left(-\frac{5}{3}\right)\right|$.
6. $|2-7| - |7-2|$.
7. $|x| < 3$.
8. $|x| < 10$.
9. $|2-\sqrt{5}|$.
10. $|\sqrt{5}-2|$.

11. Using the absolute value symbol, express the fact that

    a. x is strictly within 3 units of 7,
    b. x differs from 2 by less than 3,
    c. x is no more than 5 units from 7,
    d. the distance between 7 and x is 4,
    e. x + 4 is strictly within 2 units of 0,
    f. x is strictly between - 3 and 3,
    g. $x < -6$ or $x > 6$,
    h. $x - 6 > 4$ or $x - 6 < -4$,
    i. the number x of hours that a machine will operate efficiently differs from 105 by less than 3,
    J. the average monthly income x (in dollars) of a family differs from 850 by less than 100.

12. Use absolute value notation to indicate that x and μ differ by no more than σ.

13. Use absolute value notation to indicate that the prices p_1 and p_2 of two products may differ by no more than 2 (dollars).

14. Find all values of x such that $|x - \mu| \leq 2\sigma$.

In Problems 15 - 36, solve the given equation or inequality.

15. $|x| = 7.$

16. $|-x| = 2.$

17. $\left|\frac{x}{3}\right| = 2.$

18. $\left|\frac{4}{x}\right| = 8.$

19. $|x - 5| = 8.$

20. $|4 + 3x| = 2.$

21. $|5x - 2| = 0.$

22. $|7x + 3| = x.$

23. $|7 - 4x| = 5.$

24. $|1 - 2x| = 1.$

25. $|x| < 4.$

26. $|-x| < 3.$

27. $\left|\frac{x}{4}\right| > 2.$

28. $\left|\frac{x}{3}\right| > \frac{1}{2}.$

29. $|x + 7| < 2.$

30. $|5x - 1| < -6.$

31. $\left|x - \frac{1}{2}\right| > \frac{1}{2}.$

32. $|1 - 3x| > 2.$

33. $|5 - 2x| \leq 1.$

34. $|4x - 1| \geq 0.$

35. $\left|\frac{3x - 8}{2}\right| \geq 4.$

36. $\left|\frac{x - 8}{4}\right| \leq 2.$

4 - 7 Linear Programming.

Linear programming is a mathematical optimization technique, a method which attempts to maximize or minimize some objective, e. g., maximize profits, minimize costs, etc. In any linear programming problem certain decisions need to be made.

The basic structure of a linear programming problem is either to maximize or to minimize an *objective function* while satisfying a set of constraining conditions, or *constraints.* The objective function is a mathematical representation of the overall goal stated in terms of the decision variables. The objective function may represent goals such as profit level, total revenue, total cost, pollution levels, percent return on investments, and so forth. The set of constraints represents conditions which must be satisfied in determining levels for the decision variables. For example, in attempting to maximize profits from the production and sale of a group of products, sample constraints might reflect limited labor resources, limited raw materials, and limited demand for the products.

These problems are called linear programming problems because the objective time function and constraints are all linear.

An optimal solution to any linear programming problem is found at a corner point.

Graphical Solution of Linear Programming Problems.

Exmaple 1:

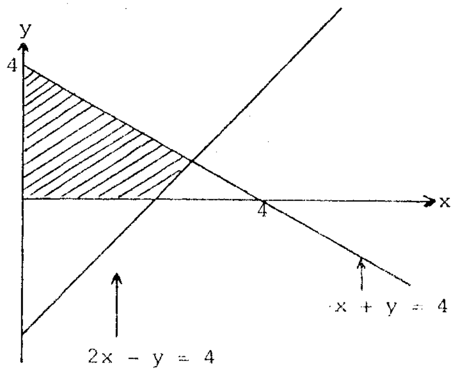
Maximize : y

Subject to : $x + y \leqslant 4$

$2x - y \leqslant 4$

$x \geqslant 0$

$y \geqslant 0$

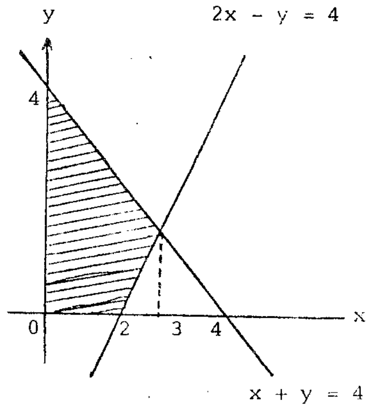


We are asked to maximize y

$\therefore$ Find the highest point in the solution space. The maximum value of y is 4 Ans

Example 2:

$$\begin{aligned} \text{Maximize} &: \; x \\ \text{Subject to} &: \; x + y \leqslant 4 \\ & \; 2x - y \leqslant 4 \\ & \; x \geqslant 0 \\ & \; y \geqslant 0 \end{aligned}$$



$\therefore$ The highest point in x in the solution space is $x = 2\frac{2}{3}$ Ans

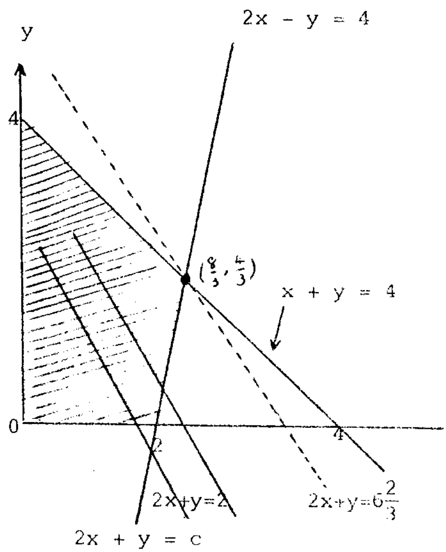
Example 3:

$$\begin{aligned} \text{Maximize} &: \; 2x + y \\ \text{Subject to} &: \; x + y \leqslant 4 \\ & \; 2x - y \leqslant 4 \\ & \; x \geqslant 0 \\ & \; y \geqslant 0 \end{aligned}$$

Test the direction of increase of the objective function $2x + y$

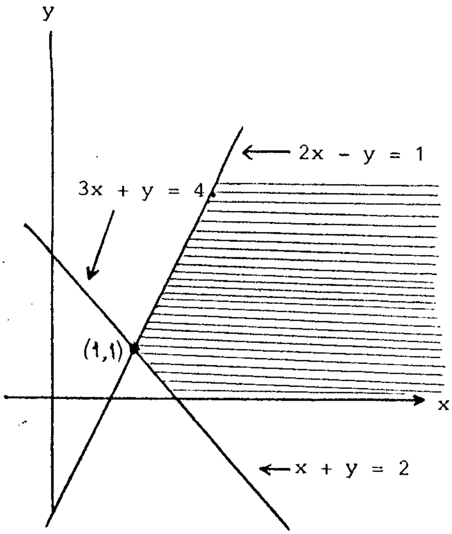
Let $2x + y = c$ and test by changing the value of c until the line touches the corner-point which is the maximum in that direction. (last point in the solution space)

$\therefore 2x + y = 6\frac{2}{3}$ And



Example 4:

$$\text{Minimize} \quad : \quad 3x + y$$
$$\text{Subject to} \quad : \quad x + y \geq 2$$
$$2x - y \geq 1$$
$$x \geq 0$$
$$y \geq 0$$



The lines lead to a decrease of the objective function and move down to the left.

The optimal solution occurs at the corner point $x = 1$, $y = 1$

$\therefore$ $3x + y = 3 + 1 = 4$ Ans

Note:

To find the optimal solution by testing the value of c is quite troublesome before we get the required value. The other alternative is to substitute the coordinates of the corner-points into the objective function. The point that gives the maximum or minimum value, depending on the problem, will be the answer.

Example 5:

A firm is planning production for the next week. It is making products, x and y each of which requires certain foundry, machining and finishing capacities. The following number of hours is available in each area during the week planned: Foundry = 110 hrs., machining = 150 hrs., and Finishing = 60 hrs. Number of hours needed in each process per unit is given in the table.

Products	Hours per unit		
	Foundry	Machining	Finishing
x	6	3	4
y	6	6	2

The firm seeks to determine the combination of products x and y which is most profitable. Assume that each unit of x produced results in a profit of ฿2 and each unit of y in a profit of ฿3. What is the optimum combination of products to produce?

Solution:

Let x = units of product X

y = units of product Y

The objective function is to maximize profit and is defined by

$$p = 2x + 3y$$

Constraints are:

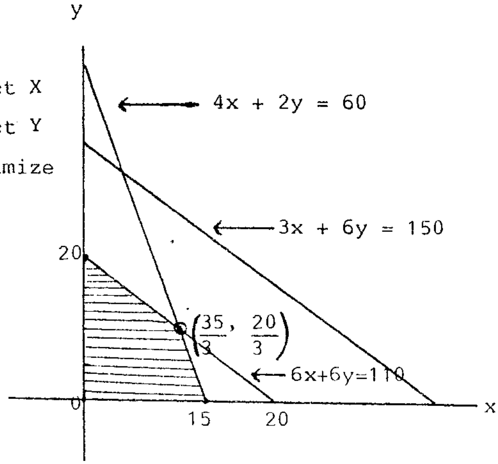
$$6x + 6y \leqslant 110$$

$$3x + 6y \leqslant 150$$

$$4x + 2y \leqslant 60$$

$$x \geqslant 0$$

$$y \geqslant 0$$



Substitute the coordinates of the corner-points in the objective function.

At $(0,0)$, $p = 2x + 3y = 0$

$(15,0)$, $p = 2(15) + 3(0) = 30$

$\left(\frac{35}{3}, \frac{20}{3}\right)$, $p = 2\left(\frac{35}{3}\right) + 3\left(\frac{20}{3}\right) = \frac{70}{3} + 20 = \frac{130}{3} = 43\frac{1}{3}$

$(0,20)$, $p = 2(0) + 3(20) = 60$

The optimum combination of products to be produced is

$x = \frac{35}{3}$, $y = \frac{20}{3}$ Ans

Example 6:

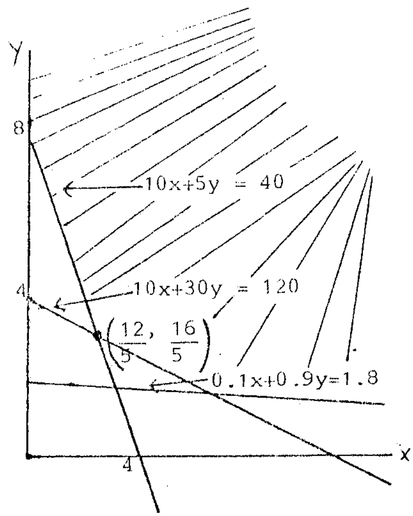
An animal feed is to be a mixture of two foodstuffs, each unit of which contains protein, fat and carbohydrate in the number of grams given in table. Each bag of the resulting mixture is to contain at least 40 grams of protein, 1.8 grams of fat, and 120 grams of carbohydrate.

	Foodstuff	
	1	2
Protein	10	5
Fat	0.1	0.9
Carbohydrate	10	30

a) *Graph the system of inequalities which meet these requirements.*

b) *The manufacturer of animal feed seeks to determine the combination of foodstuffs 1 and 2 which will satisfy the specified nutrient requirements at the lowest possible cost. Assume that each unit of foodstuff 1 costs 60 satang and each unit of foodstuff 2 costs 40 satang. What is the optimum feedmix?*

Solution:



Let x = Number of units of foodstuff 1 employed in the mix.

y = Number of units of foodstuff 2 employed in the mix.

The objective function is the lowest cost.

$$c = 0.6x + 0.4y$$

Constraints: $10x + 5y \geqslant 40$

$0.1x + 0.9y \geqslant 1.8$

$10x + 30y \geqslant 120$

$x \geqslant 0$

$y \geqslant 0$

Test all the coordinates of the corner-points. We find out that the minimum total cost is obtained at $\left(\frac{12}{5}, \frac{16}{5}\right)$

$$C = 0.6\left(\frac{12}{5}\right) + 0.4\left(\frac{16}{5}\right) = \frac{7.2}{5} + \frac{6.4}{5} = \frac{13.6}{5}$$

$$= ฿2.72 \quad \underline{\text{Ans}}$$

Example 7:

Solve the following linear programming problem by the corner-point method.

Maximize : $Z = 20x_1 + 15x_2$

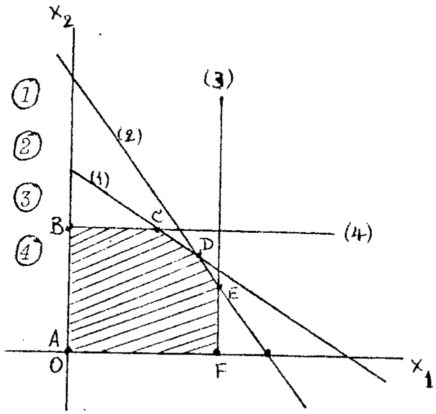
Subject to : $3x_1 + 4x_2 \leq 60$ —— ①

$4x_1 + 3x_2 \leq 60$ —— ②

$x_1 \leq 10$ —— ③

$x_2 \leq 12$ —— ④

$x_1, x_2 \geq 0$



Solution:

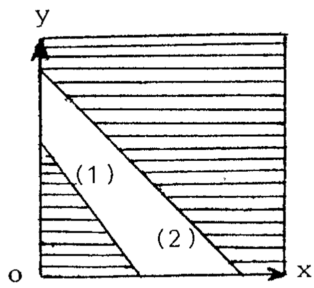
Corner-point (x_1, x_2)		$Z = 20x_1$		$+ 15x_2$		
A (0,0)	:	20(0)	+	15(0)	=	0
B (0,12)	:	20(0)	+	15(12)	=	180
C (4,12)	:	20(4)	+	15(12)	=	260
D $\left(\frac{60}{7},\frac{60}{7}\right)$		$20\left(\frac{60}{7}\right)$	+	$15\left(\frac{60}{7}\right)$	=	300
E $\left(10,\frac{20}{3}\right)$		20(10)	+	$15\left(\frac{20}{3}\right)$	=	300
F (10,0)		20(10)	+	15(0)	=	200

There is a tie for the highest value of Z between points D and E. If you compute the slope of the objective function, you will find that it is the same as for constraint ②. Thus there are alternative optimal solution along DE.

Example 8:

No Feasible Solution:

The system of constraints in a linear programming problem may not have any points which satisfy all the constraints. In such cases, there are no points in the solution set, and the linear programming problem is said to have no feasible *solution.*



Solution:

As shown in the figure, a problem has no feasible solutoion. Constraint (1) a 'less than or equal to' type while constraint (2) a 'greater than or equal to' type. A problem can certainly have both types of constraints. In this case the set of points satisfying one constraint includes none of the points satisfying the other.

EXERCISE: 4 - 3a

1. Maximize

$P = 10x + 12y$

subject to

$$\begin{aligned} x + y &\leqslant 60, \\ x - 2y &\geqslant 0, \\ x, y &\geqslant 0. \end{aligned}$$

2. Maximize

$P = 5x + 6y$

subject to

$$\begin{aligned} x + y &\leqslant 80, \\ 3x + 2y &\leqslant 220, \\ 2x + 3y &\leqslant 210, \\ x, y &\geqslant 0. \end{aligned}$$

3. Maximize

$Z = 4x - 6y$

subject to

$$\begin{aligned} y &\leqslant 7, \\ 3x - y &\leqslant 3, \\ x + y &\geqslant 5, \\ x, y &\geqslant 0. \end{aligned}$$

4. Minimize

$Z = x + y$

subject to

$$\begin{aligned} x - y &\geqslant 0, \\ 4x + 3y &\geqslant 12, \\ 9x + 11y &\leqslant 99, \\ x &\leqslant 8, \\ x, y &\geqslant 0. \end{aligned}$$

5. Maximize
$$Z = 4x - 10y$$
subject to
$$\begin{aligned} x - 4y &\geq 4, \\ 2x - y &\leq 2, \\ x, y &\geq 0. \end{aligned}$$

6. Minimize
$$Z = 20x + 30y$$
subject to
$$\begin{aligned} 2x + y &\leq 10, \\ 3x + 4y &\leq 24, \\ 8x + 7y &\geq 56, \\ x, y &\geq 0. \end{aligned}$$

7. Minimize
$$Z = 7x + 3y$$
subject to
$$\begin{aligned} 3x - y &\geq -2, \\ x + y &\leq 9, \\ x - y &= -1, \\ x, y &\geq 0. \end{aligned}$$

8. Maximize
$$Z = .5x - .3y$$
subject to
$$\begin{aligned} x - y &\geq -2, \\ 2x - y &\leq 4, \\ 2x + y &= 8, \\ x, y &\geq 0. \end{aligned}$$

9. Minimize
$$C = 2x + y$$
subject to
$$\begin{aligned} 3x + y &\geq 3, \\ 4x + 3y &\geq 6, \\ x + 2y &\geq 2, \\ x, y &\geq 0. \end{aligned}$$

10. Minimize
$$C = 2x + 2y$$
subject to
$$\begin{aligned} x + 2y &\geq 80, \\ 3x + 2y &\geq 160, \\ 5x + 2y &\geq 200, \\ x, y &\geq 0. \end{aligned}$$

11. Maximize
$$Z = 10x + 2y$$
subject to
$$\begin{aligned} x + 2y &\geq 4, \\ x - 2y &\geq 0, \\ x, y &\geq 0. \end{aligned}$$

12. Minimize
$$Z = y - x$$
subject to
$$\begin{aligned} x &\geq 3, \\ x + 3y &\geq 6, \\ x - 3y &\geq -6, \\ x, y &\geq 0. \end{aligned}$$

13. A toy manufacturer preparing his production schedule for two new toys, widgets and wadgits, must use the information concerning their construction times given in Table 16-3. For example, each widget requires 2 hours on Machine A. The available

	MACHINE A	MACHINE B	FINISHING
Widgets	2 hr	1 hr	1 hr
Wadgits	1 hr	1 hr	3 hr

employee hours per week are as follows: for operating machine A, 70 hours; for B, 40 hours; for finishing, 90 hours. If the profits on each widget and wadgit are \$4 and \$6, respectively, how many of each toy should be made per week in order to maximize profit? What would the maximum profit be?

14. A manufacturer produces two types of barbecue grills, Old Smokey and Blaze Away. During production the grills require the use of two machines, A and B. The number of hours needed on both are indicated in Table 16-4. If each machine can be used 24 hours a day, and the profits on the Old Smokey and Blaze Away models are \$4 and \$6, respectively, how many of each type of grill should be made per day to obtain maximum profit? What is the maximum profit?

	MACHINE A	MACHINE B
Old Smokey	2 hr	4 hr
Blaze Away	4 hr	2 hr

15. A diet is to contain at least 16 units of carbohydrates and 20 units of protein. Food A contains 2 units of carbohydrates and 4 of protein; Food B contains 2 units of carbohydrates and 1 of protein. If Food A costs \$1.20 per unit and Food B costs \$0.80 per unit, how many units of each food should be purchased in order to minimize cost? What is the minimum cost?

16. A produce grower is purchasing fertilizer containing three nutrients: A, B, and C. The minimum weekly requirements are 80 units of A, 120 of B, and 240 of C. There are two popular blends of fertilizer on the market. Blend I, costing \$4 a bag, contains 2 units of A, 6 of B, and 4 of C. Blend II, costing \$5 a bag, contains 2 units of A, 2 of B, and 12 of C. How many bags of each blend should the grower buy each week to minimize the cost of meeting the nutrient requirements?

17. A company extracts minerals from ore. The number of pounds of minerals A and B that can be extracted from each ton of ores I and II are given in Table 16-5 together with the costs per ton of the ores. If the company must produce at least 3000 lb of A and 2500 lb of B, how many tons of each ore should be processed in order to minimize cost? What is the minimum cost?

	ORE I	ORE II
Mineral A	100 lb	200 lb
Mineral B	200 lb	50 lb
Cost per ton	\$50	\$60

EXERCISE: 4 - 3b

Solve the following linear programming problems using the corner-point method.

1. Minimize : $Z = 2x_1 + 4x_2$

   Subject to : $3x_1 + 2x_2 \geqslant 24$

   $4x_1 + 5x_2 \geqslant 60$

   $x_1, x_2 \geqslant 0$

2. Minimize : $Z = 5x_1 + 3x_2$

   Subject to : $2x_1 + x_2 \geqslant 0$

   $x_1 + 3x_2 \geqslant 15$

   $x_1 \leqslant 0$

   $x_2 \leqslant 8$

   $x_1, x_2 \geqslant 0$

3. Maximize : $Z = 6x_1 + 4x_2$

   Subject to : $x_1 + x_2 \leqslant 5$

   $3x_1 + 2x_2 \leqslant 12$

   $x_1, x_2 \geqslant 0$

4. Minimize : $Z = 20x_1 + 5x_2$

   Subject to : $x_1 + x_2 \geqslant 4$

   $4x_1 + x_2 \geqslant 8$

   $x_1 \geqslant 1$

   $x_2 \leqslant 6$

   $5x_1 + 4x_2 \leqslant 40$

   $x_1, x_2 \geqslant 0$

5. Maximize : $Z = 15x_1 + 20x_2$

Subject to : $x_1 + x_2 \geqslant 12$

$6x_1 + 9x_2 \leqslant 54$

$15x_1 + 10x_2 \leqslant 90$

$x_1 , x_2 \geqslant 0$

6. A producer of machinery wishes to maximize the profits from producing two products, product A and product B. The major inputs for each product are steel, electricity, and work-hours. The table given summarizes the inputs per unit, available resources, and profit margin per unit.

   a) Formulate the linear programming model for this situation:

	Product A	Product B	Monthly Total Available
Energy	200 kWh	400 kWh	20,000 kWh
Steel	100 lb	120 lb	10,000 lb
Labor	5 h	8 h	400 h
Profit per unit	฿400	฿1000	

   b) Assume that energy costs ฿10 per kilowatthour, steel costs ฿80 per pound, and labor costs ฿160. An order for 10 units of product A must be filled using this month's production, and combined production for the two products cannot be less than 35 units. If the objective is to minimize total cost, formulate the linear programming model.

7. Maximize : $Z = 4x_1 + 3x_2$

Subject to : $x_1 = x_2$

$2x_1 + 5x_2 \leqslant 20$

$x_1 , x_2 \geqslant 0$

8. Graphically determine the solution, if one exists, for this system of inequalities:

$$x_1 - x_2 \geqslant 5$$

$$x_1 + x_2 \leqslant 15$$

$$x_1 \leqslant 20$$

$$x_1 , x_2 \geqslant 0$$

9. A company manufactures and sells five products. Cost per unit and selling price are given in table. If the objective is to maximize total profit, formulate programming model having the following constraints:

Products	A	B	C	D	E
Costs per unit Selling price	฿1,000 ฿1,400	฿1,600 ฿1,800	฿6,000 ฿7,000	฿ 500 ฿1,000	฿200 ฿240

Constraints: at least 20 units of product A and at least 10 units of product B must be produced; sufficient raw materials are not available for total production in excess of 75 units; the number of units produced of products C and E must be equal

10. A president of a construction company is interested in comparing costs involved in purchasing or renting a piece of machinery needed for excavation. If he were to purchase it, his fixed annual cost would be ฿80,000 and daily operation and maintenance costs would be ฿1,600 for each day it is used. But he finds that he can rent the same machinery for ฿12,000 per month (on a yearly basis). If the machinery were rented, the daily cost would be ฿1,200 for each day it is used. Neglecting any other considerations, determine the least number of days he would have to use the machinery each year to justify renting the equipment rather than purchasing it.

11. A company manufactures a product that has a unit selling price of ₱400 and a unit cost of ₱300. If fixed costs are ₱12,000,000; determine the least number of units that must be sold for the company to have a profit.

12. To produce one unit of a new product, a company determines that the cost for material is \$2.50 and the cost of labor is \$4. The costant overhead, regardless of sales volume, is \$5,000. If the cost to a wholesaler is \$7.40 per unit, determine the least number of units that must be sold by the company to realize a profit.

13. For business purposes Mr. Michael Joseph wants to determine the difference between the costs of owning and renting an automobile. He can rent a compact for \$135 per month (on an annual basis). Under this plan his cost per mile (gas and oil) is \$0.05. If he were to purchase the car, his fixed annual expense would be \$1,000 and other costs would amount to \$0.10 per mile. . What is the least number of miles he would have to drive per year to make renting no more expensive than purchasing?

14. A shirt manufacturer produces N shirts at a total labor cost (in dollars) of 1.2N and a total material cost of .3N. The constant overhead for the plant is \$6,000. If each shirt sells for \$3, how many must be sold by the company to realize a profit?

15. The cost of publication of each copy of a magazine is \$0.65. It is sold to dealers for \$0.60 each, and the amount received for advertising is 10 percent of the amount received for all magazines issued beyond 10,000. Find the least number of magazines that can be published without loss, that is, such that profit $\geqslant 0$. (Assume that all issues will be sold)

16. A company produces alarm clocks. During the regular work week the labor cost for producing one clock is \$2.00. However, if a clock is produced in overtime the labor cost is \$3.00. Management has decided to spend no more than a total of \$25,000 per week for labor. The company must produce 11,000 clocks this week. What is the minimum number of clocks that must be

produced during the regular work week?

17. A company invests a total of \$30,000 of surplus funds at two annual rates of interest: 5 percent and 6¾ percent. It wishes an annual yield of no less than 6½ percent. What is the least amount of money that it must invest at the 6¾ percent rate?

18. The current ratio of Precision Machine Products is 3.8. If their current assets are \$570,000 what are their current liabilities? To raise additional funds, what is the maximum amount they can borrow on a short-term basis if they want their current ratio to be no less than 2.6?

19. A manufacturer presently has 2500 units of his product in stock. The product is now selling at \$4 per unit. Next month the unit price will increase by \$0.05. The manufacturer wants the total revenue received from the sale of the 2500 units to be no less than \$10,750. What is the maximum number of units that can be sold this month?

20. Suppose that consumers will purchase x units of a product at a price of $\frac{100}{x} + 1$ dollars per unit. What is the minimum number of units that must be sold in order that sales revenue be greater than \$5,000?

EXERCISE: 4 - 3c

In Problems **1–10** *solve the given inequality or system of inequalities.*

1. $-3x + 2y > -6.$

2. $x - 2y + 6 \geq 0.$

3. $2y \leq -3.$

4. $-x < 2.$

5. $\begin{cases} y - 3x < 6, \\ x - y > -3. \end{cases}$

6. $\begin{cases} x - 2y > 4, \\ x + y > 1. \end{cases}$

7. $\begin{cases} x - y < 4, \\ y - x < 4. \end{cases}$

8. $\begin{cases} x > y, \\ x + y < 0. \end{cases}$

9. $\begin{cases} 3x + y > -4, \\ x - y > -5, \\ x \geq 0. \end{cases}$

10. $\begin{cases} x - y > 4, \\ x < 2, \\ y < -4. \end{cases}$

In Problems **11–23**, *do not use the simplex method.*

11. Maximize
$$Z = x - 2y$$
subject to
$$\begin{aligned} y - x &< 2, \\ x + y &< 4, \\ x &< 3, \\ x, y &> 0. \end{aligned}$$

12. Maximize
$$Z = 4x + 2y$$
subject to
$$\begin{aligned} x + 2y &< 10, \\ x &< 4, \\ y &> 1, \\ x, y &> 0. \end{aligned}$$

13. Minimize
$$Z = 2x - y$$
subject to
$$\begin{aligned} x - y &> -2, \\ x + y &> 1, \\ x - 2y &< 2, \\ x, y &> 0. \end{aligned}$$

14. Minimize
$$Z = x + y$$
subject to
$$\begin{aligned} x + 3y &< 15, \\ 3x + 2y &< 17, \\ x - 5y &< 0, \\ x, y &> 0. \end{aligned}$$

15. Minimize
$$Z = 4x - 3y$$
subject to
$$\begin{aligned} x + y &< 3, \\ 2x + 3y &< 12, \\ 5x + 8y &> 40, \\ x, y &> 0. \end{aligned}$$

16. Minimize
$$Z = 2x + 2y$$
subject to
$$\begin{aligned} x + y &> 4, \\ -x + 3y &< 18, \\ x &< 6, \\ x, y &> 0. \end{aligned}$$

17. Maximize
$$Z = 9x + 6y$$
subject to
$$\begin{aligned} x + 2y &< 8, \\ 3x + 2y &< 12, \\ x, y &> 0. \end{aligned}$$

18. Maximize
$$Z = 4x + y$$
subject to
$$\begin{aligned} x + 2y &> 8, \\ 3x + 2y &> 12, \\ x, y &> 0. \end{aligned}$$

19. Maximize
$$Z = 4x_1 + 5x_2$$
subject to
$$\begin{aligned} x_1 + 6x_2 &< 12, \\ x_1 + 2x_2 &< 8, \\ x_1, x_2 &> 0. \end{aligned}$$

20. Maximize
$$Z = 18x_1 + 20x_2$$
subject to
$$\begin{aligned} 2x_1 + 3x_2 &< 18, \\ 4x_1 + 3x_2 &< 24, \\ x_2 &< 5, \\ x_1, x_2 &> 0. \end{aligned}$$

21. Minimize
$$Z = x_1 + x_2$$
subject to
$$\begin{aligned} 3x_1 + 4x_2 &> 24, \\ x_2 &> 3, \\ x_1, x_2 &> 0. \end{aligned}$$

22. Maximize
$$Z = x_1 + 2x_2$$
subject to
$$\begin{aligned} x_1 + x_2 &< 12, \\ x_1 + x_2 &> 5, \\ x_1 &< 10, \\ x_1, x_2 &> 0. \end{aligned}$$

23. Minimize

$$Z = 2x_1 + x_2$$

subject to

$$x_1 + 2x_2 < 6,$$
$$x_1 + x_2 > 1,$$
$$x_1, x_2 > 0.$$

4.8 Applications of Equations:

An equation is a statement that two expressions are equal. They are separated by the equality sign '='

Example: $x + 2 = 3$

a) A linear equation in the variable x is an equation that can be written in the form $ax + b = 0$
   Where a and b are constants and $a \neq 0$
   A linear equation is also called a first-degree equation or an equation of degree one.
b) A quadratic equation in the variable x is an equation that can be written in the form

$$ax^2 + bx + c = 0$$

Where a, b and c are constants and $a \neq 0$

– A guadratic equation is also called a second-degree equation or an equation of degree two.

Since you have learnt how to solve linear equations and quadratic equations, we'll go straight to applications of equations.

In most cases, to solve practical problems you must translate the relationships stated in the problems into mathematical symbols. This is called modelling. The following examples illustrate basic techniques and concepts. We shall refer to some business terms relative to a manufacturing firm.

Fixed Cost (or overhead): is the sum of all costs that are independent of the level of production, such as rent, insurance, etc....This cost must be paid whether or not output is produced.

Total Cost: is the sum of variable cost and fixed cost.

Total cost = Variable Cost + Fixed Cost.

Total revenue: is the price per unit of output times the number of units sold.

Total Revenue = (Price per unit) (Number of units sold)

Profit: is total revenue minus total cost.

Profit = Total Revenue – Total Cost.

Variable Cost: is the sum of all costs that are dependent on the level of output, such as labor and material.

Example 1:

A company produces product A for which the variable cost per unit is ฿120 and fixed cost is ฿1,600,000. Each unit has a selling price of ฿200. Determine the number of units that must be sold for the company to earn a profit of ฿1,200,000.

Solution:

Let x be the number of units which must be sold.

The variable cost is	= 120x
Fixed cost is	= 1,600,000
$\therefore$ Total cost	= 120x + 1,600,000
Total revenue is	= 200x
Profit	= Total Revenue - Total cost
$\therefore$ 1,200,000	= 200x -(120x + 1,600,000)
1,200,000	= 200x - 120x - 1,600,000
80x	= 2,800,000
x	= 35,000

Thus 35,000 units must be sold to earn a profit of ฿1,200,000 Ans

Example 2:

A company manufactures women's sportswear and is planning to *sell its new line of slacks sets to retail outlets. The cost to the retailer will be ฿660 per set. As a convenience to the retailer, the manufacturer will attach a price tag to each set. What amount should be marked on the price tag so that the retailer may reduce this price by 20 percent during a sale and still make a profit of 15 percent on the cost?*

Solution:

Let p be the tag price per set in baht

The retailer receives = p - 0.2p
and this amount p - 0.2p must be equal to his cost, 660 plus his profit, (0.15)(660)

$$p - 0.2p = 660 + (0.15)(660)$$
$$0.8p = 759$$
$$p = \frac{759}{0.8} = 948.75$$

$\therefore$ The price tag should be at ฿948.75 Ans

Example 3:

A total of ฿300,000 was in vested in two business ventures, A and B. At the end of the first year, A and B yielded returns of 6 percent and 8 percent, respectively, of the original investments. How was the original amount allocated if the total amount earned was ฿21600?

Solution:

Let ฿ x be the amount invested at 6%

The amount invested at 8% $= 300,000 - x$ baht

The interest obtained from ฿ x at 6% $= \frac{6x}{100}$

The interest obtained from (300,000 - x) $= (300,000 - x)\left(\frac{8}{100}\right)$

But the total amount earned = 21600

$$\therefore \frac{6}{100}x + (300,000 - x)\left(\frac{8}{100}\right) = 21600$$

$$\left(\frac{6x}{100}\right) - \left(\frac{8x}{100}\right) = 21600 - 24000$$

$$2x = 240000$$

$$x = ฿120,000$$

Thus ฿120,000 was invested at 6% and 300,000 - 120,000
= ฿180,000 was invested at 8% <u>Ans</u>

Example 4:

A small side resort hotel consists of 70 apartments. At ฿5,000 per month every apartment can be rented. However, for each ฿200 per month increase there will be two vacancies with no possibility of filling them. The firm wants to receive ฿359,600 per month from rents. What rent should be charged for each apartment?

Solution:

Suppose n is the number of ฿200 increases. Then the increase in rent per apartment will be ฿200n and there will be 2n vacancies.

Since:

Total rent = (rent per apartment)(number of apartments rented
we have $359600 = (5000 + 200n)(70 - 2n)$

$$359600 = 350000 - 400n^2 + 14000n - 10000n$$

$$400n^2 - 400n + 9600 = 0$$

$$n^2 - 10n + 24 = 0$$

$$(n-6)(n-4) = 0$$

$$n = 4,\ 6$$

Thus, the rent charged should be 5000 + 200(6) = ฿6200
or 5000 + 200(4) = ฿5800 <u>Ans</u>

EXERCISE: 4 - 4

1. A company produces Product ~~Z~~ at a variable cost per unit of ฿44. If fixed costs are ฿1,900,000 and each unit sells for ฿60, how many units must be sold for the company to have a profit of ฿1,000,000?

2. A company management would like to know the total sales units that are required for the company to earn a profit of ฿2,000,000. The following data are available: unit selling price of ฿400,

variable cost per unit of ฿300; total fixed cost of ฿6,000,000. From these data determine the required sales units.

3. A person wishes to invest ฿400,000 in two enterprises so that the total income per year will be ฿28,800. One enterprise pays 6 percent annually, the other has more risk and pay 7½ percent annually. How much must be invested in each?

4. A person invests ฿400,000: part at an interest rate of 6 percent annually and the remainder at 7 percent annually. The total interest at the end of one year is eqaivalent to an annual 6¾ percent rate on entire ฿400,000. How much was invested at each rate?

5. The cost of a product to a retailer is ฿68. If he wishes to make a profit of 20 percent on the selling price, at what price should the product be sold?

6. In two years a company will require ฿22,472,000 in order to retire some bonds. If the company now invests ฿20,000,000 for this purpose, what annual rate of interest, compounded annually, must it receive on this amount in order to retire the bonds?

7. In two years company will begin an expansion program. It has decided to invest ฿40,000,000 now so that in two years the total value of the investment will be ฿43,264,000 the amount required for the expansion. What is the annual rate of interest, compounded annually, that the company must receive to achieve its purpose?

8. A company finds that if it produces and sells g units of a product, its total sales revenue is ฿$100\sqrt{g}$. If the variable cost per unit is ฿40 and the fixed cost is ฿24,000 find the values of g for which:

Total Sales Revenue = Variable Cost + Fixed Cost.That is, profit is Zero.

9. Suppose that consumers will purchase g units of a product when the price is $\frac{80-g}{4}$ baht each. How many units must be sold

in order that sales revenue be ฿8,000?

10. How long would it take to double an investment at simple interest with a rate of 5 percent per year?

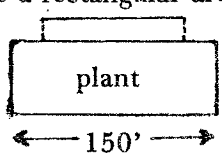
11. The investor of a new toy offers the Kiddly Toy Company exclusive rights to manufacture and sell his product for a lump-sum payment of ฿500,000. After estimating that future sales possibilities beyond one year are nonexistent, the company management is reviewing the following alternate proposal: to give him a lump-sum payment of ฿40,000 plus a royalty of ฿10.00 for each unit sold. How many units must be sold the first year to make this alternative as economically attractive to the investor as his original request? (Hint: Determine when his incomes under both proposals are the same)

12. You are the chief financial advisor to a corporation which owns an office complex consisting of 50 suites. At ฿8,000 per month every suite can be rented. However, for each ฿400 per month increase there will be two vacancies with no possibility of filling them. The corporation wants to receive a total of ฿404,800 per month from rents in the complex. You are asked to determine the rent that should be charged for each suite.

13. The monthly revenue R of a certain Company is given by $R = 800p - 7p^2$, where p is the price in Baht of the product they manufacture. At what price will the revenue be ฿200,000 if the price must be greater than ฿1,000?

14. When the price of a product is p baht each, suppose that a manufacturer will supply $2p - 8$ units of the product to the market and that consumers will demand to buy $300 - 2p$ units. At the value of p for which supply equals demand, the market is said to be in equilibrium. Find this value of p.

15. For security reasons a company will enclose a rectangular area of 11,200 ft^2. in the rear of its plant. One side will be bounded by the building and the other three sides by fencing as shown in the figure. If



300 ft. of fencing will be used, what will be the dimensions of the rectangular area?

16. A company manufactures products A and B. The cost of producing each unit of A is ฿40 more than that of B. The costs of production of A and B are ฿30,000 and ฿20,000 respectively and 25 more units of A are produced than of B. How many of each are produced?

17. A land investment company purchased a parcel of land for ฿144,000. After having sold all but 20 rai at a profit of ฿600 per rai, the entire cost of the parcel had been regained. How many rai were sold?

18. A machine company has an incentive plan for its salespeople. For each machine that a salesperson sells, the commission is ฿400. The commission for every machine sold will increase by ฿0.40 for each machine sold over 600. For example, the commission on each of 602 machines sold is ฿400.80. How many machines must a salesperson sell in order to earn ฿308,000?

CHAPTER 5

FUNCTION

Function: can be defined in different connotations as,

Definition: (1) A function is a mathematical relationship in which the values of a single dependent variable are determined from the values of one or more independent variables.

or **Definition:** (2) A function is a rule that assigns to each input number exactly one output number. The set of all input numbers to which the rule applies is called *the domain of the function.* The set of all output numbers is called *the range.*

or **Definition:** (3) A function is a relation in which each element in the domain D, there corresponds *one and only one* element in the range R.

Functions suggest that the value of something depends upon the values of other things. There are uncountable numbers of functional relationships in the world about us. The size of the crowds at a beach may depend upon the temperature, quantities sold of a product may depend on the price charged for the product, grades may depend upon the amount of time that a student studies, city tax rates may depend upon the level of municipal spending, and the length of a person's hair may depend upon time.

In mathematics, variables that are functionally related can be put in the form of equation

$$y = f(x)$$

to denote a functional relationship between x and y. It is read as 'y equals f of x' or 'y is a function of x'. When we say that y is a function of x, we mean that the value of the variable y depends upon and is uniquely determined by the value of the variable x. This is the reason why variable x is called *'independent variable'* and the variable y is called the *'dependent variable'*. 'f' is the name of the function or rule which allows one to determine the unique value of y, given a value of x.

Although y usually represents the dependent variable, x the independent variable, and f the name of the function, any letter may be used to represent the dependent and independent variables and the function name. For example, the equation

$$u = g(v)$$

where u is a dependent variable and u is determined by an independent variable v. And the name of the function or rule relating the two variables is g.

Usually, letters such as f, g, h, F, G etc, are used to name functions. Suppose we let f represent the function defined by $y = x + 2$. Then the notation f(x), which is read 'f of x', is used to represent the output number corresponding to the input number x.

$$\begin{array}{c} \text{input} \\ \downarrow \\ f(x) \\ \downarrow \\ \text{output} \end{array}$$

Thus f(x) is the same as y. But since $y = x + 2$ we may write $f(x) = x + 2$.

A functional relationship can be presented in various forms depending on its utility. For example,

(i) In form of statement
(ii) In form of table
(iii) In form of equation
(iv) In form of graph

Suppose that you have taken a job as a salesman. Your employer has stated that *your salary will depend upon the number of units you sell each week.*

The functional relationship in italics is in the form of statement.

Now if we let

$$y = \text{weekly salary in baht}$$
$$x = \text{number of units sold each week}$$

then, the dependency stated can be represented by the equation

$$y = f(x)$$

where f is the name of the salary function. This stated relationship given by this equation is in a general form. There is nothing specific about the relationship.

Suppose your employer has given you an equation for determining your weekly salary and it is

$$y = 3x + 25$$

Then, this equation presents a specific rule of relationship since

$$y = f(x) = 3x + 25$$

For instance, if we want to compute your weekly salary when you sell 100 units, substitution of $x = 100$ in the equation

$$\begin{aligned} y &= 3(100) + 25 \\ &= ฿\, 325 \end{aligned}$$

In general, $y = f(x) = ax + b$

The input number x or independent variable is called the *domain* of the function. The output number y or dependent variable is called the *range*.

Note:

Not all equations define y as a function of x.

Example:

$$y^2 = x \text{ if } x = 9$$
$$\therefore y = \pm 3$$

Thus, with the input number 9 there are assigned not one but two output numbers, + 3 and – 3. Hence y is not a function of x.

5.1 Types of Function.

Special names are given to functions having particular forms:

1. Linear function: It has the form $f(x) = ax + b$, where a and b are constants and $a = 0$. Its domain is all real numbers.

2. A Constant function: is one of the form $g(x) = c$, where c is a fixed real number.

3. A quadratic function: is one of the form $h(x) = ax^2 + bx + c$, wher a, b and c are constants and $a = 0$. The domain of h is all real numbers, and y is a function of x.

4. A polynomial function is one of the form

$$f(x) = C_n x^n + C_{n-1} x^{n-1} + C_{n-2} x^{n-2} + \dots\dots\dots + C_1 x + C_0.$$

n is a positive integer and $C_0, C_1 \dots\dots\dots, C_n$ are constants with $C_n = 0$
The domain of a polynomial function is all real numbers.

5. Piece-meal function: is the one which is defined in different parts

Example:

$$F(S) = \begin{cases} 1, & if \; -1 \leq s < 1, \\ 0, & if \;\; 1 \leq s \leq 2, \\ s-3, & if \;\; 2 < s \leq 3 \end{cases}$$

Here s represent input numbers and the domain of F is all s such that $-1 \leqslant s \leqslant 3$. The value of an input number determines which part to use.

6. Absolute value funciton: is the one expressed in absolute term.

Example:

$$f(x) = |x|$$

Functions may be classfied into the following groups.

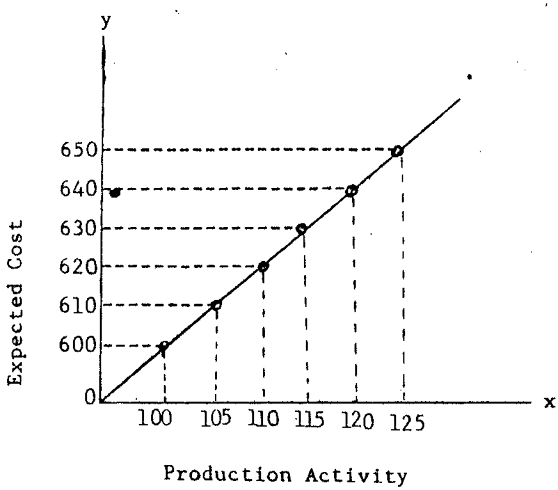
1. Inform of Table

Possible levels of Production Activity (Units/hr)	Expected Variable Unit Cost. (baht)
100	600
105	610
110	620
115	630
120	640
125	650

2. In form of Graph.

Set of pairs (100,600), (105,610), (110,620), (115,630), (120,640), (125,650). This set of pairs can be put in the form of graph by putting variable y or f(x) on the vertical scale and variable x on the horizontal scale.

∴ The graph tells us everything about the function we need to know.



3. In form of Equation:

Demand equation: is any equation that describes the relationship between the price and quantity of a certain product.

Example:

Suppose $p = \frac{100}{q}$ *describes the relationship between the price per unit, P, of a certain product and the number of units, q, of that product that consumers will buy (that is, demand) per week. This equation is called a* demand equation *for the product.*

If q an input number, then to each value of q there is assigned exactly one output number p.

$$\text{input } q \longrightarrow \frac{100}{q} \quad \underline{\text{output}} \longrightarrow \quad = p$$

$$\text{input } 20 \longrightarrow \frac{100}{q} \quad \underline{\text{output}} \longrightarrow \quad = 5$$

Thus price p is a function of quantity demanded, q. Here q is the independent variable and p is the dependent variable. This function is called *a demand function.*

5.2 Combinations of functions.

Functions can be combined to create new functions. They can be added, substracted, multiplied or divided

If $f(x) = x^2$ and $g(x) = x + 1$

$f(x) + g(x) = x^2 + (x + 1)$

The new function $H(x) = f(x) + g(x) = x^2 + x + 1$

Similarly, we can create other functions:

$f(x) - g(x) = x^2 - (x + 1)$

$f(x) .g(x) = x^2(x + 1)$

$\frac{f(x)}{g(x)} = \frac{x^2}{x+1}$, if $g(x) \neq 0$

We assume that x is in the domains of both f and g.

It is possible to combine functions in yet another way.

Definition.

If f and g are functions, then the composition of f with g is the function h defined by

$$h(x) = f[g(x)].$$

Where the domain of h is the set of all x in the domain of g such that g(x) is in the domain of f.

Example:

Let $f(x) = \sqrt{x}$ *and* $g(x) = x + 1$. *Find*

a) $f[g(x)]$

b) $g[f(x)]$

Solution:

a) $f[g(x)] = f(x + 1)$

$f(x) = \sqrt{x}$

$\therefore f(x + 1) = \sqrt{x + 1}$

The domain of g is all real numbers x, and the domain of f is all nonnegative reals. Hence, the domain of the composition is all x for which $g(x) = x + 1$ is nonnegative. That is, the domain is all $x > -1$

b) $g[f(x)] = g(\sqrt{x})$

But $g(x) = x + 1$

$\therefore g(\sqrt{x}) = \sqrt{x} + 1$

The domain of f is all $x > 0$ and the domain of g is all reals. Hence, the domain of the composition is all $x \geq 0$ for which $f(x) = \sqrt{x}$ is real, namely all $x > 0$.

5.3 Applications of linear functions.

Definitions:

Total Variable costs vary with the level of output and are computed as the product of variable cost per unit of output and the level of output.

Revenue = The money which flows into an organization from either selling products or providing services. Total revenue = (price) (quantity sold)

Profit for an organization is the difference between total revenue and total cost. Profit = total revenue – total cost.

A linear function involving one independent variable x and a dependent variable y has the general form

$$y = f(x) = ax + b$$

Linear Cost Functions.

Total cost is usually defined in terms of two components, namely, total variable cost and total fixed cost.

Example 1:

A small city police department is contemplating the purchase of an additional patrol car. Police analysts estimate the purchase cost of a fully equipped car to be \$18,000. They also have estimated an average operating cost of \$0.40 per mile.

a) *Determine the mathematical function which represents the total cost C of owning and operating the car in terms of the number of miles x it is driven.*

b) *What are total projected costs if the car is driven 50,000 miles during its lifetime?*

c) *If it is driven 100,000 miles?*

Solution:

a) Total cost(c) = Variable cost + fixed cost

$$= \text{Total operating cost} + \text{purchase cost}$$

$$C = 0.40x + 18{,}000 \quad \underline{\text{Ans}}$$

b) If the car is driven 50,000 miles

$$\begin{aligned} C &= f(x) \\ &= f(50{,}000) \\ &= 0.40(50{,}000) + 18{,}000 \\ &= \$38{,}000 \quad \underline{\text{Ans}} \end{aligned}$$

c) Similarly at 100,000 miles

$$\begin{aligned} C &= f(x) = 0.04x + 18{,}000 \\ &= f(100{,}000) \\ &= 0.04(100{,}000) + 18{,}000 \\ &= \$58{,}000 \quad \underline{\text{Ans}} \end{aligned}$$

Example 2:

A firm which produces a single product is interested in determining the function that expresses annual total cost y as a function of the number of units produced x. Accountants indicate that fixed expenditures each year are \$50,000. They also have extimated that raw material costs for each unit produced are \$5.50, and labor costs per unit are \$1.50 in the assembly department, \$0.75 in the finishing room, and \$1.25 in the packaging and shipping department. Find the total cost function.

Solution:

The total cost function = total variable cost + total fixed cost.

$$\begin{aligned} y &= C(x) \\ y &= \text{total raw material cost} + \text{total labor cost} + \\ &\quad \text{total fixed cost} \\ &= 5.50x + (1.50x + 0.75x + 1.25x) + 50{,}000 \\ y &= 9x + 50{,}000 \quad \underline{\text{Ans}} \end{aligned}$$

1. Cost function:

Organizations are concerned with costs because they reflect the amount of money flowing out of the organization. The total cost can be defined in terms of two components: Total variable cost and total fixed cost.

Total cost = Total variable cost + total fixed cost.

- *Variable costs* are costs that vary with the level of output and are computed as the product of variable cost per unit of output and the level of output.

Fixed Costs: Costs incurred in a given time period, regardless of output.

2. Revenue Functions:

The money which flows into an organization from either selling products or providing services is often referred to as *revenue.*

Total revenue = (price) (Quantity sold)

3. Profit Functions:

—Profit is the difference between total revenue and total cost.

Profit = total revenue – total cost.

—When total revenue exceeds total cost, profit is positive. In such cases the profit may be referred to as a net gain, or net profit. When total cost exceeds total revenue, profit is negative and it may be called a net loss, or deficit.

If Total revenue = $R(x)$

Total cost = $C(x)$

Profit = $P(x)$

$\therefore P(x) = R(x) - C(x)$

4. Straight-line Depreciation:

When organizations purchase equipment vehicles, buildings, and so forth, accountants usually allocated the cost of the item over the period the item is used. The cost allocated to any given period is called depreciation.

Although there are a variety of depreciation methods, one of the simplest is *straight-line depreciation.* Under this method the rate of depreciation is constant. This implies that the book value declines as a linear function over time.

Straigh-line Depreciation with Salvage Value.

Many assets have a resale, or salvage value even after they have served the purposes for which they were originally purchased. In such cases the cost allocated to each time period is the difference between the purchase cost and the salvage value divided by the useful life.

$$\text{Annual Depreciation} = \frac{\text{Purchase cost - salvage value}}{\text{Useful life (in years)}}$$

Example:

A truck costing \$10,000 and having a useful life of 5 years, accountants might allocate \$2,000 a year as a cost of owning the truck.

The cost \$2,000 allocated is called depreciation. Assume that the truck has a useful life of 5 years, after which it can be sold for \$1,000.

If the truck is to be depreciated on a straight-line basis, the annual depreciation will be.

$$\frac{\text{Purchase cost - salvage value}}{\text{Useful life (in years)}} = \frac{\$10,000 - \$\ 1,000}{5}$$
$$= \frac{\$9,000}{5}$$
$$= \$1,800$$

The function which expresses the 'book' value (Current value) V as a function of time t is

$$V = f(t) = 10{,}000 - 1{,}800t. \quad 0 \leqslant t \leqslant 5$$

5. Demand function:

A demand function is a mathematical relationship expressing the way in which the quantity demanded of an item varies with the price charged for it.

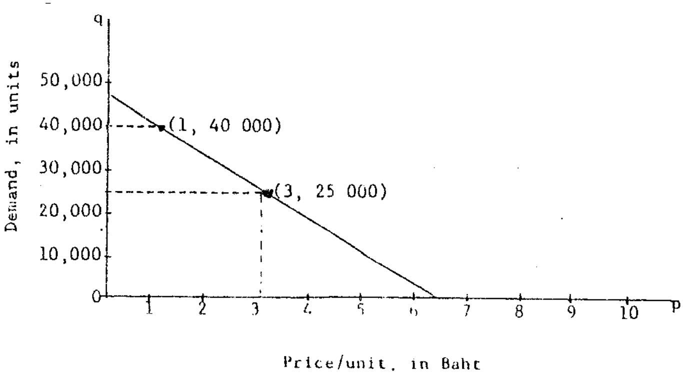
The relationship between these two variables___quantity demanded and price per unit___is usually inverse. That is, as one variable increases, the other decreases. There are exceptions to this behavior. The demand for products or services which are considered necessities is likely to fluctuate less with moderate changes in price.

Although most demand functions are nonlinear, there are situations in which the demand relationship either is, or can be approximated by, a linear function. Most view of the demand relationship is in the form

$$\text{Quantity demanded} = f(\text{price per unit})$$

That is, consumers respond to price

$$q = f(p) = 47{,}500 \quad 7{,}500p$$



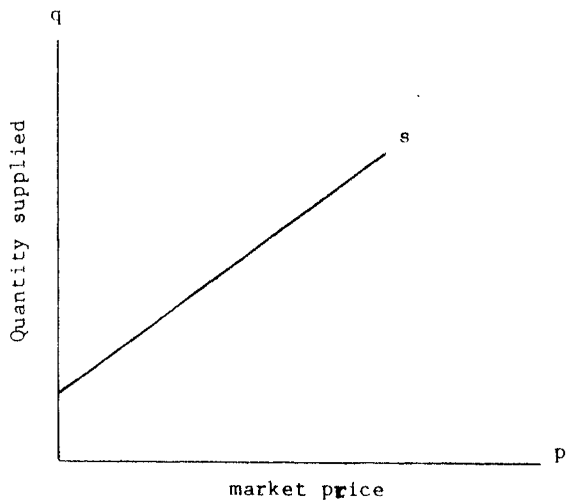
6. Supply Functions:

A supply function relates market price to the quantities that suppliers are willing to produce and sell.

The implication of supply functions is that what is brought to the market depends upon the price people are willing to pay. As opposed to the inverse nature of price and demand in demand functions, the quantity which suppliers are willing to provide usually varies directly with the market price. All other things being equal, the higher the market price, the more a supplier would like to produce and sell; and the lower the price people are willing to pay, the less the incentive to produce and sell.

Supply functions can be approximated sometimes by using linear functions.

Quantity supplied = f(market price)



EXERCISE: 5 - 1

1. A piece of machinery is purchased for \$50,000. Accountants have decided to use a straight-line depreciation method with the machine being fully depreciated after 8 years. Letting V equal the book value of the machine and t the age of the machine, determine the function V = f(t). (Assume no salvage value.)

2. In Exercise 1 assume that the machine will have a salvage value of \$6,000 at the end of 8 years. Determine the function V = f(t) for this situation.

3. An airline claims that it is using straight-line depreciation on one of its 747s. The initial purchase cost was \$5.5 million. Company records indicate that after the first year the book value of the plane was \$4,950,000. After the third year the value was \$3,850,000, and after the sixth year the value was \$2,200,000. Assume no salvage value.

   (i) Are they using straight-line depreciation?
   (ii) If so, what is the function V = f(t)?
   (iii) When will the 747 be fully depreciated?

4. Assume that the 747 in Exercise 3 has a useful life of 8 years and at this stage it can be salvaged for \$750,000. Determine the function V = f(t). What is the restricted domain for the function?

5. A metropolitan police department projects that average weekly levels of serious crime decrease in direct proportion to the number of patrol cars assigned to preventive patrol. Analysts estimate that the number of serious crimes which would occur each week with no patrol cars is 800. They also believe that the linear relationship is valid only for allocations of up to 120 patrol cars. At this level of preventive patrol, crime levels are expected to be 260 serious crimes per week. Analysts believe that any additional allocations would have no effect on the level of crime.

   (i) Determine the linear function relating number of serious crimes to the number of patrol cars.
   (ii) Define the restricted range and domain of the function.

(iii) What is the marginal effect of each additional patrol car for this function?

6. Two points (p, q) on a linear demand function are (\$2.50, 20,000) and (\$2.75, 18,000)

   (i) Determine the demand function q = f(p).
   (ii) What price would result in demand of 12,000 units?
   (iii) Interpret the slope of the function.
   (iv) Sketch the function.

7. Two points (p, q) on a linear demand function are (\$50, 35,000) and (\$47.50, 45,000).

   (i) Determine the demand function q = f(p.).
   (ii) What price would result in demand of 60,000 units?
   (iii) Interpret the slope of the function.
   (iv) Sketch the function.

8. Two points (p, q) on a linear supply function are (\$3, 60,000) and (\$2.50, 52,000).

   (i) Determine the supply function q = f(p).
   (ii) What price would result in suppliers offering 80,000 units for sale?
   (iii) Interpret the slope of the function.
   (iv) Sketch the function.

9. Two points (p, q) on a linear supply function are (\$12.50, 75,000) and (\$15, 90,000)

   (i) Determine the supply function q = f(p)
   (ii) What price would cause suppliers to offer 50,000 units for sale?
   (iii) Interpret the slope of the function.
   (iv) Sketch the function.

10. Since 1960 there has been a seemingly linear increase in the percentage of the population who are alcoholics in one European country. In 1960 the percentage of the population who were alcoholics was 15.6 percent. In 1970 the percentage had risen

to 21.2 percent. Let p equal the percentage of the population who are alcoholics and t represent time measured in years since 1960 ($t = 0$ for 1960).

(i) Determine the linear growth function $p = f(t)$
(ii) Interpret the meaning of the slope.
(iii) If the pattern of growth continues, forecast the percentage of alcoholics in 1985. What is the forecasted percentage for 1990.

11. Since 1973 there has been evidence of grade inflation at a large Midwestern university. The cumulative grade-point average for all undergraduate students was 2.42 in 1973. In 1977 the average was 2.66. Assuming the trend is linear, let g represent the cumulative grade-point average for all undergraduate students and t represent time measured in years since 1973.

(i) Determine the function $g = f(t)$
(ii) According to this function, when will the grade-point average reach 3.0?
(iii) Forecast the grade-point average for 1980.
(iv) Interpret the meaning of the slope in this function.

7. Break-even models:

Break-even analysis focuses upon the profitability of a firm. Of specific concern in break-even analysis is identifying the level of operation or level of output that would result in a zero profit. This level of operation or output is called the *break-even point*. It represents the level of operation where total revenue equals total cost.

$$\text{Total revenue} = \text{total cost.}$$

Any changes from this level of operation will result in either a profit or a loss.

The break-even point may be expressed in terms of (1) volume of output, (2) total dollar sales, or possibly (3) percentage of production capacity.

For example, it might be stated that a firm will break even at 100,000 units of output, when total sales equal \$2.5 million or when the firm is operating at 60 percent of its plant capacity.

Example:

A group of engineers is interested in forming a company to produce smoke detectors. They have gone through a design stage and estimate that variable costs per unit, including materials, labor, and marketing costs, are \$22.50. Fixed costs associated with the formation, operation, and management of the company and the purchase of equipment and machinery total \$250,000. They estimate that the selling price will be \$30 per detector.

a) *Determine the number of smoke detector which must be sold in order for the firm to break-even on the venture.*
b) *Preliminary marketing data indicate that the firm can expect to sell approximately 30,000 smoke detectors over the life of the project if they price the detectors at \$30. Determine expected profits at this level of output.*

Solution:

a) Let x be the number of smoke detectors sold to break-even

$\therefore$ total revenue $R(x) = 30x$

total cost $C(x) = 22.50x + 250{,}000$

The break-even condition occurs when total revenue equals total cost.

$$\begin{aligned} R(x) &= C(x) \\ \therefore\ 30x &= 22.50x + 250{,}000 \\ 7.50x &= 250{,}000 \\ x &= \frac{250{,}000}{7.50} \\ &= 33{,}333.33 \text{ units} \end{aligned}$$

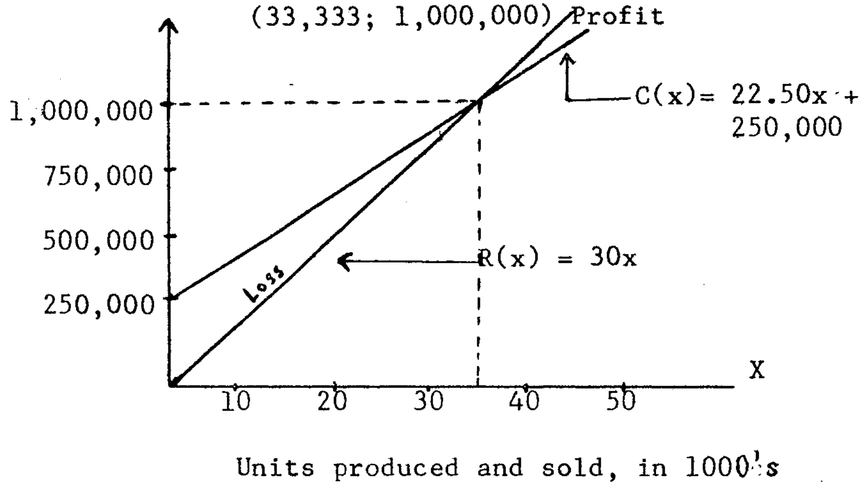
$\therefore$ The firm must sell 33,333.33 units in order to break-even

Ans

b) Graphical approach

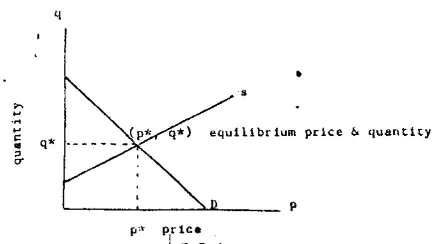
The two functions are graphed on the same set of axes. The only point where the two functions intersect represents the one level of output where total revenue and total cost are equal. This is the break-even point.

Note that for all points to the left of the break-even point the cost function C(x) has a value greater than the revenue function R(x). For any level of output below 33,333 units, the vertical distance separating the two functions represents the loss which would occur. To the right of x = 33,333, R(x) is higher than C(x). And, when total revenue exceeds total cost, a profit exists.



8. Equilibrium between Supply and Demand:

One of the concerns of economists is whether the consumers and suppliers will ever have a meeting of the minds and reach agreement about the quantities which will be supplied and purchased and the market price at which this will occur. When agreement is reached, the market is said to be in *equilibrium.* That is, at the equilibrium price the amount demanded by consumers is exactly equal to the quantity suppliers are willing to bring to the market. This equilibrium condition is defined by the coordinates of the point where the supply and demand functions intersect.



Example:

The demand function for a particular product is $q_d = 10{,}000 - 50p$, where p is price stated in dollars and q_d is quantity demanded in thousands of units. The supply function is $q_s = 2000 + 30p$, where p is defined as before and q_s is the quantity sypplied, in thousands of units. Determine the equilibrium price and quantity. Sketch the two functions.

Solution:

Equilibrium occurs if there is a price which equates supply and demand, or when

$$q_s = q_d$$

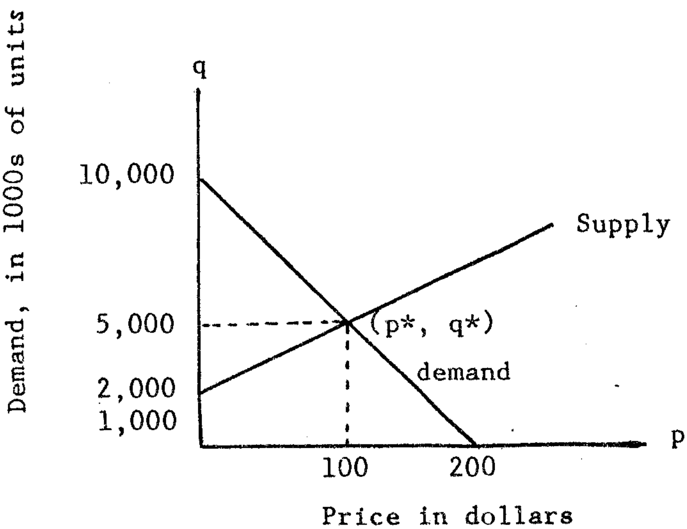
$$\text{or} \quad 2000 + 30p = 10{,}000 - 50p$$

Solving for p, we find equilibrium occurs when

$$80p = 8000$$

$$p = 100$$

At a price of 100 the quantity supplied and demanded equals 5,000 thousands, or 5 million units <u>Ans</u>



9. Defining Mathematical Functions:

Simultaneous equations are one approach to defining mathematical functions. A linear function involving two variables has the form

$$y = ax + b$$

In order to define a linear function, values must be determined for the parameters a and b. If the coordinates of two points which lie on the function are known, the values of a and b can be determined by substituting each pair of coordinates into the above equation ($y = ax + b$), forming two equations. These two equations can then be solved simultaneously to determine a and b.

Example:

Using simultaneous equations, determine the linear function which contains (4, 8) and (7, –4).

Solution:

At (4, 8) $\quad y = ax + b$

$8 = 4a + b$ ....(1)

At (7, –4) $\quad -4 = 7a + b$ ....(2)

From the 2 equations solving for a and b, we get

$a = -4$ and

$b = 24$

The function can be specified as $y = f(x) = -4x + 24$ <u>Ans</u>

EXERCISE: 5 - 2

1. Determine the equilibrium price and quantity if

$$q_d = 25{,}000 - 50p \quad \text{and} \quad q_s = 4{,}000 + 20p$$

Where p is stated in dollars.

2. Determine the equilibrium price and quantity if

$$q_d = 100{,}000 - 1{,}500p \quad \text{and} \quad q_s = 10{,}000 + 3{,}000p$$

Where p is stated in dollars.

3. A firm sells a product for \$50 per unit. Raw material costs are \$22.50 per unit, labor costs are \$13.50 per unit, and annual fixed costs are \$70,000.

   a) Determine the profit function $P(x)$ where x equals the number of units sold.
   b) How many units would have to be sold to earn an annual profit of \$35,000?

4. A firm produces three products which sell, respectively, for \$10, \$15, and \$8.50 . Labor requirements for each product are respectively, 2.5, 3.5, and 2 hours per unit. Assume labor costs are \$3 per hour and annual fixed costs are \$50,000.

   a) Construct a joint total revenue function for the sales of the three products.
   b) Determine an annual total cost function for production of the three products.
   c) Determine the profit function for the three products.
   d) What is annual profit if 20,000, 10,000 and 30,000 units are sold, respectively, of the three products?

5. A city has purchased a new fire truck for \$100,000. The city comptroller states that the fire truck will be depreciated by using a straight-line method. At the end of 12 years, the truck will be sold with an expected salvage value of \$22,000.

   a) Determine the function $V = f(t)$ which expresses the book value of the truck V as a function of the age of the truck t.
   b) What is the book value expected to be when the truck is 6 years old?

6. The birthrate in a particular country has been declining linearly in recent years. In 1970 the birthrate was 25.6 births per 1,000 people. In 1978 the birthrate was 23.6 births per 1,000 people. Assume R equals the birthrate per 1,000 and t equals time

measured in years since 1970 ($t = 0$ for 1970)

a) Determine the linear birthrate function $R = f(t)$.
b) Interpret the meaning of the slope.
c) If the linear pattern continues, what is the expected birthrate in 1986?
d) What is the restricted domain for this function?

7. Two points (p, q) on a linear demand function are ($4.50, 45,000) and ($5, 36,000)

   a) Determine the demand function $q = f(p)$
   b) What price would result in a demand of 58,500 units?
   c) Determine the y intercept and interpret its meaning.
   d) Determine the x intercept and interpret its meaning.

8. Two points (p, q) on a linear supply function are ($10, 450,000) and ($15, 750,000)

   a) Determine the supply function $q = f(p)$
   b) What price would result in suppliers offering 100,000 units for sale?
   c) Interpret the slope of the supply function.
   d) What is the x intercept? Interpret the meaning of this point.

9. Annual sales for a company have been increasing linearly at a rate of $4.5 million per year since 1972. Sales in 1975 were $260 million. Assume S equals annual sales in millions of dollars and t equals time in years (measured since 1972)

   a) Determine the function $S = f(t)$
   b) What were annual sales in 1972?
   c) What are annual sales expected to equal in 1990?

10. A publisher has a fixed cost of $90,000 associated with the production of a college mathematics book. The contribution to profit and fixed cost from the sale of each book is $2.50.

    a) Determine the number of books which must be sold in order to break-even.
    b) What is the expected profit if 30,000 books are sold?

11. A local college basketball team has added a national power to next year's schedule. The other team has agreed to play the game for a guaranteed fee of \$10,000 plus 20 percent of the gate receipts. Assume ticket prices are \$5.

   a) Detemine the number of tickets which must be sold to recover the \$10,000 guaranty.
   b) If college officials hope to net a profit of \$25,000 from the game, how many tickets must be sold?

12. A firm has two equipment alternatives it can choose from in producing a new product. One automated piece of equipment costs \$100,000 and produces items at a cost of \$2.50 per unit. Another semiautomated piece of equipment costs \$40,000 and produces items at a cost of \$3 per unit.

   a) What volume of output makes the two pieces of equipment equally costly?
   b) If 100,000 units are going to be produced, which piece of equipment is the least costly? What is the minimum cost?

13. Determine the equilibrium price and quantity if

$$q_d = 50{,}000 - 75p \quad \text{and} \quad q_s = 10{,}000 + 25p$$

14. Using simultaneous equations, determine the linear function which contains the points (7.5, 4) and (6.0, 6)

15. Using simultaneous equations, determine the quadratic function which contains the points (–1, 2), (2, 8), and (4, 32).

16. A company is combining peanuts, cashews, and almonds to form 1000 pounds of a mixed-nut blend. Peanuts cost \$0.60 per pound, cashews \$1.10 per pound, and almonds \$1.40 per pound. The number of pounds of cashews used in the blend should equal the number of pounds of almonds. If the blend is to cost \$695, determine the number of pounds of each type of nut which should be used?

17. A company sells a product for \$65 per unit. Raw material costs are \$25 per unit, labor costs are \$20 per unit, shipping costs are

$5 per unit, and annual fixed costs are $75,000.

a) Determine the profit function $P = f(x)$, where x equals the number of units sold.
b) How many units must be sold in order to earn an annual profit of $150,000?

18. A piece of equipment had a book value of $60,000 when it was 2 years old and a book value of $37,500 when it was 5 years old. Assume that it is being deprectiated by a straight-line method.
    a) What was the original book value?
    b) At what annual rate is the piece of equipment being depreciated?
    c) When will the book value equal 0?

19. A student organization is planning a ski week during semester break. It has arranged a package deal with a ski resort which provides for meals, equipment, lodging, and lift tickets at a cost of $180 per person plus a fixed cost of $2,000 for arranging special facilities and activities. Transportation cost are expected to equal $20 per person. Assume the student organization charges each student $225 for the complete package (transportation included).
    a) How many students will be required in order to break even?
    b) If the school subsidizes the trip by contributing $600, how many students will be required?

20. Determine the market equilibrium price and quantity if

$$q_d = 60{,}000 - 250p \quad \text{and} \quad q_s = 150p + 20{,}000$$

21. A company produces three products, each of which must be processed through three different departments. Table below summarizes the hours required per unit of each product in each department as well as the weekly capacities in each department. Formulate, but do not solve, the system of equations which when solved, would indicate whether there are any combinations of the three products that would consume the weekly labor availability in all departments.

Table No 21

	Department 1	Department 2	Department 3
Product A	6	7	5
Product B	2	4	5
Product C	2	1	3
Hours available/wk	80	60	100

5.4 Inverse Functions:

Consider the case of a function,

$$y = f(x)$$

the value of the variable y depends upon and is uniquely determined by the value of the variable x. If it is possible to determine x as a function of y, g(y), then g(y) is the inverse function of $y = f(x)$, that is

$$x = g(y)$$

In this case of inverse function, x becomes dependent variable and y becomes independent variable.

Therefore, $x = g(y)$ is termed the inverse of $y = f(x)$ mathematically, it is written as $f(x) \longrightarrow f^{-1}(x)$

Example 1:

$$y = f(x) = 400 + 2x$$

The inverse function is

$$x = g(y) = 0.5y - 200$$

Ans

Example 2:

$$y = 4x + 10$$

The inverse of this function is $x = 0.25y - 2.5$.

Thus the inverse was found by solving the function for x in terms of y; that is, $x = g(y)$. Ans

Example 3:

Suppose that sales as a function of advertising are reflected by the equation

$$y = \begin{cases} x, & 0 \leqslant x < 4 \\ 2 + 0.5x, & 4 \leqslant x \leqslant 6 \end{cases}$$

Write the equation for the inverse function which describes advertising as a function of sales.

Solution:

If $y = x, \quad 0 \leqslant x < 4$

Find x in term of y

$\therefore \quad x = y, \quad 0 \leqslant y < 4$

If $y = 2 + 0.5x, \quad 4 \leqslant x \leqslant 6$

$$0.5x = y - 2$$

$$x = \frac{y - 2}{0.5}, \quad 4 \leqslant y \leqslant 5$$

$$x = 2y - 4, \quad 4 \leqslant y \leqslant 5$$

Note: that the range of y is between 4 and 5 since the permissible value of x is between 4 and 6.

$\therefore$ The required inverse function is

$$x = \begin{cases} y, & 0 \leqslant y < 4 \\ 2y - 4, & 4 \leqslant y \leqslant 5 \end{cases} \Bigg\} \text{ Ans}$$

Functions such as y = 4x − 3, in which the dependent variable is clearly designated as y and the dependent variable as x, are termed *explicit functions.* A function can also be defined *implicitly.* As an example, the equation 5x − 2y = 14 implicitly defines x in terms of y.

If y is allowed to assume a certain value, then x is implicitly defined by the equation. Similarly, y is an implicit function of x in that if x is permitted to take on certain values, the value of y is implicitly established by the equation. If we are given the equation

$h(x, y) = 0$, then $y = f(x)$ and $x = g(y)$ are inverse functions.

Example 4:

Consider the implicit function $3x + 4y - 6 = 0$. Determine $y = f(x)$ and $x = g(y)$ for this function.

Solution:

The explicit function of y in terms of x is

$$\begin{aligned} 3x + 4y - 6 &= 0 \\ 4y &= -3x + 6 \\ y &= \frac{-3x}{4} + \frac{6}{4} \\ y &= -0.75x + 1.5 \end{aligned}$$ <u>Ans</u>

and the explicit function of x in terms of y is

$$\begin{aligned} 3x + 4y - 6 &= 0 \\ 3x &= -4y + 6 \\ x &= \frac{-4y}{3} + \frac{6}{3} \\ x &= -1.33y + 2 \end{aligned}$$ <u>Ans</u>

Example 5:

For the implicit function $x + 2y - 10 = 0$, determine the explicit functions $y = f(x)$ and $x = g(y)$

Solution:

To find the explicit function $y = f(x)$, consider

$$\begin{aligned} x + 2y - 10 &= 0 \\ 2y &= -x + 10 \\ y &= -\frac{1}{2}x + \frac{10}{2} \\ y &= -0.5x + 5 \end{aligned}$$ <u>Ans</u>

and the explicit function or the inverse $x = g(y)$ is

$$x + 2y - 10 = 0$$
$$x = 10 - 2y \qquad \underline{\text{Ans}}$$

Example 6:

Determine the inverse of the explicit function $y = x^2$.

Solution:

To find the inverse of $y = x^2$, express x into terms of y.

$$y = x^2$$
$$x = \pm\sqrt{y}$$

They are both inverse functions of $y = x^2$. <u>Ans</u>

EXERCISE: 5 - 3

1. A certain chemical is produced as it passes through two production processes. In the first process, there is a shrinkage of 10 percent, and in the second there is a shrinkage of 20 percent of the material entering the second process. Develop a functional relationship between the gallons of input as the independent variable, x, and the gallons of output as the dependent variable, y.
   a) Give the rule for the function in equation form and indicate its domain (that is, the values of x for which the function is valid).
   b) Also give the equation for the relationship that allows one to find the number of gallons needed to product a given output. This is called the inverse function of the first.

2. Define the inverse of each of the function rules below. If the inverse cannot be defined, give the reason.

a) $y = 5 + 2x$ for $0 < x \leqslant 5$ and $5 \leqslant y < 15$.

b) [(1, 6), (2, 7), (3, 9), (4, 9), (5, 10)] for x the set of positive integers less than 6 and y the set of positive integers greater than 6 and less than 11.

c) $y = x^I$: all real x and y, where I is an integer.

d) $y = x^2$: all real and positive x and y.

3. A bank charges interest of 'a' percent per year on the original balance of a 36-month consumer credit loan. Equal payments are made each month.

   a) Express the monthly-payment amount, y, as a function of the amount borrowed, x.

   b) If a borrower can afford to pay back a maximum of y per month, what is the maximum credit that he should be granted?

4. A firm has a demand for its product which is a function of the product's price. The function is $x = 10{,}000 - p$ for x and p real and greater than zero. The cost of production, c, is equal to $4000 + 5x$ for x greater than zero. Express the firm's profit, y, as a function of only the price it charges and given the range for which the function holds.

5.5 Exponential Function:

An exponential function is one of the form

$$y = f(x) = a^x$$

Where $a > 0$, $a \neq 1$ is a real constant and x is any real number.

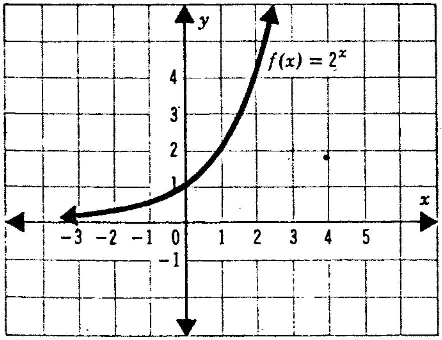
It involves a constant raised to a variable power.

Consider $$y = f(x) = 2^x$$

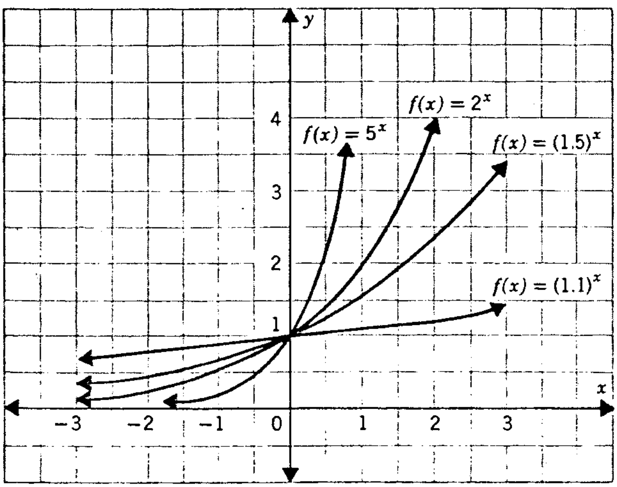
x	-10	-5	-4	-3	-2	-1	0	1	2	3	5	10
$y=2^x$	0.00098	0.031	0.0625	0.125	0.25	0.5	1.0	2.0	4.0	8.0	32	1024

For increasingly negative values of x, the value of 2^x is very close to zero, but positive. For large positive values of x, the value of 2^x is very large and positive.

As show in the table, it can be concluded that the domain of $f(x) = 2^x$ is the set of real numbers and the range is the set of positive real numbers.



The graph of $f(x) = 2^x$ is typical of all exponential functions whose base is larger than one. Such functions are positive everywhere and are increasing functions on their domain. Each of their graph passes through the point (0, 1) and thereafter rises rapidly as x increases as shown in the graphs below



Exponential functions, as shown above, are often used as model for describing such phenomena in business as the growth of bacteria, population growth, and compound interest. We refer to such curves as *growth curves.*

Consider $y = f(x) = \left(\frac{1}{2}\right)^x$

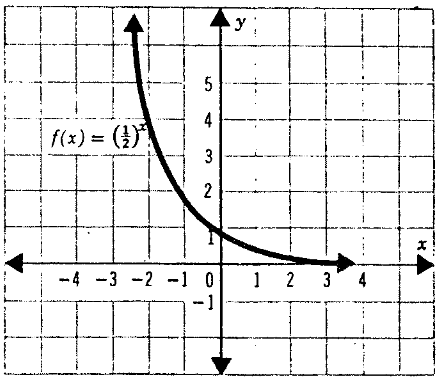
Plot certain points on the graph

x	-10	-5	-4	-3	-2	-1	0	1	2	3	4	5	10
$y=\left(\frac{1}{2}\right)^x$	1024	32	16	8	4	2	1	0.5	0.25	0.125	0.0625	0.031	0.00098

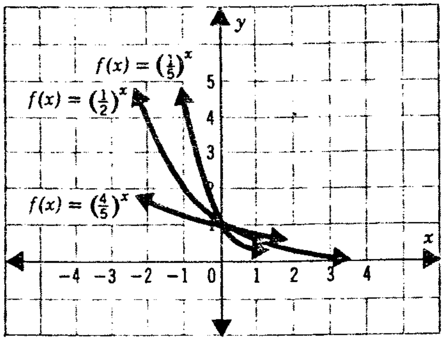
$$y = \left(\frac{1}{2}\right)^x$$

or $$y = 2^{-x}$$

For increasingly negative values of x, the value of $\left(\frac{1}{2}\right)^x$ becomes large and positive; for large positive values of x, $\left(\frac{1}{2}\right)^x$ is close to zero and positive.



The graph of $f(x) = \left(\frac{1}{2}\right)^x$ is typical of all exponential functions whose base is smaller than one. Such functions are positive everywhere and are decreasing on their domain. Each of their graphs passes through the point (0, 1) and thereafter the graph decreases as x increases.



Exponential functions of this type are often used as models for describing phenomena in business such as depreciation, and price-demand curves. We refer to such curves as decay curves.

Example 1:

A man deposits \$100 in a bank that pays 5 percent interest per annum compounded annually. How much is in his account after x years? We assume the interest paid is left in the account.

Solution:

The initial amount $P_o = \$100$

If P_1 = amount after one year, then

$$P_1 = 100 + 0.05(100) = 105$$

or $$P_1 = P_o + 0.05(P_o) = (1 + 0.05)\ P_o$$

If P_2 = amount at the end of the $2^{\underline{nd}}$ year, then

$$P_2 = P_1 + 0.05(P_1) = (1 + 0.05)(1 + 0.05)\ P_o$$

$$P_o\ (1 + 0.05)^2$$

$$P_x = P_x + 0.05P_x = P_x(1 + 0.05) = P_o(1 + 0.05)^x$$

Thus, the balance after x years is given as an exponential function whose base is larger than one.

Example 2:

A ฿450,000 car depreciates in such a way that the car's value each year is $\frac{2}{3}$ of its value a year earlier. What is the worth of the car after 18 months? What is it worth after 40 months? After t months?

Solution:

There is a relationship between the worth (W) of the car and the time (t) of ownership.

$$W = f(t)$$

where W = worth of the car in baht and

t = time in months.

At t = 0 the car is worth its original cost ฿450,000

Thus $450{,}000 = f(0)$ ———— (1)

After 12 months (t = 12), the value is $\frac{2}{3}$ of ฿450,000 or ฿300,000. That is,

$$\frac{2}{3}(450{,}000) = f(12)$$

$$\text{or} \quad 300{,}000 = f(12) \text{ ———— } (2)$$

After 12 more months, the new value is

$$\frac{2}{3}(300{,}000) = f(24)$$

$$\text{or} \quad 200{,}000 = f(24)$$

$$\text{or} \quad \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)(450{,}000) = f(2\times 12)$$

$$\left(\frac{2}{3}\right)^2(450{,}000) = f(2\times 12)$$

After 12 more months,

$$\frac{2}{3}(200,000) = f(36)$$

$$\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)(450,000) = f(3\times 12)$$

$$\left(\frac{2}{3}\right)^3 (450,000) = f(3\times 12)$$

In general, we conclude that

$$\left(\frac{2}{3}\right)^n (450,000) = f(n.12)$$

This formula is accurate for multiples of 12 months. But we want to know the value after 18 months.

$$\text{Set} \quad t = 12n$$

$$\text{or} \quad n = \frac{t}{12}$$

$$\therefore \quad W = f(t) = 450,000\left(\frac{2}{3}\right)^{\frac{t}{12}}$$

$$\left.\begin{aligned} \text{Then} \quad W &= f(18) = 450,000\left(\frac{2}{3}\right)^{\frac{18}{12}} = ฿244,949 \\ W &= f(40) = 450,000\left(\frac{2}{3}\right)^{\frac{40}{12}} = ฿116,477 \end{aligned}\right\} \quad \underline{\text{Ans}}$$

Example 3:

A new car costing ฿450,000 depreciates in such a way that each year it is worth $\frac{2}{3}$ of what it was worth a year earlier. What is the cost of the car due to depreciation after 18 months? After t months?

Solution.

Cost due to depreciaton is = ฿450,000 – worth of a car. Use the result obtained in example 2, we get

a) Cost due to depreciation after t months is

$$= ฿450,000 - 450,000\left(\frac{2}{3}\right)^{\frac{t}{12}}$$

$$= 450,000\left[1 - \left(\frac{2}{3}\right)^{\frac{t}{12}}\right]$$

b) The cost due to depreciation after 18 months is

$$= ฿450,000 - 244,949$$

$$= ฿205,051 \qquad \underline{\text{Ans}}$$

Example 4:

The predicted population P(t) of a city is given by

$$P(t) = 100,000^{e0.05t}$$

Where t is the number of years after 1980. Predict the population in the year 2000.

Solution:

From 1980 − 2000 = t = 20 years.

$$\therefore P(20) = 100,000e^{0.05(20)}$$

$$= 100,000e^{1}$$

Since e ≈ 2.71828, the predicted population 20 years after 1980 is approximately is

$$P(20) = 100,000(2.71828)$$

$$= 271,828 \qquad \underline{\text{Ans}}$$

In general, the models of the curves of the exponential functions can be summarized as follows:

In case of $y = a^x$ (where $a > 0$)

1. If $a > 1$, the curve rises to the right and approaches the x-axis to the left.

2. If $0 < a < 1$, the curve rises to the left and approaches the x-axis to the right.

3. Let a and b be numbers such that $1 < a < b$, than

   a) for $x > 0$, the curve corresponding to $y = b^x$ is above the curve corresponding to $y = a^x$.

   b) for $x < 0$, the curve $y = b^x$ is below the curve $y = a^x$.

   In case of $y = ma^x$ (where $a > 0$)

4. If $m > 0$, the function lies above the x-axis.

5. If $m < 0$, curve lies below the x-axis.

6. The domain of an exponential functions is all real numbers, and the range is all positive numbers.

5.7 Logarithmic Function:

Logarithmic function is related to exponential function. The logarithmic function base b, denoted $\log_b$, is defined by

$$y = \log_b x$$

if and only if $$b^y = x$$

The domain of $\log_b$ is all positive numbers and its range is all real numbers.

The logarithmic function reverses the action of the exponential function. Because of this we say that the logarithmic function is the inverse of the exponential function.

Example:

$$\log_b x = y \qquad b^y = x$$

In this sense, the logarithmic of a number is an exponent.
To draw logarithmic function, express it in the exponential form. Then choose convenient values of y and find the corresponding values of x to plot the graph.

ln x	=	$\log_e x$ = natural (or Naperiams) logarithm
e	=	irrational number ≃ 2.71828
ln e	=	1, since $e^1 = e$
ln 1	=	0, since $e^0 = 1$
ln 0	=	it is not defined since there is no number "m" for which $e^m = 0$
ln(−1)	=	undefined since there is no "m" for which $e^m = -1$

Negative numbers do not have logarithms

$$\log_a M = \frac{\log_b M}{\log_b a}$$

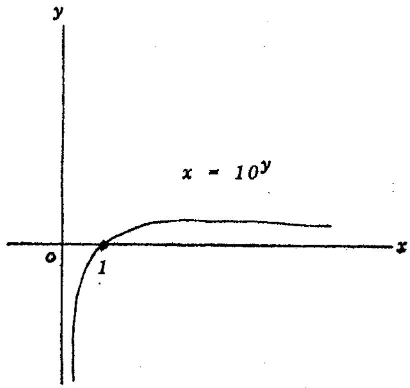
Example 1:

Graph the function $y = \log_{10} x$

Solution:

First express the logarithmic function into the exponential form

$\therefore x = 10^y$

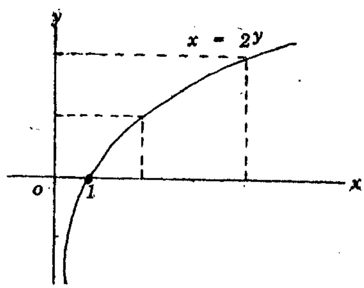


Example 2:

Draw $y = \log_2 x$

Solution:

$y = \log_2 x \qquad x = 2^y$



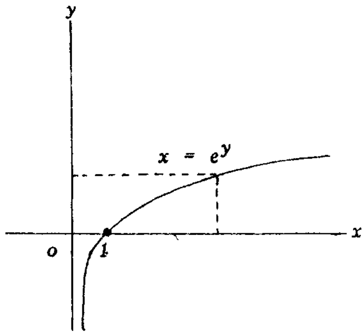
Example 3:

Draw $y = \ln x$

Solution:

In exponential form we get,

$$x = e^y$$



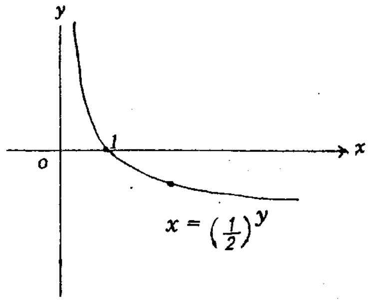
Example 4:

Draw $y = \log_{1/2} x$

Solution:

In term of exponential function

$$x = \left(\frac{1}{2}\right)^y = 2^{-y}$$

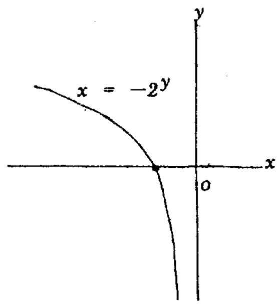


Example 5:

Draw $y = \log_2(-x)$

Solution:

In exponential form $x = -2^y$



In general,

1. In case of $y = \log_a(\pm x)$

   or $x = a^y$

   $x = -a^y$

   If $x = a^y$, the entire curve is to the right of y—axis.
   If $x = -a^y$, then the entire curve is to the left of y—axis.

2. In the equation $y = \log_a x$. if $a > 1$, the curve rises toward the right; if $a < 1$, the curve goes downward toward the right.

Since logarithmic functions is the reverse of the exponential function, then nature of various curves would be similarly to the general rule given in the exponential section.

EXERCISE: 5 - 4

In Problems 1 - 6, graph each function.

1. $y = f(x) = 4^x$.
2. $y = f(x) = (\frac{1}{3})^x$.
3. $y = f(x) = 2(4^{-x})$.
4. $y = f(x) = \frac{1}{2}(3^{x/2})$.
5. $y = f(x) = \log_3 x$.
6. $y = f(x) = \log_{\frac{1}{2}} x$.

In Problems 7 - 14, use the tables in Appendices to find the approximate value of each expression.

7. $e^{1.5}$.
8. $e^{3.4}$.
9. $e^{-.4}$.
10. $e^{-\frac{3}{4}}$.
11. ln 5.
12. ln 3.12.
13. ln 7.39.
14. ln 9.98.

In Problem 15 - 26 express each logarithmic form exponentially and each exponential form logarithmically.

15. $25^{\frac{1}{2}} = 5$.
16. $2 = \log_{12} 144$.
17. $10^4 = 10,000$.
18. $\log_{\frac{1}{2}} 4 = -2$.
19. $\log_2 64 = 6$.
20. $8^{2/3} = 4$.
21. $\log_2 x = 14$.
22. $10^{.48302} = 3.041$.

23. $e^2 = 7.3891$.

24. $e^{.33647} = 1.4$.

25. $\ln 3 = 1.0986$.

26. $\log 5 = .6990$.

In Problems 27 - 50, find x.

27. $\log_3 x = 2$.

28. $\log_2 x = 4$.

29. $\log_5 x = 3$.

30. $\log_4 x = 0$.

31. $\log x = -1$.

32. $\ln x = 1$.

33. $\ln x = 2$.

34. $\log_x 100 = 2$.

35. $\log_x 8 = 3$.

36. $\log_x 3 = \frac{1}{2}$.

37. $\log_x \frac{1}{6} = -1$.

38. $\log_x y = 1$.

39. $\log_4 16 = x$.

40. $\log_3 1 = x$.

41. $\log 10{,}000 = x$.

42. $\log_2 \frac{1}{16} = x$.

43. $\log_{25} 5 = x$.

44. $\log_9 9 = x$.

45. $\log_3 x = -4$.

46. $\log_x (2x - 3) = 1$.

47. $\log_x (6 - x) = 2$.

48. $\log_8 64 = x - 1$.

49. $2 + \log_2 4 = 3x - 1$.

50. $\log_3 (x + 2) = -2$.

In Problems 51 - 58, find x and express your answer in terms of logarithms.

51. $2^x = 5$.

52. $4^{x+3} = 7$.

53. $e^{3x} = 2$.

54. $\frac{8}{3^x} = 4$.

55. $5(3^x - 6) = 10$.

56. $.1e^{.1x} = .5$.

57. $e^{2x-5} + 1 = 4$.

58. $3e^{2x} - 1 = \frac{1}{2}$.

59. The predicted population P of a city is given by $P = 125{,}000(1.12)^{t/20}$, where t is the number of years after 1980. Predict the population in 2000.

60. For a certain city the population P grows at the rate of 2 percent per year. The formula $P = 1{,}000{,}000(1.02)^t$ gives the population t years after 1980. Find the population in (a) 1980 and (b) 1982.

61. Interest is said to be compounded continuously if the accumulated amount S of a principal P after n years at an annual rate of r (expressed as a decimal) is given by the formula $S = Pe^{rn}$. Find the amount that \$1,000 will become after eight years with interest compounded continuously at an annual rate of .05.

62. The formula $A = Pe^{-rn}$ gives the amount at the end of n years of a principal P which depreciates at a rate of r (expressed as a decimal) per year compounded continuously. What is the value at the end of ten years of \$60,000 of machinery which depreciates at a rate of 8 percent, compounded continuously? Give your answer to the nearest dollar.

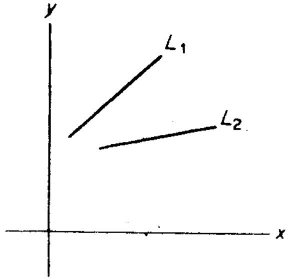
63. The demand equation for a certain product is $q = 80 - 2^p$. Sketch its graph and choose q for the horizontal axis.

CHAPTER 6

STRAIGHT LINE

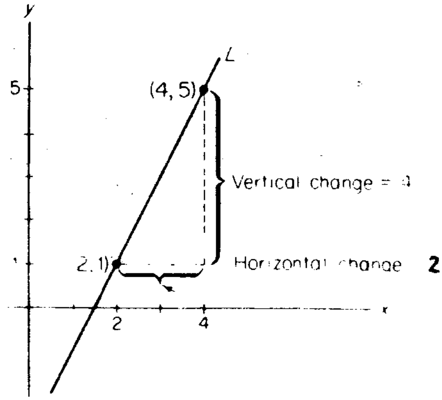
6.1 Lines:

Many relationship in economics can be represented by straight lines. One feature of a straight line is its '*steepness*'. Take for example, the line L_1 and L_2 as shown in the figure, L_1 rises faster as it goes from left to right than line L_2. In this sense it is steeper.



To measure the steepness of a line, we introduce the notion of *slope*.

Consider two points (2, 1) and (4, 5) on the line L. As the x-coordinate increases from 2 to 4, the y-coordinate increases from 1 to 5. The average rate of change of y with respect to x is the ratio.



$$\frac{\text{change in } y}{\text{change in } x} = \frac{5-1}{4-2} = \frac{4}{2} = 2.$$

Therefore, 2 is the slope of L. This means that for each 1-unit increase in x, there is a 2-unit increase in y.

In general, if (x_1, y_1) and (x_2, y_2) are two different points on a nonvertical line, the slope m of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1}\left(= \frac{\text{vertical change}}{\text{horizontal change}}\right).$$

Case 1:

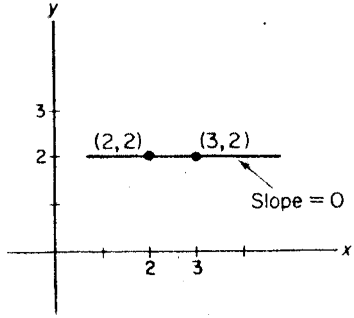
Find the slope of the horizontal line through (2, 2) and (3, 2).

Solution:

$$m = \frac{\text{vertical change}}{\text{horizontal change}}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-2}{3-2} = \frac{0}{1} = 0.$$

Therefore, the slope of horizontal line is 0. Ans



Case 2:

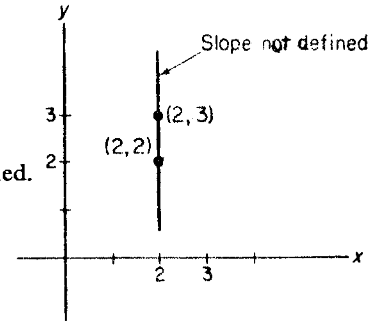
Find the slope of the vertical line through (2, 2) and (2, 3).

Solution:

$$m = \frac{\text{vertical change}}{\text{horizontal change}}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-2}{2-2} = \frac{1}{0},$$ which is not defined.

Thus, the slope of other vertical line is not defined. Ans



Case 3:

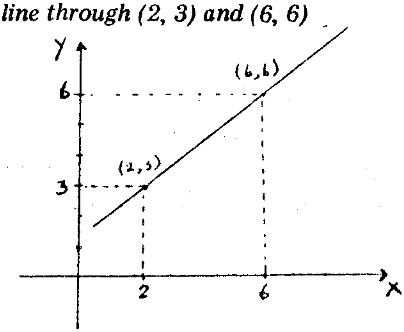
Find the slope of the line through (2, 3) and (6, 6)

Solution:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{6-3}{6-2}$$

$$= \frac{3}{4}$$

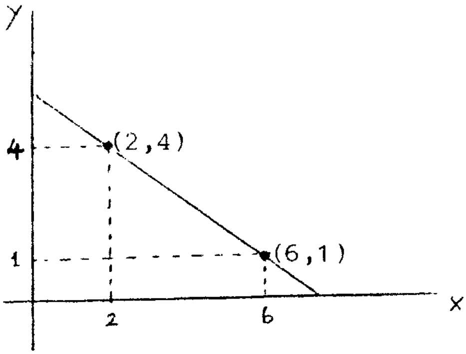


Therefore, the slope is $= \frac{3}{4}$ Ans

Case 4:

Find the slope of the line through (2, 4) and (6, 1).

Solution:

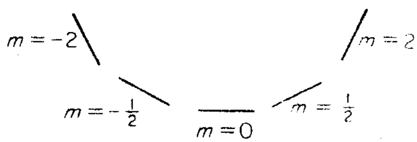


$$m = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{1 - 4}{6 - 2} = -\frac{3}{4}$$

Hence, the slope is $= -\frac{3}{4}$ Ans

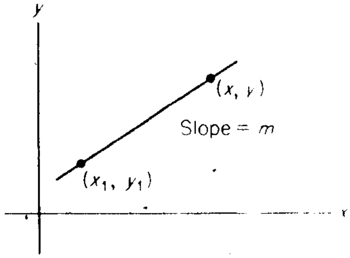
In summary



1. The slope of every horizontal line is *zero.*
2. The slope of every vertical line *is not defined* (no slope).
3. The slope of the line rising from left to right is *positive.*
4. The slope of the line falling from left to right is *negative.*

6.2 Point - Slope Form:

Suppose that line L has slope m, that it passes through (x_1, y_1), and that (x, y) is any other point on L.



$$\text{Then } m = \frac{y - y_1}{x - x_1}$$

$$y - y_1 = m(x - x_1).$$

Therefore, every point on L satisfies the equation and any point satisfying the equation must lie on L.

Hence $y - y_1 = m(x - x_1)$ is the *point-slope form* of an equation of the line through (x_1, y_1) and having slope m.

Example 1:

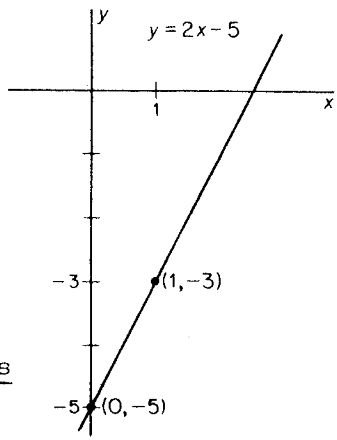
Determine an equation of the line that has slope 2 and passes through (1, −3) and sketch the graph.

Solution:

Here m = 2 and $(x_1, y_1) = (1, -3)$

Using a point-slope form, we have

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - (-3) &= 2(x - 1) \\
y + 3 &= 2x - 2 \\
y &= 2x - 5 \qquad \underline{\text{Ans}}
\end{aligned}
$$



Example 2:

Determine an equation of the line passing through (−3, 8) and (4, −2)

Solution:

First, find the slope of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Let $(x_1, y_1) = (-3, 8)$ and $(x_2, y_2) = (4, -2)$

$$\therefore m = \frac{-2 - 8}{4 - (-3)} = \frac{-10}{7}$$

Then use a point-slope form with either point as (x_1, y_1) Here we take $(-3, 8)$ as (x_1, y_1)

$$y - y_1 = m(x - x_1)$$

$$y - 8 = \frac{10}{7}[x - (-3)]$$

$$y - 8 = -\frac{10}{7}(x + 3) \qquad \underline{\text{Ans}}$$

6.3 Slope-Intercept Form:

A point $(0, b)$ where a graph intersects the y-axis is called a y-intercept. If the slope and y-intercept of a line L are known, an equation for L can be found by using a point-slope form.

$$\text{Since} \quad (x_1, y_1) = (0, b)$$

$$\therefore y - y_1 = m(x - x_1)$$

$$y - b = m(x - 0)$$

$$y - b = mx$$

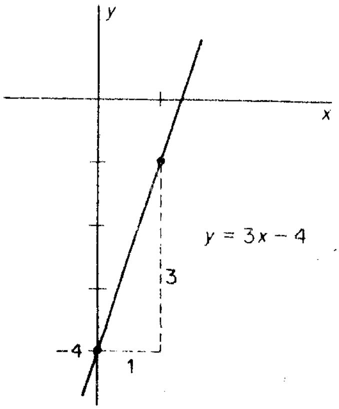
$$\text{or} \quad y = mx + b$$

Therefore $y = mx + b$ is the *slope-intercept form* of an equation of the line with slope m and y-intercept $(0, b)$

Example 1:

Determine the equation of the line with slope 3 and y-intercept (0, −4)

Solution:



$$y = mx + b$$

$$y = 3x + (-4)$$

$$y = 3x - 4 \qquad \underline{\text{Ans}}$$

Example 2:

Find the slope-intercept form of an equation of the line passing through (−3, 8) and (4, −2)

Solution:

Find the slope of the line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Put $(x_1, y_1) = (-3, 8)$ and $(x_2, y_2) = (4, -2)$

$$\therefore \quad m = \frac{-2 - 8}{4 - (-3)} = \frac{-10}{7}$$

Then use a point-slope form:

$$y - y_1 = m(x - x_1)$$

$$y - 8 = -\frac{10}{7}[x - (-3)]$$

$$y - 8 = \frac{-10}{7}x - \frac{30}{7}$$

$$y = \frac{-10x}{7} + \frac{26}{7} \qquad \underline{\text{Ans}}$$

Example 3:

From the equation $y = \frac{3}{2}x - 6$, *point out the slope and y—intercept of the line.*

Solution:

$$\text{Here} \quad m = \frac{3}{2}$$

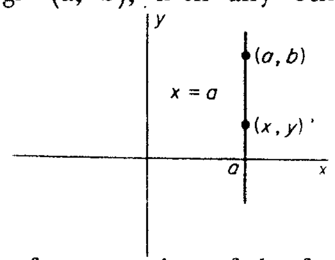
$$\text{and y-intercept is} = (0, -6) \qquad \underline{\text{Ans}}$$

6.4 Linear Function:

Definition: A function f is a *linear function* if and only if f(x) can be written in the form $f(x) = ax + b$, where a and b are constant and $a \neq 0$.

Suppose f is a linear function and we let $y = f(x)$. Then $y = ax + b$, which is an equation of a straight line with slope 'a' and y-intercept (0, b). Thus the graph of a linear function is a straight line.

If a *vertical line* L passes through (a, b), then any other point (x, y) lies on L if and only if $x = a$. There is no restriction on y. Hence an equation of L is $x = a$. Similarly, an equation of the *horizontal line* passing through (a, b) is $y = b$. There is no restriction on x.



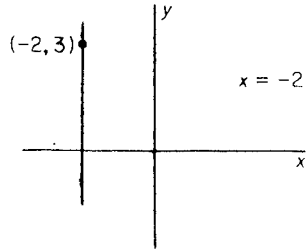
Every straight line is the graph of an equation of the form $Ax + By + C = 0$, where A, B, and C are constants and A and B are not both zero. And it is called a *general linear equation.*

Example 1:

Sketch the graph of the vertical line through (—2, 3)

Solution:

The vertical line is $x = -2$. Ans



Example 2:

Find the equations of the vertical and horizontal lines passing through the origin.

Solution:

$$\left.\begin{array}{ll} & \text{An equation of the x-axis is } y = 0 \\ \text{and} & \text{an equation of the y-axis is } x = 0 \end{array}\right\} \quad \underline{\text{Ans}}$$

Example 3:

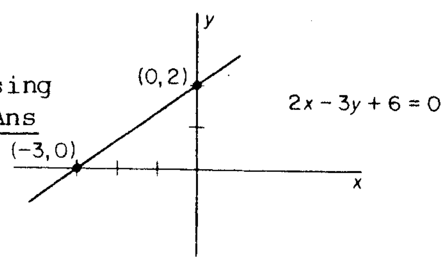
Sketch the graph of $2x - 3y + 6 = 0$

Solution:

Since it is an equation of a straight line, only two points on the line are needed to sketch the graph.

If $x = 0$, then $y = 2$
If $y = 0$, then $x = -3$

Now draw the line passing through $(0,2)$ and $(-3,0)$ Ans



Example 4.

Suppose a manufacturer has 100 kilograms of material from which he can produce two products, A and B, which require 4 kilograms and 2 kilograms of material per unit respectively. Write the equation and sketch the graph.

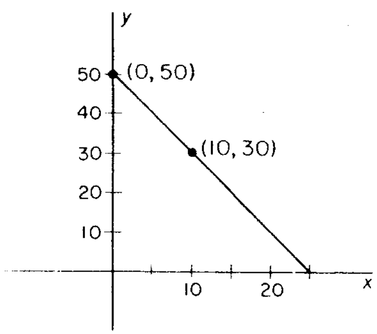
Solution:

If x and y denote the number of units produced of A and B, respectively, then all levels of production are given by the combinations of x and y that satisfy the equation.

$$4x + 2y = 100 \quad \text{where } x, y > 0$$

In the slope-intercept form, we get

$$y = -2x + 50$$



EXERCISE: 6 - 1

In Problems 1-8, find the slope of the straight line which passes through the given points.

1. (1, 2), (4, 8).
2. (−1, 9), (1, 5).
3. (6, −3), (−7, 5).
4. (2, −4), (3, −4).
5. (−2, 4), (−2, 8).
6. (0, −6), (3, 0).
7. (5, −2), (4, −2).
8. (1, −6), (1, 0).

In Problems 9-22, determine a general linear equation ($Ax + By + C = 0$) of the straight line that has the indicated properties and sketch each line.

9. Passes through (1, 2) and has slope 6.

10. Passes through origin and has slope -5.

11. Passes through $(-2, 5)$ and has slope $-\frac{1}{4}$.

12. Passes through $(\frac{1}{2}, 6)$ and has slope $\frac{1}{3}$.

13. Passes through (1, 4) and (8, 7).

14. Passes through (7, 1) and $(7, -5)$.

15. Passes through $(3, -1)$ and $(-2, -9)$.

16. Passes through (0, 0) and (2, 3).

17. Passes through $(-2, 5)$ and (3, 5).

18. Passes through (4, 3) and (2, 0).

19. Passes through $(2, -8)$ and is vertical.

20. Passes through (7, 4) and is horizontal.

21. Passes through $(-1, 3)$ and is parallel to the line $y = 4x - 5$.

22. Passes through (2, 1) and is parallel to the line $y = 3 + 2x$.

In Problems 23-36, find, if possible, the slope and y-intercept of the given linear function and sketch the graph.

23. $y = 2x - 1$.

24. $x - 1 = 5$.

25. $3x - 8y = 8$.

26. $(x - 1) + (y - 2) = 0$.

27. $x + 2y - 3 = 0$.

28. $x + 4 = 7$.

29. $x = -5$.

30. $x - 1 = 5y + 3$.

31. $y = 3x$.

32. $y - 7 = 3(x - 4)$.

33. $y = 1$.

34. $2y - 3 = 0$.

35. $\frac{x}{5} - 8y = 4$.

36. $y + 7 = 0$.

In Problems 37-46, determine a general linear form and the slope-intercept form of the given equation.

37. $x = -2y + 4$.

38. $3x + 2y = 6$.

39. $4x + 9y - 5 = 0$.

40. $2(x - 3) - 4(y + 2) = 8$.

41. $\frac{3}{4}x = \frac{7}{3}y + \frac{1}{4}$.

42. $\frac{y}{-2} + \frac{x}{3} = 1$.

43. $\frac{x}{2} - \frac{y}{3} = -4$.

44. $y = \frac{1}{300}x + 8$.

45. $3x + 4y - 7 = 2x + 3y - 6$.

46. $3x - 4y = 13$.

In Problems 47-50, determine the slope and y-intercept of the given linear function and sketch the graph.

47. $f(x) = x + 1$.

48. $f(x) = x$.

49. $f(x) = -3x + 5$.

50. $f(x) = 2x - 3$.

In Problems 51-54, determine f(x) if f is a linear function that has the given properties.

51. slope $= 5$, $f(3) = 1$.

52. $f(0) = 4$, $f(2) = -6$.

53. $f(2) = 3$, $f(-1) = 12$.

54. slope $= -6$, $f(\frac{1}{2}) = -2$.

55. A straight line passes through (1, 2) and (–3, 8). Find the point on it that has a first coordinate of 5.

56. A straight line has slope –3 and passes through (4, –1). Find the point on it that has a second coordinate of –2.

57. Suppose q and p are related linearly such that p = 12 when q = 40, and p = 18 when q = 25. Find an equation that satisfies these conditions. Find p when q = 30. Hint: The given data can be represented in a q, p-coordinate plane by the points (40, 12) and (25, 18).

58. Suppose the cost to produce 10 units of a product is \$40 and the cost of 20 units is \$70. If cost c is linearly related to output q, find a linear equation relating c and q. Find the cost to produce 35 units.

59. In production analysis, an isocost line is a line whose points represent all combinations of two factors of production that can be purchased for the same amount. Suppose a farmer has allocated \$20,000 for the purchase of x tons of fertilizer (costing \$200 per ton) and y acres of land (costing \$2000 per acre). Find an equation of the isocost line which describes the various

combinations that can be purchased for \$20,000. Observe that neither x nor y can be negative.

60. Suppose the value of a piece of machinery decreases each year by 10 percent of its original value. If the original value is \$8000, find an equation that expresses the value v of the machinery after t years of purchase where $0 \leqslant t \leqslant 10$. Sketch the equation, choosing t as the horizontal axis and v as the vertical axis. What is the slope of the resulting line? This method of considering the value of equipment is called straight-line depreciation.

61. A manufacturer produces products X and Y for which the profits per unit are \$4 and \$6, respectively. If x units of X and y units of Y are sold, then the total profit P is given by $P = 4x + 6y$, where $x, y \geqslant 0$. (a) Sketch the graph of this equation for $P = 240$. The result is called an isoprofit line and its points represent all combinations of sales that produce a profit of \$240. (b) Determine the slope for $P = 240$. (c) If $P = 600$, determine the slope. (d) What conclusion can you draw concerning isoprofit lines for products X and Y?

6.5 Systems of Linear Equations:

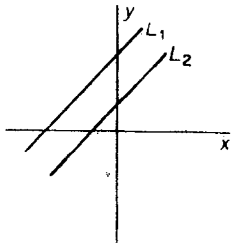
In this section, we shall discuss methods of solving a 'set of equations'. The solution will consist of values of the variables for which all the equations in the set are satisfied simultaneously.

Consider a system of two linear equations in the variables (or unknowns) x and y.

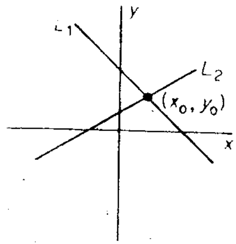
$$a_1x + b_1y = c_1, \quad \ldots\ldots\ldots\ldots (1)$$
$$a_2x + b_2y = c_2 \quad \ldots\ldots\ldots\ldots (2)$$

Let L_1 represent the line of the Equation (1) and L_2 represent the line of Equation (2). Since the coordinates of any point on a line satisfy the equation of that line, the coordinates of any point of intersection of L_1 and L_2 will satisfy both equations. Hence a point of intersection will give a solution of the system, and vice versa. If L_1 and L_2 are sketched on the same plane, there are three possibilites as to their relative orientations:

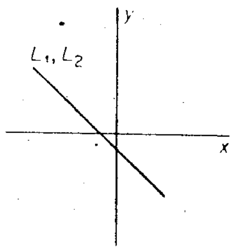
(1) L_1 and L_2 may be parallel and have no points in common. Thus there is no solution.



(2) L_1 and L_2 may intersect at exactly one point (x_0, y_0). Thus the system has the solution at that point.



(3) L_1 and L_2 may coincide. Thus the coordinates of any points on L_1 are a solution of the system and so there are infinitely many solutions. In this case the given equations must be equivalent.



Case 1:

Solve the equations

$$2x + y = 1 \quad \ldots\ldots\ldots(1)$$

$$4x + 2y = 4 \quad \ldots\ldots\ldots(2)$$

Solution:

$(1) \times 2 \quad 4x + 2y = 2 \quad \ldots\ldots\ldots(3)$

$(2) - (3) \qquad 0 = 2$

Which is never true. So there is no solution to the system.

In term of slope-intercept form, we have

$$y = -2x + 1 \qquad \text{and}$$

$$y = -2x + 2$$

These equations represent straight lines havings slopes of -2 but different y-intercepts, (0, 1) and (0, 2).

Case 2:

Solve the system

$$x + y = 7 \quad \ldots\ldots\ldots\ldots (1)$$

$$x - y = 1 \quad \ldots\ldots\ldots\ldots (2)$$

Solution:

(1) + (2) $\qquad 2x = 8$

$$x = 4$$

Substitute the value of x in (1)

$$4 + y = 7$$

$$y = 3$$

The solution is $x = 4$, $y = 3$ Ans

Case 3:

Solve

$$2x + 2y = 6 \quad \ldots\ldots\ldots\ldots (1)$$

$$3x + 3y = 9 \quad \ldots\ldots\ldots\ldots (2)$$

Solution:

From (1) $\qquad x + y = 3$

$$x = 3 - y \quad \ldots\ldots\ldots (3)$$

Substitute eq.(3) in eq.(2)

$$3(3 - y) + 3(y) = 9$$

$$9 - 3y + 3y = 9$$

$$9 = 9$$

Any solution of eq. (1) is a solution of the system In their slope-intercept forms, we get

$$2x + 2y = 6$$

$$\text{or} \quad y = -x + 3$$

$$\text{and} \quad 3x + 3y = 9$$

$$y = -x + 3$$

in which both equations represent the same line. Hence the lines coincide. The coordinates of any point on the line $y = -x + 3$ are a solution, and so there are infinitely many solutions.

The same principles given above can be applied to a system of more than two linear equations.

Example 1:

Solve

$$2x + y + z = 3 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

$$-x + 2y + 2z = 1 \quad \ldots\ldots\ldots\ldots\ldots (2)$$

$$x - y - 3z = -6 \quad \ldots\ldots\ldots\ldots\ldots (3)$$

(2) + (3) $\quad y - z = -5 \quad \ldots\ldots\ldots\ldots\ldots (4)$

(2) × 2 $\quad -2x + 4y + 4z = 2 \quad \ldots\ldots\ldots\ldots\ldots (5)$

(1) + (5) $\quad 5y + 5z = 5$

$$y + z = 1 \quad \ldots\ldots\ldots\ldots\ldots (6)$$

(4) + (6) $\quad 2y = -4$

$$y = -2$$

Substitute the value of y in (6)

$$-2 + z = 1$$

$$z = 3$$

Substitute the values of y and z in (1)

$$2x + (-2) + 3 = 3$$

$$x = 1$$

The solution is $x = 1$, $y = -2$, and $z = 3$ Ans

Example 2:

A chemical manufacturer wishes to fill a request for 500 litres of a 25 percent acid solution (25 percent by volume is acid). If solutions of 30 and 18 percent are available in stock, how many litres of each must be mixed to fill the order?

Solution:

Let x be the number of litres of the 30 percent solution and y be the number of litres of the 18 percent solution.

Then $$x + y = 500$$

or $$x = (500 - y) \quad \ldots\ldots\ldots\ldots\ldots (1)$$

In 500 litres of a 25 percent solution, there will be

$$0.25(500) = 125 \text{ litres of acid}$$

This acid comes from two sources: 0.30x litres of acid come from the 30 percent solution and 0.18y litres of acid come from the 18 percent solution.

Hence

$$0.30x + 0.18y = 125 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Substitute the value of x in (2)

$$0.30(500 - y) + 0.18y = 125$$

$$\left.\begin{aligned} y &= 208\tfrac{1}{3} \text{ litres} \\ \text{and} \quad x &= 291\tfrac{2}{3} \text{ litres} \end{aligned}\right\} \underline{\text{Ans}}$$

6.6 Nonlinear Systems:

A system of equations in which at least one equation is not linear is called a *nonlinear system.* The solutions of such systems may often be found algebraically as was done with linear systems.

Example:

Solve

$$x^2 - 2x + y - 7 = 0 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

$$3x - y + 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots (2)$$

Solution:

From (2) $\qquad y = 3x + 1 \quad \ldots\ldots\ldots (3)$

Substitute(3) in (1)

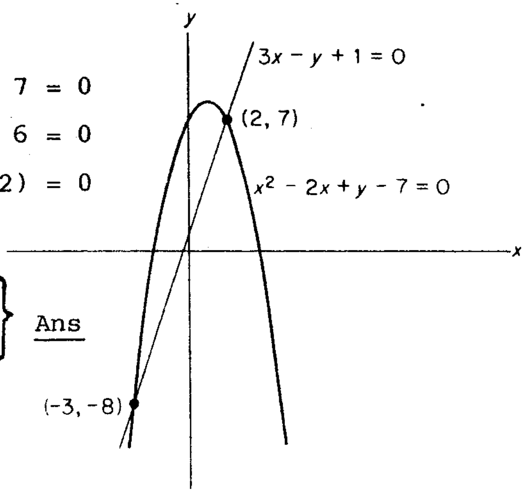
$$x^2 - 2x + (3x + 1) - 7 = 0$$

$$x^2 + x - 6 = 0$$

$$(x + 3)(x - 2) = 0$$

$$x = 2, \text{ or } x = -3$$

$$\left.\begin{array}{l} \text{If } x = -3, \text{ then } y = -8; \\ \text{If } x = 2, \text{ then } y = 7 \end{array}\right\} \underline{\text{Ans}}$$

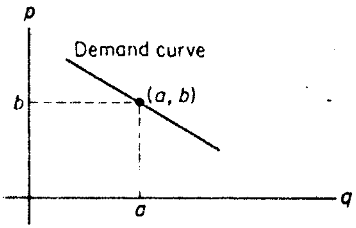


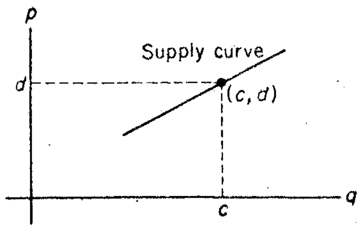
6.7 Some Applications of Systems of Equations:

For each price level of a product there is a corresponding quantity of the product that consumers will demand (that is, purchase) during some time period. Usually, the higher the price, the smaller the quantity demanded; as the price falls, the quantity demanded increases.

On the other hand, in response to various prices, there is a corresponding quantity of output of a product that producers are willing to place on the market during some time period. Usually, the higher the price per unit, the larger the quantity that producers are willing to supply; as the price falls, so will the quantity supplied.

The quantities of a product that will be demanded or supplied per unit of time at all possible alternative prices can be indicated geometrically on a coordinate plane by a *demand or supply curve.*



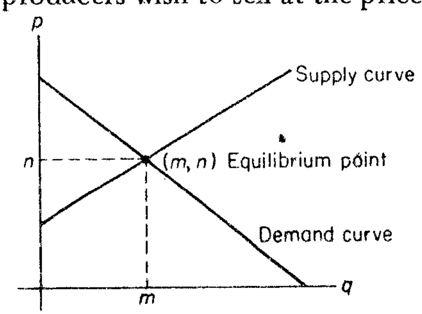


The point (a, b) from figure (a) indicates that at a price of 'b' dollars per unit, consummers will demand 'a' units per week. Similarly, in figure (b), the point (c, d) indicates that at a price of 'd' dollars each, producers will supply 'c' units per week.

In most cases a demand curve falls from left to right. This reflects the relationship that consumers will buy more of a product as its price goes down. A supply curve usually rises from left to right. This indicates that a producer will supply more of a product at higher prices.

An equation that relates price per unit and quantity demanded (supplied) is called a *demand equation (supply equation).*

When both the demand and supply curves of a product are reprèsented on the same coordinate plane, the point (m, n) at which the curves intersect is called the *point of eqilibrium.* The price n, called the *equilibrium price,* is the price at which consumers will purchase the same quantity of a product that producers wish to sell at the price. In short, n is the price at which stability in the producer-consumer relationship occurs. The quantity m is called the *equilibrium quantity.*



Example 1:

Let $= \frac{8}{100}q + 50$ *be the supply equation for a certain manufacturer. Suppose the demand per week for his product is 100 units when the price is \$58 per unit, and 200 units per week at \$51 each.*

a) *Determine the demand equation, assuming that it is linear.*

b) *If a tax of \$1.50 per unit is to be imposed on the manufacturer, how will the original equilibrium price be affected if the demand remains the same?*

c) *Determine the total revenue obtained by the manufacturer at the equilibrium point both before and after the tax.*

Solution:

a) Since the demand equation is linear, the demand curve must be a straight line. From the given data we conclude that the points (100, 58) and (200, 51) lie on this line and thus its slope is

$$m = \frac{51 - 58}{200-100} = -\frac{7}{100}$$

An equation of the line is

$$p - p_1 = m(q - q_1)$$

$$p - 58 = -\frac{7}{100}(q - 100)$$

$$\text{or} \quad p = -\frac{7}{100}q + 65 \qquad \underline{\text{Ans}}$$

b) Before the tax, the equilibrium price is obtained by solving the system

$$p = \frac{8}{100}q + 50 \quad \ldots\ldots\ldots\ldots(1)$$

$$p = -\frac{7}{100}q + 65 \quad \ldots\ldots\ldots\ldots(2)$$

$$\therefore \quad -\frac{7}{100}q + 65 = \frac{8}{100}q + 50$$

$$15 = \frac{15}{100}q$$

$$q = 100$$

Substitute value of q in (1)

$$p = \frac{8}{100}(100) + 50$$

$$= 58$$

Thus, \$58 is the original equilibrium price

Before the tax the manufacturer supplies q units at a price of $p = \frac{8}{100}q + 50$ per unit. After the tax he will sell the same q units for an additional \$1.50 per unit.

The price per unit will then be $= \left(\frac{8}{100}q + 50\right) + 1.50$

Thus the new supply equation will be $p = \frac{8}{100}q + 51.50$

Solving the system $p = \frac{8}{100}q + 51.50$ ....(1)

$$p = -\frac{7}{100}q + 65 \quad \text{............(2)}$$

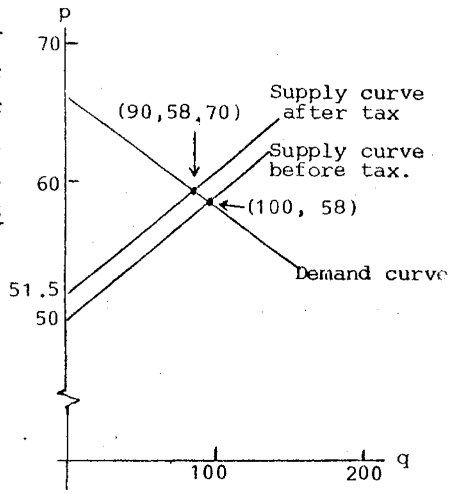
(1) = (2) Then, $\frac{8}{100}q + 51.50 = -\frac{7}{100}q + 65$

$$\frac{15}{100}q = 13.50$$

$$q = 90$$

and $p = \frac{8}{100}(90) + 51.50 = 58.70$

The tax of \$1.50 per unit increases consequently, the equilibrium price by \$0.70. There is also a decrease in the equilibrium quantity from q = 100 to q = 90. <u>Ans</u>



(c) If q units of a product are sold at a price of p dollars each, then the total revenue (y_{TR}) is given by

$$y_{TR} = pq$$

Before the tax the revenue at (100, 58) is

$$y_{TR} = (100)(58) = \$5800 \quad \underline{\text{Ans}}$$

After the tax it is

$$y_{TR} = (90)(58.70) = \$5283 \quad \underline{\text{Ans}}$$

Example 2:

Find the equilibrium point if the supply and demand equations of a product are

$$P = \frac{q}{40} + 10 \text{ and } p = \frac{8000}{q}, \text{ respectively.}$$

Solution:

Solve the system:

$$p = \frac{q}{40} + 10 \quad \ldots\ldots\ldots (1)$$

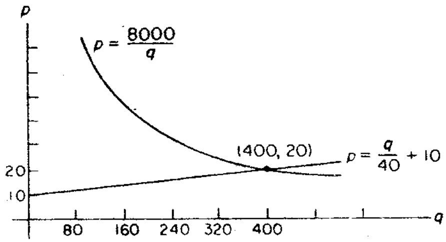
$$p = \frac{8000}{q} \quad \ldots\ldots\ldots (2)$$

(1) = (2) Then, $\frac{8000}{q} = \frac{q}{40} + 10$

$$q^2 + 400q - 320{,}000 = 0$$

$$(q + 800)(q - 400) = 0$$

$$q = -800 \quad \text{or} \quad q = 400$$



Since q represents quantity, it should be positive number.

$$\left.\begin{array}{ll} \text{Therefore,} & q = 400 \\ \text{and} & p = 20 \end{array}\right\} \underline{\text{Ans}}$$

Profit (or Loss) = Total Revenue – Total Cost.

Total Cost: is the sum of total variable costs and total fixed costs.

Fixed Costs: are those costs that under normal conditions do not depend on the level of production; that is, over some period of time they remain costant at all levels of output. Examples are rent, officers' salaries and normal maintenance.

Variable Costs: are those costs that vary with the level of production (such as cost of materials, labor, maintenance due to wear and tear, etc.)

The break-even point: is the point at which Total Revenue = Total Cost. It occurs when the level of production and sales result in neither a profit nor a loss to the manufacturer.

Example:

Suppose a manufacturer produces product A and sells it at \$8.00 per unit. If the cost of production per unit is $\$\frac{22}{9}$, and fixed cost is \$5000.

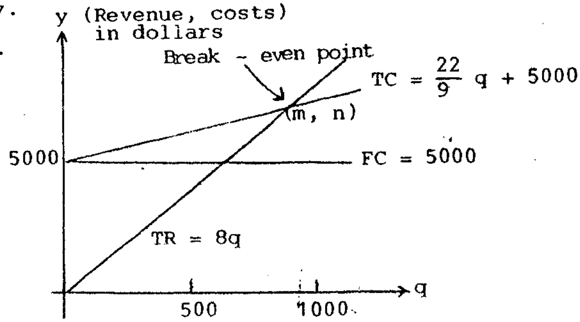
Then total revenue TR = 8q...........(1)

$$\text{Total Cost} = \text{variable costs} + \text{fixed costs}$$
$$= \frac{22}{9}q + 5000 \ldots\ldots(2)$$

The break-even point is where the graphs of total revenue (TR) and total cost (TC) intersect, as shown in the figure.

m = break-even quantity.

n = break-even revenue.



Example 1:

A manufacturer sells his product at \$8 per unit, selling all that he produces. His fixed cost is \$5000 and the variable cost per uhit is $\frac{22}{9}$ (dollars). Find

a) *the total out put and revenue at the break-even point.*

b) *the profit when 1800 units are produced.*

c) *the loss when 450 units are produced*

d) *the output required to obtain a profit of \$10,000.*

Solution:

a) At an output level of q units, the variable cost is $\frac{22}{9}$ q and the total revenue = 8q.

Hence,

$$TR = 8q$$

$$TC = VC + FC$$
$$= \frac{22}{9} q + 5000$$

At the break-even point, total revenue = total cost.

$$TR = TC$$
$$8q = \frac{22}{9} q + 5000$$
$$\frac{50}{9} q = 5000$$
$$q = 900$$

Thus the desired output is 900 units, resulting in a total revenue of = 8(900) = \$7200 <u>Ans</u>

b) To find the profit when 1800 units are produced:

$$\text{Profit} = \text{Total revenue} - \text{Total Cost}$$
$$= (8)(1800) - [\frac{22}{9}(1800) + 5000]$$
$$= \$5000 \quad \underline{\text{Ans}}$$

c) To determine the loss when 450 units are produced:

$$\text{The Loss} = \text{Total revenue} - \text{Total Cost}$$
$$= 8(450) - \left[\frac{22}{9}(450) + 5000\right]$$
$$= -\$2500 \quad \underline{\text{Ans}}$$

(negative profit = loss)

d) To determine to output required to obtain a profit of \$10,000.

$$\text{Profit} = TR - TC$$
$$10{,}000 = 8q - \left(\frac{22}{9} q + 5000\right)$$
$$15{,}000 = \frac{50}{9} q$$
$$q = 2700$$

Thus 2700 units must be produced. <u>Ans</u>

Example 2:

Determine the break-even quantity of XYZ Manufacturing Co. given the following data: total fixed cost, \$1200; variable cost per unit, \$2, total revenue for selling q units, TR = 100 $\sqrt{q}$.

Solution:

$$\text{Total Revenue TR} = 100\sqrt{q}.$$

$$\text{Total Cost. TC} = \text{Variable costs} + \text{Fixed Costs}$$

$$= 2q + 1200$$

To find the break-even quantity we have to determine the point of intersection between the total revenue and total cost

To solve the system: TR = TC

$$\therefore \quad 100\sqrt{q} = 2q + 1200$$

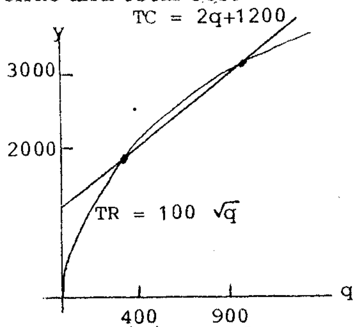
$$50\sqrt{q} = q + 600$$

$$\text{or } 2500q = q^2 + 1200q + 360{,}000$$

$$q^2 - 1300q + 360{,}000 = 0$$

$$(q - 400)(q - 900) = 0$$

$$q = 400 \quad \text{or} \quad q = 900$$



Thus q = 400 and q = 900 are both break-even quantities <u>Ans</u>

Note that there will always be a loss when $q > 900$. Thus producting more than the break-even quantity does not necessarily guarantee a profit.

EXERCISE: 6 - 2

1. Suppose a manufacturer of shoes will place on the market 50 (thousand pairs) when the price is 35 (dollars per pair) and 35 when the price is 30. Find the supply equation, assuming that it is linear.

2. Suppose consumers will demand 20 (thousand) pairs of shoes when the price is 35 (dollars per pair) and 25 pairs when the price is 30. Find the demand equation, assuming that it is linear.

In Problem 3-10, the first equation is a supply equation and the second is a demand equation for a product. If p represents price per unit in dollars and q represents the number of units per unit of time, find the equilibrium point. In Problem 3 and 4 sketch the system.

3. $p = \frac{3}{100}q + 2,$

$p = -\frac{7}{100}q + 12.$

4. $p = \frac{1}{2000}q + 3,$

$p = -\frac{1}{2500}q + \frac{42}{5}.$

5. $35q - 2p + 250 = 0,$

$65q + p - 537.5 = 0.$

6. $246p - 3.25q - 2460 = 0,$

$410p + 3q - 14{,}452.5 = 0.$

7. $p = 2q + 20,$

$p = 200 - 2q^2.$

8. $p = (q + 10)^2,$

$p = 388 - 16q - q^2.$

9. $p = \sqrt{q + 10},$

$p = 20 - q.$

10. $p = \frac{1}{5}q + 5,$

$p = \frac{3000}{q + 20}.$

In Problems 11-16, TR represents total revenue in dollars and TC represents total cost in dollars for a manufacturer. If q represents both the number of units produced and the number of units sold, find the break-even quantity. Sketch a break-even chart in Problems 11 and 12.

11. $y_{TR} = 3q,$

$y_{TC} = 2q + 4500.$

12. $y_{TR} = 14q,$

$y_{TC} = \frac{40}{3}q + 1200.$

13. $y_{TR} = .05q,$

$y_{TC} = .85q + 600.$

14. $y_{TR} = .25q,$

$y_{TC} = .16q + 360.$

15. $y_{TR} = 100 - \frac{1000}{q + 10},$

$y_{TC} = q + 40.$

16. $y_{TR} = .1q^2 + 7q,$

$y_{TC} = 2q + 500.$

17. The supply and demand equations for a certain product are $3q - 200p + 1800 = 0$ and $3q + 100p - 1800 = 0$, respectively, where p represents the price per unit in dollars, and q represents the number of units per time period.

a) Find the equilibrium price algebraically, and derive it graphically.

b) Find the equilibrium price when a tax of 27 cents per unit is imposed on the supplier.

18. A manufacturer of a product sells all that he produces. His total revenue is given by $TR = 7q$ and his total cost is given by $TC = 6q + 800$, where q represents the number of units produced and sold.

    a) Find the level of production at the break-even point and draw the break-even chart.
    b) Find the level of production at the break-even point if the total cost increases by 5 percent.

19. A manufacturer sells his product at \$8.35 per unit, selling all he produces. His fixed cost is \$2116 and his variable cost is \$7.20 per unit. At what level of production will he have a profit of \$4600? At what level of production will he have a loss of \$1150? At what level of production will he break even?

20. The market equilibrium point for a product occurs when 13,500 units are produced at a price of \$4.50 per unit. The producer will supply no units at \$1 and the consumers will demand no units at \$20. Find the supply and demand equations if they are both linear.

21. A manufacturer of widgets will break even at a sales volume of \$200,000. Fixed costs are \$40,000 and each unit of output sells for \$5. Determine the variable cost per unit.

22. The Footsie Sandal Co. manufactures sandals for which the material cost it \$0.80 per pair and the labor cost is \$0.90 per pair. Additional variable costs amount to \$0.30 per pair. Fixed costs are \$70,000. If each pair sells for \$2.50, how many pairs must be sold for the company to break even?

23. Find the break-even point for Company Z, which sells all it produces, if the variable cost per unit is \$2, fixed costs are \$1050, and $TR = 50\sqrt{q}$, where q is the number of units of output.

24. A company has determined that the demand equation for its product is $p = 1000/q$, where p is the price per unit for q units in some time period. Determine the quantity demanded when the price per unit is: (a) \$4; (b) \$2; (c) \$0.50. For each of these prices, determine the total revenue that the company will receive. What will be the revenue regardless of the price? (Hint: Find the revenue when the price is p dollars.)

25. A firm produces a product which sells at a price of \$25 per unit. Variable costs are estimated to be \$18.75 per unit, and fixed costs are \$50,000.

   a) Determine the break-even level of output.
   b) Compute total cost and total revenue at the break-even point.
   c) What will profit equal if demand equals 7,500 units?

26. A firm produces a product which sells at a price of \$150 per unit. Variable cost per unit is estimated at \$130, and fixed costs are \$250,000.

   a) Determine the break-even level of output.
   b) Compute total cost and total revenue at the break-even point.
   c) What will profit equal if 12,000 units are demanded?

27. A local charity organization is planning a chartered flight and one-week vacation to the Caribbean. The venture is a fund-raising effort. A package deal has been worked out with a commercial airline in which the charity will be charged a fixed cost of \$10,000 plus \$300 per person. The \$300 covers the flight cost, transfers, hotel, meals, and tips. The organization is planning to price the package at \$450 per person.

   a) Determine the number of persons necessary to break even on the venture.
   b) The goal of the orgainzation is to net a profit of \$10,000. How many people must participate for the goal to be realized?

28. Assume that the organization in Exercise 27 has received pledges guaranteeing that the trip will be subscribed to the capacity of 150 people. Assume the organization wished only to break even on the venture.

    a) What price should they charge each person?
    b) What price would enable the organization to realize its profit goal of $10,000?

29. The management of a local civic center is negotiating a contract with the rock and roll group The Windy City. The Windy City commands a fee of $25,000 plus 37.5 percent of gate receipts. Promoters expect to charge $8 per ticket for the performance.

    a) Determine the number of tickets which must be sold in order to break even.
    b) If the promoters hope to clear a profit of $20,000, how many tickets must be sold?

30. Assume that promoters in Exercise 29 believe that the show will be a sellout of 10,000 fanatics.

    a) What ticket price would allow them to break even?
    b) What ticket price would allow them to realize the profit goal of $20,000?

31. A firm is developing a TV advertising campaign. Development costs (fixed costs) are $100,000, and the firm must pay $10,000 per minute for television slots. The firm estimates that for each minute of advertising additional sales of $50,000 result. Of this $50,000, $37,500 is absorbed to cover the variable costs of producing the items and $10,000 must be used to pay for the minute of advertising. Any remainder is the contribution to fixed cost and profit.

    a) How many minutes of advertising are necessary to recover the development costs of the advertising campaign?
    b) If the firm uses this campaign for 60 one-minute slots, determine total revenues, total costs (production and advertising), and total profit (or loss) resulting from the

campaign.

32. A car leasing agency purchases new cars each year for use in the agency. The cars cost \$5,000 new. They are used for two years, after which they are sold for \$1,800. The owner estimates that the variable costs of operating the cars, exclusive of gasoline, are \$0.18 per mile. The cars are leased at a flat rate of \$0.23 per mile.

    a) What is the break-even mileage for the 2–year period?
    b) What are total revenue, total cost, and total profit for the 2–year period if a car is leased for 50,000 miles?

33. In Exercise 32, it is expected that the average car will be leased for 50,000 miles during a 2–year period.

    a) What rate per mile needs to be charged in order to break even?
    b) If the dealer wishes to earn a profit of \$1,000 per car over its 2–year lifetime, what rate must be charged per mile?

CHAPTER 7

LIMITS AND CONTINUITY

7.1 Limits of Functions

In the calculas, there is often a concern about the limiting value of a function as the independent variable approaches some specific value. This limiting value, when it exists, is called a limit.

The notation used to express the limiting value of a function is

$$\boxed{\lim_{x \to a} f(x) = L}.$$

A key point with the limit concept is that we are not interested in the behavior of the values of f(x) as x comes closer and closer to a value of 'a'.

There are different procedures for determining the limit of a function. The temptation is simply to substitute the value x = a into f(x) and determine f(a). This is actually a valid way of determining the limit for many but not all functions.

One approach that can be used is to substitute values of the independent variable into the function while observing the behavior of f(x) as the value of x comes closer and closer to 'a'. An important point in this procedure is that the value of the function is observed as the value of 'a' is approached from both sides of 'a'.

Note:

$\lim_{x \to a^-} f(x) \longrightarrow$ represents the limit of f(x) as x approaches 'a' from the left. (left-hand limit) or from below.

$\lim_{x \to a^+} f(x) \longrightarrow$ represents the limit of f(x) as x approaches 'a' from the right (right-hand limit) or from above.

– If the value of the function approaches the same number L as x approaches 'a' from either direction, then the limit exists.

If $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

Then $\lim_{x \to a} f(x) = L$

Example 1:

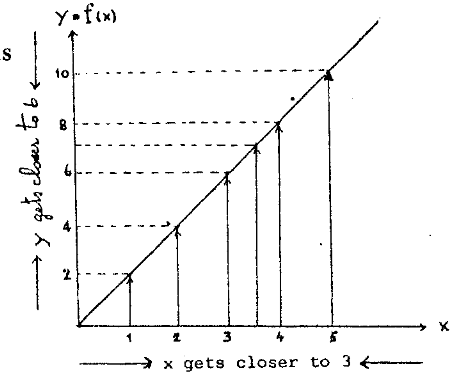
If $y = f(x) = 2x$, let us consider a specific value of x, say $x = 3$, and ask whether the function approaches a limit as x approaches 3. Intuitively, if we take x values closer and closer to 3, then we can get as closer as to a number $L = 6$, for $f(3) = 2(3) = 6$

Values of $x < 3$	Values of $y = 2x$	Values of $x > 3$	Values of $y = 2x$
1	2	5	10
2	4	4	8
2.5	5	3.5	7
2.9	5.8	3.1	6.2
2.99	5.98	3.01	6.02
2.999	5.998	3.001	6.002

The table suggests that we can get as close as we like to 6 by taking x values close enough to 3 on either side of 3 therefore, the function given by $f(x) = 2x$ has a limit of 6 as x approaches 3.

Formal way of writing is

$$\lim_{x \to 3} 2x = 6$$



If the limiting values of f(x) are different when x approaches from each direction, then the function does not approach a limit as x approaches 'a'.

Example 2:

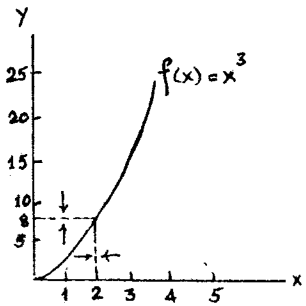
Determine $\lim_{x \to 2} x^3$

Solution:

Values of $x < 2$	Values of $y = x^3$	Values of $x > 2$	Values of $y = x^3$
1	1	3	27
1.5	3.375	2.5	15.625
1.9	6.858	2.1	9.261
1.95	7.415	2.05	8.615
1.99	7.881	2.01	8.121
1.995	7.94	2.005	8.060
1.999	7.988	2.001	8.012

The table indicates that as the value of x = 2 has been approached from both the left and the right, f(x) is approaching the same value of 8.

$$\lim_{x\to 2^-} x^3 = \lim_{x\to 2^+} x^3 = \therefore \lim_{x\to 2} x^3 = 8$$



The closer we get to a value of 2, the closer the value of f(x) comes to 8.

Note: that limit could have been determinded by simply substitution x = 2 into f(x)

Example 3:

Determine $\lim_{x\to 4} f(x)$

where $f(x) = \begin{cases} 2x & \text{where } x \leqslant 4 \\ 2x + 3 & \text{where } x > 4 \end{cases}$

Solution:

Values of x $x \leqslant 4$	Values of $y = 2x$	Values of x $x > 4$	Values of $y = 2x + 3$
3	6.0	5	13.0
3.5	7.0	4.5	12.0
3.8	7.6	4.3	11.6
3.9	7.8	4.1	11.2
3.95	7.9	4.05	11.1
3.99	7.98	4.01	11.02

This function is defined in two parts. From this table, as x approaches a value of 4 from the left, f(x) approaches a value of 8 or

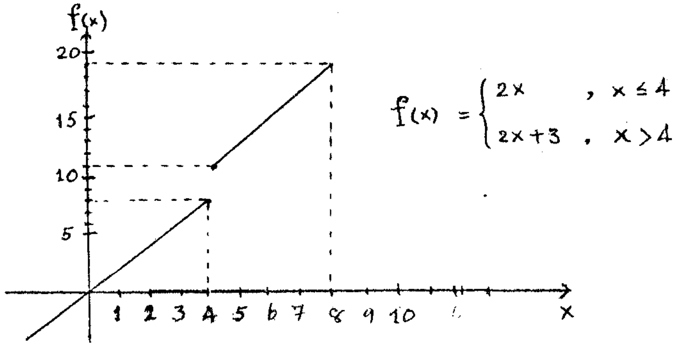
$$\lim_{x\to 4^-} f(x) = 8$$

As x approaches 4 from the right, f(x) approaches a value of 11, or

$$\lim_{x \to 4^+} f(x) = 11$$

Since $\lim_{x \to 4^-} f(x) \neq \lim_{x \to 4^+} f(x)$

the function does not approach a limiting value as $x \to 4$, and $\lim_{x \to 4} f(x)$ does not exist.



Note: the break in the function at x = 4 is the reason that the limit does not exist.

Example 4:

Determine $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

Solution:

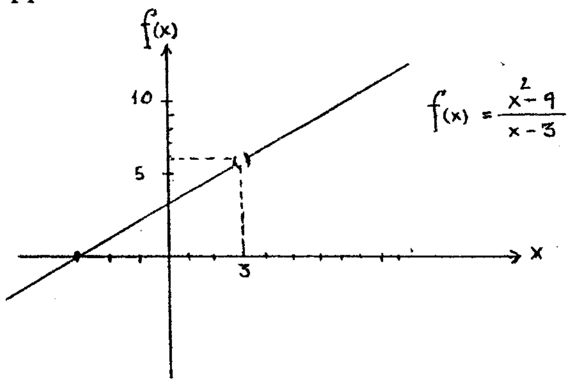
Since the denominator equals Zero when x = 3, we can conslude that the function is underfined at this point.

Values of x < 3	Values of f(x)	Values of x > 3	Values of f(x)
2	5.0	4	7.0
2.5	5.5	3.5	6.5
2.9	5.9	3.1	6.1
2.95	5.95	3.05	6.05
2.99	5.99	3.01	6.01

$\lim_{x \to 3^-} f(x) = 6$ and $\lim_{x \to 3^+} f(x) = 6$

$$\therefore \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$$

Even though the function is undefined when x = 3, the function approaches a value of 6 as the value of x comes closer to 3.



7.2 Properties of Limits:

1. If f(x) = C is a constant function, then $\lim_{x \to a} f(x) = \lim_{x \to a}(C) = C$

   Ex: $\lim_{x \to -5} 10 = 10$

2. If $f(x) = x^n$, where n is a positive integer, then $\lim_{x \to a} x^n = a^n$

   Ex: $\lim_{x \to 6} x^2 = 6^2 = 36$

3. If f(x) has a limit as $x \to a$ and C is real, then $\lim_{x \to a} C.f(x) = c.\lim_{x \to a} f(x)$

   Ex: $\lim_{x \to -2} 3x^3 = 3 \lim_{x \to -2} x^3 = 3(-2)^3 = -24$

4. If f(x) and g(x) have limits defined as $x \to a$

   $$\lim_{x \to a} \left[f(x) \pm g(x)\right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

   Ex: $\lim_{x \to 2} (x^2 + x) = \lim_{x \to 2} x^2 + \lim_{x \to 2} x$

   $= 2^2 + 2$
   $= 6$

5. If f(x) and g(x) have limits defined as $x \to a$

$$\lim_{x\to a}\left[f(x).g(x)\right] = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$$

Ex:

$$\lim_{x\to 2}\left[(x+1)(x-3)\right] = \lim_{x\to 2}(x+1).\lim_{x\to 2}(x-3)$$

$$= \left(\lim_{x\to 2} x + \lim_{x\to 2} 1\right)\left(\lim_{x\to 2} x - \lim_{x\to 2} 3\right)$$

$$= (2 + 1)(2 - 3)$$

$$= -3$$

6. If f(x) and g(x) have limits defined as $x \longrightarrow a$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{(Provided } \lim_{x\to a} g(x) \neq 0\text{)}$$

Ex:

$$\lim_{x\to 2}\frac{x^3 - 1}{x^2} = \frac{\lim_{x\to 2}(x^3 - 1)}{\lim_{x\to 2} x^2} = \frac{\lim_{x\to 2} x^3 - \lim_{x\to 2} 1}{\lim_{x\to 2} x^2}$$

$$= \frac{2^3 - 1}{2^2} = \frac{8 - 1}{4}$$

$$= \frac{7}{4}$$

7. If f(x) has a limit as $x \longrightarrow a$, and n is a positive integer, then

$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$

Ex:

a) $$\lim_{t\to 4}\sqrt{t^2 + 1} = \sqrt{\lim_{t\to 4}(t^2 + 1)} = \sqrt{\lim_{t\to 4} t^2 + \lim 1}$$

$$= \sqrt{4^2 + 1} = \sqrt{17}$$

b) $$\lim_{x\to 3}\sqrt[3]{x^2 + 7} = \sqrt[3]{\lim_{x\to 3}(x^2 + 7)} = \sqrt[3]{\lim_{x\to 3} x^2 + \lim 7}$$

$$= \sqrt[3]{3^2 + 7} = \sqrt[3]{16}$$

$$= 2\sqrt[3]{2}$$

EXERCISE: 7 - 1

In Problems **1–36**, *find the limits.*

1. $\lim_{x\to 2} 16.$

2. $\lim_{x\to 3} 2x.$

3. $\lim_{x\to 4} (x + 3).$

4. $\lim_{s\to 1} 2.$

5. $\lim_{t\to -5} (t^2 - 5).$

6. $\lim_{t\to 1/2} (3t - 5).$

7. $\lim_{x\to 0.3} (3 - 2x^2).$

8. $\lim_{x\to -3} (x^3 - 4).$

9. $\lim_{h\to 6} (h^2 - 5h - 6).$

10. $\lim_{x\to -2} (x^2 - 2x + 1).$

11. $\lim_{x\to -1} (x^3 - 3x^2 - 2x + 1).$

12. $\lim_{r\to 9} \dfrac{4r - 3}{11}.$

13. $\lim_{t\to -3} \dfrac{t - 2}{t + 5}.$

14. $\lim_{x\to -6} \dfrac{x^2 + 6}{x - 6}.$

15. $\lim_{h\to 0} \dfrac{h}{h^2 - 7h + 1}.$

16. $\lim_{h\to 0} \dfrac{h^2 - 2h - 4}{h^3 - 1}.$

17. $\lim_{p\to 4} \sqrt{p^2 + p + 5}\,.$

18. $\lim_{y\to 9} \sqrt{y + 3}\,.$

19. $\lim_{x\to -2} \sqrt{\dfrac{4x - 1}{x + 1}}\,.$

20. $\lim_{x\to -1} \sqrt[3]{x^2}\,.$

21. $\lim_{x\to 2} \dfrac{(x + 3)\sqrt{x^2 - 1}}{(x - 4)(x + 1)}.$

22. $\lim_{t\to 3} \sqrt{\dfrac{2t + 3}{3t - 5}}\,.$

23. $\lim_{t\to 1} \dfrac{t^2}{\sqrt[3]{(t^2 - 2)^2}}.$

24. $\lim_{t\to 2} \dfrac{(t + 3)(t + 7)}{(t - 1)(t + 4)}.$

25. $\lim_{x\to -1} \dfrac{x^2 + 2x + 1}{x + 1}.$

26. $\lim_{t\to 1} \dfrac{t^2 - 1}{t - 1}.$

27. $\lim_{x\to 3} \dfrac{x - 3}{x^2 - 9}.$

28. $\lim_{x\to 0} \dfrac{x^2 - 2x}{x}.$

29. $\lim_{x\to 4} \dfrac{x^2 - 9x + 20}{x^2 - 3x - 4}.$

30. $\lim_{x\to 2} \dfrac{x^2 - 2x}{x - 2}.$

31. $\lim_{x\to 1/2} \dfrac{2x^2 + 5x - 3}{4x^2 - 2x}.$

32. $\lim_{x\to -4} \dfrac{x^2 + 2x - 8}{x^2 + 5x + 4}.$

33. $\lim_{x\to 2} \dfrac{3x^2 - x - 10}{x^2 + 5x - 14}.$

34. $\lim_{x\to 0} \dfrac{(x + 2)^2 - 4}{x}.$

35. $\lim\limits_{h \to 0} \dfrac{(2+h)^2 - 2^2}{h}$.

36. $\lim\limits_{x \to a} \dfrac{x^4 - a^4}{x^2 - a^2}$.

37. Find $\lim\limits_{h \to 0} \dfrac{(x+h)^2 - x^2}{h}$ by treating x as a constant.

38. Find $\lim\limits_{h \to 0} \dfrac{2(x+h)^2 + 5(x+h) - 2x^2 - 5x}{h}$ by treating x as a constant.

39. If $f(x) = x + 5$, show that $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = 1$ by treating x as a constant.

40. If $f(x) = x^2$, show that $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = 2x$ by treating x as a constant.

7.3 Function as x approaches 0

As mentioned before, if the limiting values of f(x) are different when x approaches from each direction, then the function does not approach a limit as x approaches 'a', limits like these are called *one-sided limits.*

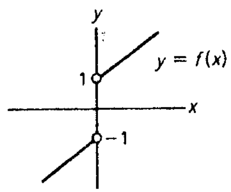
Ex: 1. *Consider the graph. As x approaches 0 from the right, f(x) approaches 1.*

$\lim\limits_{x \to 0^+} f(x) = 1$

On the other hand, as x approaches 0 from the left. f(x) approaches −1.

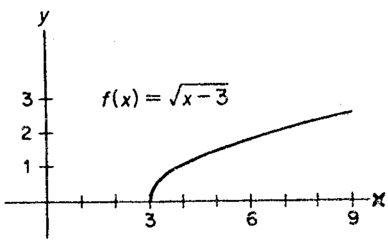
$\lim\limits_{x \to 0^-} f(x) = -1$

$\therefore$ *They are one-sided limits*



Ex: 2. *Consider $f(x) = \sqrt{x-3}$ as x approaches 3. Since f is defined only when $x > 3$, we speak of the limit as x approaches 3 from the right*

$\therefore \lim\limits_{x \to 3^+} \sqrt{x-3} = 0$



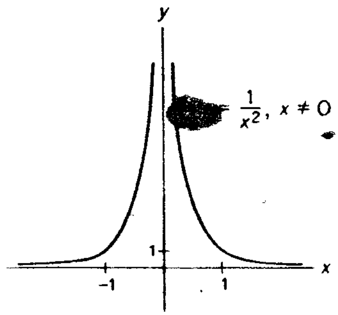
Functions where limit does not exist.

a) Function as x approaches 0 or x approaches 'a' that make the denominator of a fraction to be Zero.

1. Consider $y = f(x) = \dfrac{1}{x^2}$. As x approaches 0, both from the left and from the right, f(x) increases without bound. Hence no limit exists at 0. We say that as $x \to 0$, f(x) becomes positively infinite and symbolically we write.

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

x	$f(x)$
± 1	1
± 0.5	4
± 0.1	100
± 0.01	10,000
± 0.001	1,000,000



Note: The use of the 'equals' sign in this situation does not mean that the limit exists. On the contrary, the symbolism here (∞) is a way of saying specifically that there is no limit and it indicates why there is no limit.

2. Consider $y = f(x) = \dfrac{1}{x}$ for $x \neq 0$

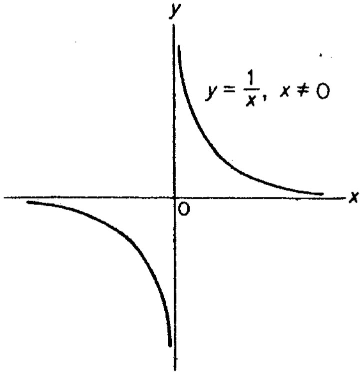
   As x approaches 0 from the right, $\frac{1}{x}$ become positively infinite; as x approaches 0 from the left, $\frac{1}{x}$ become negatively infinite. Symbolically we write,

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \qquad \text{and} \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Either one of these facts implies that

$\lim_{x \to 0} \frac{1}{x}$ does not exist

x	$f(x)$
0.01	100
0.001	1,000
0.0001	10,000
-0.01	-100
-0.001	-1,000
-0.0001	-10,000



Ex: 1. *Find* $\lim_{x \to -1^+} \frac{2}{x+1}$.

Solution:

As x approaches -1 from the right, $x + 1$ approaches 0 but is always positive. Since we are dividing 2 by positive numbers approaching 0, the results, $\frac{2}{x+1}$, are positive numbers that are becoming arbitrarily large.

$$\lim_{x \to -1^+} \frac{2}{x+1} = \infty,$$

and the limit does not exist.

Ex: 2. *Find* $\lim_{x \to 2} \frac{x+2}{x^2-4}$.

Solution:

As $x \to 2$ the numerator approaches 4 and denominator approaches 0. Thus we are dividing numbers near 4 by numbers near 0. The results are numbers that become arbitrarily large in magnitude.

$\lim_{x \to 2} \frac{x+2}{x^2-4}$ does not exist.

$$\lim_{x \to 2} \frac{x+2}{x^2-4} = \lim_{x \to 2} \frac{x+2}{(x+2)(x-2)} = \lim_{x \to 2} \frac{1}{x-2}.$$

Since $\lim_{x\to 2^+} \frac{1}{x-2} = \infty$ and $\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$,

then $\lim_{x\to 2} \frac{x+2}{x^2-4}$ is neither ∞ nor $-\infty$.

Ex: 3. $\lim_{t\to 2} \frac{t-2}{t^2-4}$.

Solution:

As $t \to 2$ both numerator and denominator approaches 0. Thus we first simplify the fraction.

$$\therefore \lim_{t\to 2} \frac{t-2}{t^2-4} = \lim_{t\to 2} \frac{t-2}{(t+2)(t-2)} = \lim_{t\to 2} \frac{1}{t+2} = \frac{1}{4}.$$

b) Function as x approaches ∞:

In a positive sense:
As x increases without bound through positive values, the values of f(x) approaches 0.

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

x	$f(x)$
1,000	.001
10,000	.0001
100,000	.00001
1,000,000	.000001

x	$f(x)$
−1,000	−.001
−10,000	−.0001
−100,000	−.00001
−1,000,000	−.000001

In a negative sense, as x decreases without bound through negative values, the values of f(x) also approaches 0.

$$\lim_{x\to -\infty} \frac{1}{x} = 0.$$

Ex: 1. *Find* $\lim_{x\to\infty} \frac{4}{(x-5)^3}$.

Solution:

As x becomes very large, so does (x − 5). Since the cube of a large number is also large, $(x-5)^3$ dividing 4 by very large numbers results in numbers near 0.

Thus $\lim_{x\to\infty} \frac{4}{(x-5)^3} = 0.$

c) Quotient of two polynomials where $x \to \infty$

1. Consider $\lim_{x\to\infty} \frac{8x^2 + 2x + 3}{2x^3 + 3x - 1}$.

As $x \to \infty$, both numerator and denominator become infinite. In such a case, divide both the numerator and denominator by the largest power of x to change the form of the quotient.

$$\text{Then } \lim_{x\to\infty} \frac{8x^2 + 2x + 3}{2x^3 + 3x - 1} = \lim_{x\to\infty} \frac{\dfrac{8x^2 + 2x + 3}{x^3}}{\dfrac{2x^3 + 3x - 1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\dfrac{8}{x} + \dfrac{2}{x^2} + \dfrac{3}{x^3}}{2 + \dfrac{3}{x^2} - \dfrac{1}{x^3}}$$

$$= \frac{0}{2} \quad = 0.$$

2. *Find* $\lim_{x\to\infty} \frac{2x + 5}{3x + 2}$

Solution:

$$\lim_{x\to\infty} \frac{2x + 5}{3x + 2} = \lim_{x\to\infty} \frac{\dfrac{2x + 5}{x}}{\dfrac{3x + 2}{x}}$$

$$= \lim_{x\to\infty} \frac{2 + \dfrac{5}{x}}{3 + \dfrac{2}{x}}$$

$$= \frac{2}{3}.$$

3. *Find* $\lim_{x \to -\infty} \frac{x^2 - 5x}{x^4 + 2x^2 + 1}$

Solution:

$$\lim_{x \to -\infty} \frac{x^2 - 5x}{x^4 + 2x^2 + 1} = \lim_{x \to -\infty} \frac{\frac{x^2 - 5x}{x^4}}{\frac{x^4 + 2x^2 + 1}{x^4}}$$

$$= \lim_{x \to -\infty} \frac{\frac{1}{x^2} - \frac{5}{x^3}}{1 + \frac{2}{x^2} + \frac{1}{x^4}}$$

$$= \frac{0}{1} \qquad = 0.$$

4. *Find* $\lim_{x \to -\infty} \frac{10x^2}{x}$.

Solution:

$$\lim_{x \to -\infty} \frac{10x^2}{x} = \lim_{x \to -\infty} 10x$$

$$= -\infty$$

and no limit exists.

'How to find limits' in Summary:

1. By substitution: $\lim_{x \to a} f(x) = f(a)$

   It is used commonly with polynomial functions

Ex: $\lim_{x \to -2} (3x^2 - 4x + 10) = 3(-2)^2 - 4(-2) + 10$

$$= 12 + 8 + 10$$

$$= 30$$

2. When substitution fails and after substitution both the numerator and denominator approach 0 as $x \to a$, then use algebraic manupulation on the original function to form a new function.

Ex: a) $\lim\limits_{h\to 0} \dfrac{(2+h)^2 - 4}{h} = \lim\limits_{h\to 0} \dfrac{(4+4h+h^2) - 4}{h}$

$$= \lim_{h\to 0} \frac{(4h+h^2)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(4+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} (4+h)$$

$$= 4$$

b) $\lim\limits_{x\to -1} \dfrac{x^2 - 1}{x+1} = \lim\limits_{x\to -1} \dfrac{\cancel{(x+1)}(x-1)}{\cancel{x+1}}$

$$= \lim_{x\to -1} (x-1)$$

$$= -2$$

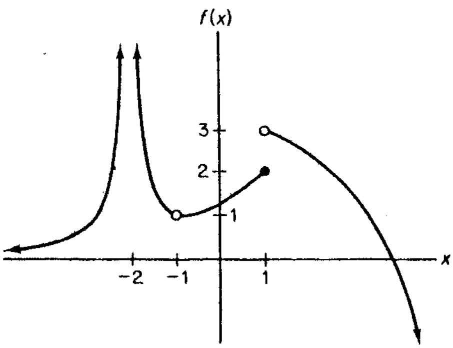
3. When f(x) is in the form of fractions and $x\to\infty$. After substitution both numerator and denominator become Zero, then divide both the numerator and denominator by the highest power of x.

EXERCISE: 7 - 2a

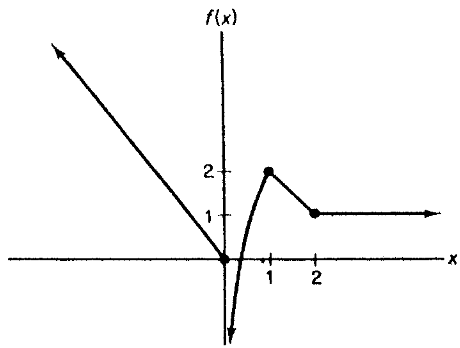
1. For the function f given in Fig.(a), find the following limits. If the limit does not exist, so state or use the symbol ∞ or $-\infty$ where approapriate.

(a) $\lim\limits_{x\to 1^-} f(x)$, (b) $\lim\limits_{x\to 1^+} f(x)$, (c) $\lim\limits_{x\to 1} f(x)$,

(d) $\lim\limits_{x\to \infty} f(x)$, (e) $\lim\limits_{x\to -2^-} f(x)$, (f) $\lim\limits_{x\to -2^+} f(x)$,



(a)



(b)

(g) $\lim_{x\to -2} f(x)$, (h) $\lim_{x\to -\infty} f(x)$, (i) $\lim_{x\to -1^-} f(x)$,

(j) $\lim_{x\to -1^+} f(x)$, (k) $\lim_{x\to -1} f(x)$.

2. For the function f given in Fig.(b) above find the following limits. If the limit does not exist, so state or use the symbol ∞ or $-\infty$ where appropriate.

(a) $\lim_{x\to 0^-} f(x)$, (b) $\lim_{x\to 0^+} f(x)$, (c) $\lim_{x\to 0} f(x)$,

(d) $\lim_{x\to -\infty} f(x)$, (e) $\lim_{x\to 1} f(x)$, (f) $\lim_{x\to 2^-} f(x)$,

(g) $\lim_{x\to 2^+} f(x)$, (h) $\lim_{x\to \infty} f(x)$.

In each of Problems 3-46, find the limit. If the limit does not exist, so state or use the symbol ∞ or $-\infty$ where approapriate.

3. $\lim_{x\to 3^+} (x - 2)$.

4. $\lim_{x\to -1^-} (1 - x^2)$.

5. $\lim_{x\to -\infty} 5x$.

6. $\lim_{x\to \infty} 3$.

7. $\lim_{x\to 0^-} \dfrac{6x}{x^4}$.

8. $\lim_{x\to 0} \dfrac{5}{x - 1}$.

9. $\lim_{x\to -\infty} x^2$.

10. $\lim_{t\to \infty} (t - 1)^3$.

11. $\lim_{h\to 0^+} \sqrt{h}$.

12. $\lim_{h\to 5^-} \sqrt{5 - h}$.

13. $\lim_{x\to 5} \dfrac{3}{x - 5}$.

14. $\lim_{x\to 0^-} 2^{1/2}$.

15. $\lim_{x\to 1^+} (4\sqrt{x - 1})$.

16. $\lim_{x\to 2^+} (x\sqrt{x^2 - 4})$.

17. $\lim_{x\to \infty} \dfrac{7}{2x + 1}$.

18. $\lim_{x\to -\infty} \dfrac{1}{(4x - 1)^3}$.

19. $\lim_{x\to \infty} \dfrac{x + 2}{x + 3}$.

20. $\lim_{x\to \infty} \dfrac{2x - 4}{3 - 2x}$.

21. $\lim_{x\to -\infty} \dfrac{x^2 - 1}{x^3 + 4x - 3}$.

22. $\lim_{r\to \infty} \dfrac{r^3}{r^2 + 1}$.

23. $\lim_{t\to \infty} \dfrac{5t^2 + 2t + 1}{4t + 7}$.

24. $\lim_{x\to -\infty} \dfrac{2x}{3x^6 - x + 4}$.

25. $\lim_{x\to \infty} \dfrac{3 - 4x - 2x^3}{5x^3 - 8x + 1}$.

26. $\lim_{x\to \infty} \dfrac{7 - 2x - x^4}{9 - 3x^4 + 2x^2}$.

27. $\lim_{x\to 3^-} \dfrac{x + 3}{x^2 - 9}$.

28. $\lim_{x\to -2^+} \dfrac{2x}{4 - x^2}$.

29. $\lim_{w\to\infty} \dfrac{2w^2 - 3w + 4}{5w^2 + 7w - 1}$.

30. $\lim_{x\to\infty} \dfrac{4 - 3x^3}{x^3 - 1}$.

31. $\lim_{x\to -5} \dfrac{2x^2 + 9x - 5}{x^2 + 5x}$.

32. $\lim_{t\to 2} \dfrac{t^2 + 2t - 8}{2t^2 - 5t + 2}$.

33. $\lim_{x\to 1} \dfrac{x^2 - 3x + 1}{x^2 + 1}$.

34. $\lim_{x\to -1} \dfrac{3x^3 - x^2}{2x + 1}$.

35. $\lim_{x\to 1^+} \left[1 + \dfrac{1}{x - 1}\right]$.

36. $\lim_{x\to -\infty} \dfrac{x^3 + 2x^2 + 1}{x^3 - 4}$.

37. $\lim_{x\to 0^+} \dfrac{2}{x + x^2}$.

38. $\lim_{x\to\infty} \left(x + \dfrac{1}{x}\right)$.

39. $\lim_{x\to 1} x(x - 1)^{-1}$.

40. $\lim_{x\to 1/2} \dfrac{1}{2x - 1}$.

41. $\lim_{x\to 0^+} \left(-\dfrac{3}{x}\right)$.

42. $\lim_{x\to 0} \left(-\dfrac{3}{x}\right)$.

43. $\lim_{x\to 0} |x|$.

44. $\lim_{x\to 0} \left|\dfrac{1}{x}\right|$.

45. $\lim_{x\to -\infty} \dfrac{x + 1}{x}$.

46. $\lim_{x\to\infty} \left[\dfrac{2}{x} - \dfrac{x^2}{x^2 - 1}\right]$.

In Problems 47-50, sketch the graphs of the functions and find the indicated limits. If the limit does not exist, so state or use the symbol ∞ or $-\infty$ where appropriate.

47. $f(x) = \begin{cases} 2, & \text{if } x < 2 \\ 1, & \text{if } x > 2 \end{cases}$; (a) $\lim_{x\to 2^+} f(x)$, (b) $\lim_{x\to 2^-} f(x)$, (c) $\lim_{x\to 2} f(x)$,

(d) $\lim_{x\to\infty} f(x)$, (e) $\lim_{x\to -\infty} f(x)$.

48. $f(x) = \begin{cases} x, & \text{if } x < 1 \\ 2, & \text{if } x > 1 \end{cases}$; (a) $\lim_{x\to 1^+} f(x)$, (b) $\lim_{x\to 1^-} f(x)$, (c) $\lim_{x\to 1} f(x)$,

(d) $\lim_{x\to\infty} f(x)$, (e) $\lim_{x\to -\infty} f(x)$.

49. $g(x) = \begin{cases} x, & \text{if } x < 0 \\ -x, & \text{if } x > 0 \end{cases}$; (a) $\lim_{x\to 0^+} g(x)$ (b) $\lim_{x\to 0^-} g(x)$, (c) $\lim_{x\to 0} g(x)$,

(d) $\lim_{x\to\infty} g(x)$, (e) $\lim_{x\to -\infty} g(x)$.

50. $g(x) = \begin{cases} x^2, & \text{if } x < 0 \\ x, & \text{if } x > 0 \end{cases}$; (a) $\lim_{x\to 0^+} g(x)$, (b) $\lim_{x\to 0^-} g(x)$, (c) $\lim_{x\to 0} g(x)$,

(d) $\lim_{x\to\infty} g(x)$, (e) $\lim_{x\to -\infty} g(x)$.

51. If c is the total cost in dollars to produce q units of a product, then the average cost per unit $\bar{c}$ for an output of q units is given by $\bar{c} = c/q$. Thus, if the total cost equation is $c = 5000 + 6q$, then $\bar{c} = (5000/q) + 6$. For example, the total cost of an output of 5 units is \$5030, and the average cost per unit at this level of production is \$1600. By finding $\lim_{q\to\infty} \bar{c}$, show that the average cost approaches a level of stability if the producer continually increases output. What is the limiting value of the average cost? Sketch the graph of the average cost function.

52. Repeat Problem 51 given that fixed cost is \$12,000 and the variable cost is given by the function $c_v = 7q$.

53. The population N of a certain small city t years from now is predicted to be

$$N = 20{,}000 + \frac{10{,}000}{(t + 2)^2}$$

Determine the population in the long run; that is, find $\lim_{t\to\infty} N$.

In Problems 54-57, find $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ by treating x as a constant.

54. $f(x) = 2x + 3$.

55. $f(x) = 4 - x$.

56. $f(x) = x^2 + x + 1$.

57. $f(x) = x^2 - 3$.

(Calculator Problems) In Problems 58 and 59, evaluate the given function when x = 1, .5, .2, .1, .01, .001, and .0001. From your results draw a conclusion about $\lim_{x\to 0} f(x)$.

58. $f(x) = x \ln x$.

59. $f(x) = x^{2x}$.

EXERCISE: 7 - 2b

For the following exercises, find the indicated limit.

1. $\lim_{x\to 0} (4x^3 - x)$

2. $\lim_{x\to -1} (3x^2 - 2x + 1)$

3. $\lim_{x\to -3} \left(\frac{x^2}{4} - \frac{x}{2} + 10\right)$

4. $\lim_{x\to 2} \frac{x - 2}{x + 6}$

5. $\lim_{x\to 4} \frac{x^2 - 5}{1 - x}$

6. $\lim_{x\to -2} (x - 4)(x^2 + 3x)$

7. $\lim_{x \to 0} (3 - x^2)(5x + 6)$

8. $\lim_{x \to 3} (-12)$

9. $\lim_{x \to 0} 250$

10. $\lim_{x \to 6} \dfrac{x^2 - 36}{x - 6}$

11. $\lim_{x \to -2} \dfrac{x^2 - 4}{x + 2}$

12. $\lim_{x \to -2} \dfrac{3x^2 + x - 10}{x + 2}$

13. $\lim_{x \to 2} \dfrac{3x^2 + x - 14}{x - 2}$

14. $\lim_{x \to a} (x^3 - 2x)$

15. $\lim_{x \to -b} (x^2 - 2x + 1)$

16. $\lim_{x \to \infty} (1/x)$

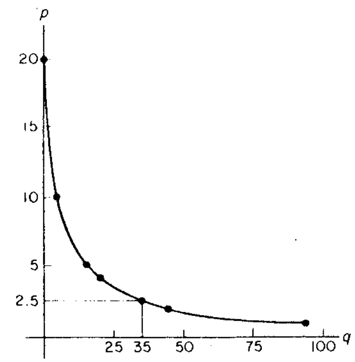
17. $\lim_{x \to -\infty} \dfrac{2x + 1}{2x}$

18. $\lim_{x \to \infty} \dfrac{x}{2x + 1}$

7.4 Continuity:

Often it is helpful to describe a situation by a contipous function.

Consider the Demand Schedule Table



Demand Schedule

PRICE/UNIT (DOLLARS) p	QUANTITY PER WEEK q
20	0
10	5
5	15
4	20
2	45
1	95

It indicates the number of units of a particular product that consumers will demand per week at various prices. This information can be given graphically by plotting each quantity-price pair as a point as shown. This graph does not represent a continuous function. Furthermore, it gives us no information as to the price at which, say, 35 units would be demanded. However, if we connect the points by a smooth curve, we get a so-called demand curve. From it we could guess that at about ฿50.00 per unit, 35 units would be demanded.

Frequently it is possible and useful to describe a graph by means of an explicit equation that defines a continuous function f. Such a function permits a convenient mathematical analysis of the nature and basic properties of the problem.

In general, it will be our desire to view practical situations in terms of continuous functions wherever possible so that we may be better able to analyze their nature.

In an informal sense, a function is described as continuous if it can be sketched without lifting your pen or pencil from the paper. Most of the functions that we will examine in the calculus will be continuous functions. A function which is not continuous is termed *discontinuous.*

A more formal definition is as follows:

I. Definition: A function f is continuous at $x = a$ if and only if the following three conditions are set:

(1) $f(x)$ is defined at $x = a$,

(2) $\lim_{x \to a} f(x)$ exists, and

(3) $\lim_{x \to a} f(x) = f(a)$

Conversely, a function is discontionuous at $x = a$ if and only if it is not continuous at $x = a$.

Ex: 1. Show that $f(x) = 5$ is continuous at $x = 7$

Solution:

Verify the three conditions.

(i) f is defined at $x = 7$

$$f(x) = 5$$

$$f(7) = 5$$

(ii) $\lim_{x\to 7} f(x)$ exists

$$\lim_{x\to 7} f(x) = \lim_{x\to 7} 5 = 5$$

(iii) $\lim_{x\to 7} f(x) = f(7)$

$$\lim_{x\to 7} f(x) = 5$$

and $f(7) = 5$

Therefore $f(x) = 5$ is continuous at $x = 7$ Ans

Ex: 2. Show that $g(x) = x^2 - 3$ is continuous at $x = -4$

Solution:

Verify the three conditions.

(i) Function g is defined at $x = -4$

$$g(x) = x^2 - 3$$
$$g(-4) = (-4)^2 - 3$$
$$= 13$$

(ii) $\lim_{x\to -4} g(x)$ exists

$$\lim_{x\to -4} g(x) = \lim_{x\to -4} (x^2 - 3)$$
$$= (-4)^2 - 3$$
$$= 13$$

(iii) $\lim_{x\to -4} g(x) = g(-4)$ as shown above

Therefore $g(x) = x^2 - 3$ is continuous at $x = -4$ Ans

Ex: 3. Find any points of discontinulity for

$$f(x) = \frac{x^4 - 3x^3 + 2x - 1}{x^2 - 4}$$

Solution:

The denominator is Zero when $x = \pm 2$

$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

Hence f is not defined at $x = \pm 2$ and is therefore discontinuous at these points. Otherwise the function is 'well-behaved' Ans

Ex: 4. Determine whether there are any discontinuities for the function $g(x) = \begin{cases} x+6, & \text{if } x \geqslant 3 \\ x^2, & \text{if } x < 3 \end{cases}$

Solution:

With such a problem, the only point of discontinuity can occur only at $x = 3$

as $x \longrightarrow 3^+$ then, $g(x) = x + 6$

$g(3) = 3 + 6$

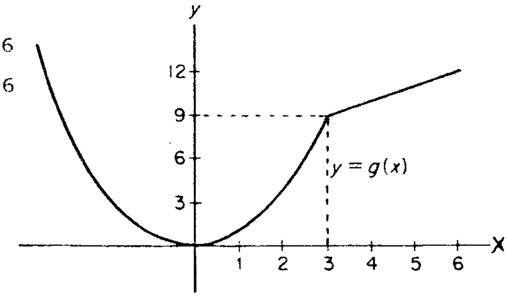
$= 9$

and as $x \longrightarrow 3^-$ then,

$g(x) = x^2$

$g(3) = 3^2$

$= 9$



Thus the function is continuous at $x = 3$ as well as at all other x

Ans

Ex: 5. Is this function $f(x) = \begin{cases} x + 2, & \text{if } x > 2 \\ x^2, & \text{if } x < 2 \end{cases}$ Continuous?

Solution:

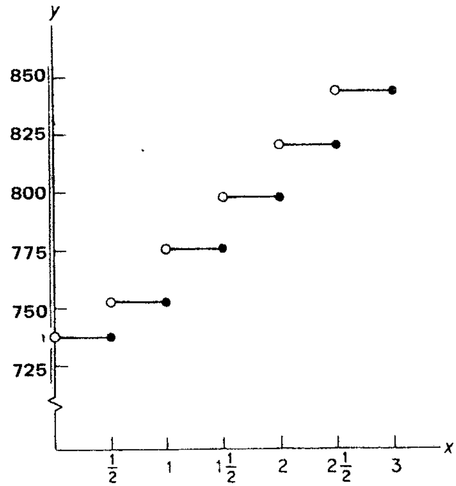
Since f(x) is not defined at $x = 2$, it is discontinuous at this point. Ans

Ex: 6. The table shows the redemptive values of ₱50x20 savings bond for the first six successive periods after the date of issue. Determine its continuity.

NUMBER OF YEARS AFTER ISSUE DATE (greater than)		(not more than)	REDEMPTION VALUE
0	–	$\frac{1}{2}$	\$37.50x20
$\frac{1}{2}$	–	1	38.10x20
1	–	$1\frac{1}{2}$	39.02x20
$1\frac{1}{2}$	–	2	39.90x20
2	–	$2\frac{1}{2}$	40.80x20
$2\frac{1}{2}$	–	3	41.76x20

Solution:

Since this is a step function it is clear then that f has discontinuities when x = ½, 1, 1½, 2 and 2½.



Remark:

A function given by f(x) is defined at the point x = a, if f(a) exists as a real number

Ex: 1.

$$f(x) = 2x \quad \text{at } x = 3$$
$$f(x) = 2(3)$$
$$= 6$$

The function is defined at the point x = 3

Ex: 2.

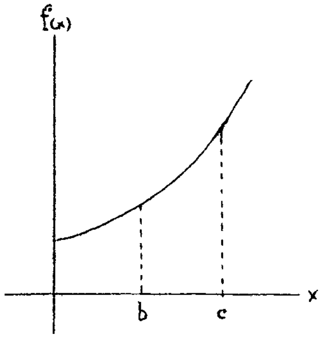
$$f(x) = \frac{2x^2}{x} \quad \text{at } x = 0$$

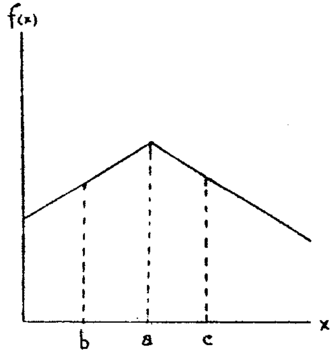
$$f(0) = \frac{2(0)^2}{0}$$

$$= \text{undefined}$$

The function is undefined at x = 0

2. **Continuity over an interval:** This is an extension of the 'continuity at a point'. A function is continuous in an interval from x = b to x = c if the function has no breaks or jumbs in the interval as shown in the figures.





Example 1:

Determine whether there are any discontinuities for the function.

$$f(x) = \frac{1}{x^3 - x}$$

Solution:

This function is not defined when

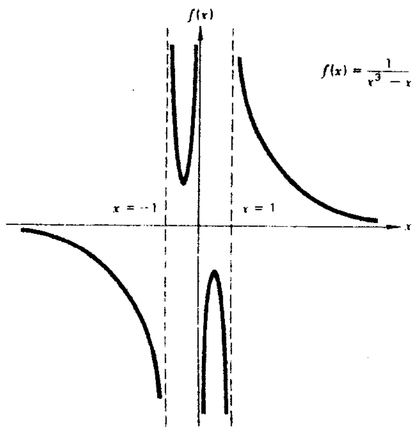
$$x^3 - x = 0$$

$$\text{or} \quad x(x^2 - 1) = 0$$

$$\text{or} \quad (x + 1)(x - 1) = 0$$

or when $x = 0$, $x = -1$, and $x = +1$.

Thus, the function is discontinuous at these three points.



Note:

1. A polynomial function is continuous at every point.

2. Discontinuities tend to arise at special points, and usually it is sufficient to check the function's behavior at these points.

   Several examples of such points are:-

   a) Where the denominator of some term in the equation becomes zero.

   b) At the end points of the function or of the separate intervals for which different equations are used to define the function.

EXERCISE: 7 - 3a

In the following exercises, determine whether there are any discontinuities and, if so, where they occur.

1. $f(x) = 3x^2 - 2x + 10$
2. $f(x) = 1/x$
3. $f(x) = (2x + 6)(x - 5)$
4. $f(x) = x^7$
5. $f(x) = \dfrac{x}{x - 3}$
6. $f(x) = \dfrac{3}{6 - x}$
7. $f(x) = |x|$
8. $g(x) = |-x|$
9. $h(x) = \dfrac{5}{-2x^2 + 9x - 9}$
10. $v(x) = \dfrac{2x - 1}{x^2 - 2x + 1}$
11. $g(x) = \dfrac{4 - x}{3x^3 - 27x}$
12. $h(x) = \dfrac{5}{2x^3 - 2x}$

EXERCISE: 7 - 3b

In Problems 1-6, use the definition of continuity to show that the given function is continuous at the indicated point.

1. $f(x) = x^3 - 5x, x = 2.$
2. $f(x) = \dfrac{x - 3}{9x}, x = -3.$
3. $g(x) = \sqrt{2 - 3x}\ , x = 0.$
4. $f(x) = \frac{1}{8}, x = 2.$
5. $h(x) = \dfrac{x - 4}{x + 4}, x = 4.$
6. $f(x) = \sqrt[3]{x}\ , x = -1.$

In Problems 7-12, determine whether the function is continuous at the given points.

7. $f(x) = \dfrac{x+4}{x-2}$; $-2, 0$.

8. $f(x) = \dfrac{x^2 - 4x + 4}{6}$; $2, -2$.

9. $g(x) = \dfrac{x-3}{x^2-9}$; $3, -3$.

10. $h(x) = \dfrac{3}{x^2+4}$; $2, -2$.

11. $F(x) = \begin{cases} x+2, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases}$; $2, 0$.

12. $f(x) = \begin{cases} \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$; $0, -1$.

In Problems 13-16, state why the functions are continuous everywhere.

13. $f(x) = 2x^2 - 3$.

14. $f(x) = \dfrac{x+2}{5}$.

15. $f(x) = \dfrac{x-1}{x^2+4}$.

16. $f(x) = x(1-x)$.

In Problems 17-34, find all points of discontinuity.

17. $f(x) = 3x^2 - 3$.

18. $h(x) = x - 2$.

19. $f(x) = \dfrac{3}{x-4}$.

20. $f(x) = \dfrac{x^2+3x-4}{x+4}$.

21. $g(x) = \dfrac{(x^2-1)^2}{5}$.

22. $f(x) = \begin{cases} 5, & \text{if } x \geq 3, \\ 2x-1, & \text{if } x < 3. \end{cases}$

23. $f(x) = \dfrac{x^2+6x+9}{x^2+2x-15}$.

24. $g(x) = \dfrac{x-3}{x^2+x}$.

25. $h(x) = \dfrac{x-7}{x^3-x}$.

26. $f(x) = \dfrac{x}{x}$.

27. $p(x) = \dfrac{x}{x^2+1}$.

28. $f(x) = \dfrac{x^4}{x^4-1}$.

29. $f(x) = \begin{cases} x^2, & \text{if } x > 2, \\ x-1, & \text{if } x < 2. \end{cases}$

30. $f(x) = \begin{cases} \dfrac{1}{x}, & \text{if } x \neq 3, \\ 5, & \text{if } x = 3. \end{cases}$

31. $f(x) = \begin{cases} \dfrac{1}{x-3}, & \text{if } x \geq 4, \\ 5-x, & \text{if } x < 4. \end{cases}$

32. $f(x) = \begin{cases} 10x-3, & \text{if } x \geq 1, \\ \dfrac{1}{x+1}, & \text{if } x < 1. \end{cases}$

33. $f(x) = \begin{cases} \dfrac{-3}{x-2}, & \text{if } x > 0, \\ 4-x, & \text{if } x \leq 0. \end{cases}$

34. $f(x) = \dfrac{5x+2}{3} - \dfrac{7}{x}$.

35. Suppose the long distance rate for a telephone call from Hazleton, Pa. to Washington, D.C. is \$1.85 for the first three minutes and \$0.30 for each additional minute or fraction thereof. If $y = f(t)$ is a function that indicates the total charge y for a call of t minutes' duration, sketch the graph of f for $0 < t \leqslant 6$. Use your graph to determine the values of t at which discontinuities occur.

36. The greatest integer function, $f(x) = [x]$, is defined to be the greatest integer less than or equal to x, where x is any real number. For example, $[3] = 3$, $[1.999] = 1$, $[\frac{1}{4}] = 0$, and $[-4.5] = -5$. Sketch the graph of this function for $-3.5 \leqslant x \leqslant 3.5$. Use your sketch to determine the values of x at which discontinuities occur.

37. Sketch the graph of

$$y = f(x) = \begin{cases} -100x + 600, & \text{if } 0 \leqslant x < 5, \\ -100x + 1100, & \text{if } 5 \leqslant x < 10, \\ -100x + 1600, & \text{if } 10 \leqslant x < 15. \end{cases}$$

A function such as this might describe the inventory y of a company at time x.

38. Sketch the 'post-office function'

$$c = f(x) = \begin{cases} 15, & \text{if } 0 < x \leqslant 1, \\ 28, & \text{if } 1 < x \leqslant 2, \\ 41, & \text{if } 2 < x \leqslant 3, \\ & \text{etc.,} \end{cases}$$

for $0 < x \leqslant 6$, Here c is the cost (in cents) of sending a parcel of weight x (ounces) in January 1980. Where do discontinuities occur?

CHAPTER 8

THE DERIVATIVE

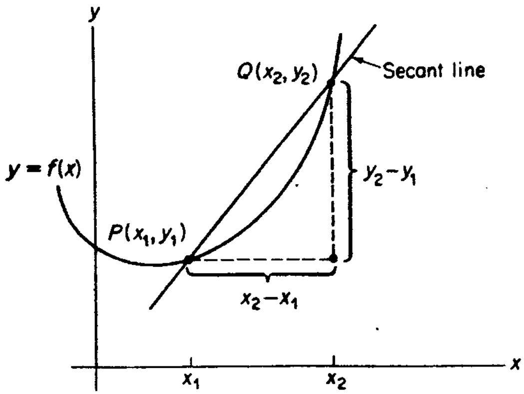
What do we mean by differentiating a function?

The objective of this chapter is not only to convey an understanding of what the so-called 'derivative' of a function is, but also to teach techniques of finding derivatives by properly applying rules.

8.1 The Derivative:

One of the main problems with which calculus deals is finding the slope of the *tangent line* on a curve.

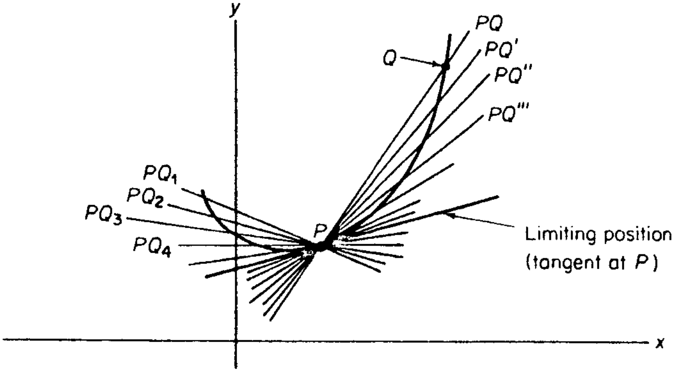
Consider the graph of the function y = f(x) in the figure below.



Here $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two different points on the curve. The line PQ passing through them is called a secant line. By the slope formula, the slope of PQ is

$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1}.$$

If Q moves along the curve and approaches P, the secant line has a limiting position as shown in the figure.



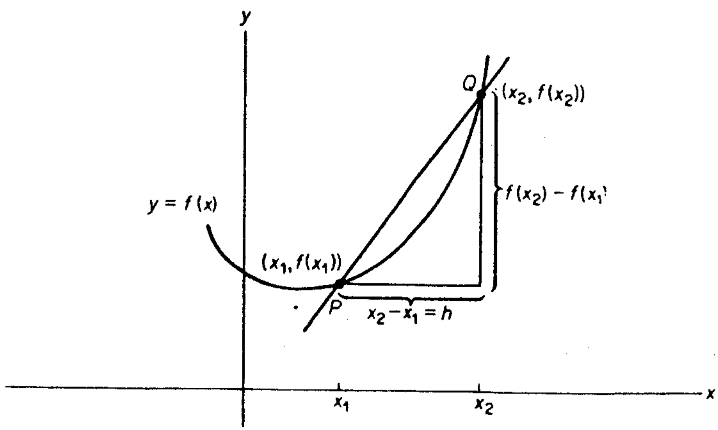
As Q approaches P from the right, the positions of the secant lines are PQ', PQ'', etc. As Q approaches P' from the left, they are PQ_1, PQ_2, etc. In both cases, the same limiting position is obtained. This common limiting position of the secant lines is called the tangent line to the curve at P.

Definition:

The slope of a curve at a point P is the slope of the tangent line at P.

Since the tangent is a limiting position of secant lines, the slope of the tangent is the limiting value of the slopes of the secant lines PQ as Q approaches P. The slope of the secant line PQ as shown in the figure is

$$m_{PQ} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$



If the difference $x_2 - x_1$ is h, then $x_2 = x_1 + h$

$$\therefore m_{PQ} = \frac{f(x_1 + h) - f(x_1)}{(x_1 + h) - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}.$$

As Q moves along the curve toward P, then $x_2 \longrightarrow x_1$. This means that h is getting closer to zero. The limiting value of the slopes of the secant line — which is the slope of the tangent line at $[x_1, f(x_1)]$ — is the limit:

$$\boxed{\lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}.} \quad \text{———— (1)}$$

Example 1:

Find the slope of the curve $y = f(x) = x^2$ at the point (1, 1)

Solution:

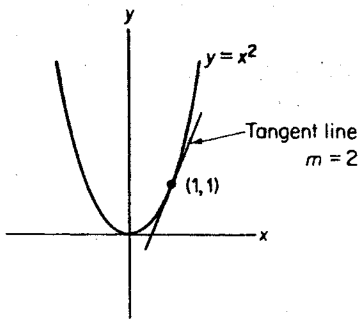
$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

at (1, 1) $$= \lim_{h \to 0} \frac{(1 + h)^2 - (1)^2}{h}$$

$$= \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h}$$

$$= \lim_{h \to 0} \frac{h(2 + h)}{h}$$

$$= \lim_{h \to 0} (2 + h) = 2.$$



Thus the tangent line to $y = x^2$ at (1, 1) has a slope of 2. Ans

We can generalize the result in (1) to any point $[x, f(x)]$ on the curve by replacing x_1 by x. We thus have the following definition, which forms the basis of differential calculus.

Definition:

If $y = f(x)$ defines a function f, the limit

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

if it exists, is called the derivative of f at x and is denoted $f'(x)$, which is read "f prime of x". The process of finding the derivative is called differentiation.

Example 2:

If $f(x) = x^2$, find the derivative of f

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} (2x + h) = 2x.$$

Ans

Other ways of denoting the derivative of y = f(x) at x

$\frac{dy}{dx}$ (pronounced "dee y, dee x"),

$\frac{d}{dx}[f(x)]$ (dee $f(x)$, dee x),

y' (y prime),

$D_x y$ (dee x of y),

$D_x[f(x)]$ (dee x of $f(x)$).

Note:

1. In all cases, the derivative of a function f is also a function, f'.

2. $\frac{dy}{dx}$ is not a fraction, but is a symbol for a derivative. There is no meaning attached to individual symbols such as dy and dx.

3. The derivative of y = f(x) at x is also referred as the derivative with respect to x.

4. If the derivative of f can be evaluated at $x = x_1$, the resulting number is called the derivative of f at x_1, denoted $f'(x_1)$. In this case we say that f is differentiable at x_1.

$$\boxed{f'(x_1) \text{ is the slope of the tangent to } y = f(x) \text{ at } (x_1, f(x_1)).}$$

5. In addition to the notation $f'(x_1)$, we can also write

$$\left.\frac{dy}{dx}\right|_{x=x_1} \quad \text{and} \quad y'(x_1).$$

6. The derivative must be evaluated at the point of tangency to determine the slope of the tangent line.

Example 3:

If $f(x) = 2x^2 + 2x + 3$, *find* $f'(1)$.

We shall find the derivative $f'(x)$ and then evaluate it at $x = 1$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left[2(x+h)^2 + 2(x+h) + 3\right] - (2x^2 + 2x + 3)}{h}$$

$$= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 + 2x + 2h + 3 - 2x^2 - 2x - 3}{h}$$

$$= \lim_{h \to 0} \frac{4xh + 2h^2 + 2h}{h} = \lim_{h \to 0} (4x + 2h + 2).$$

$$f'(x) = 4x + 2.$$

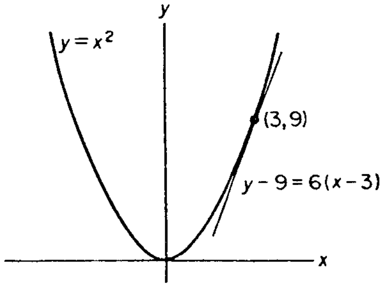
$$f'(1) = 4(1) + 2 = 6.$$

<u>Ans</u>

Example 4:

a. Find the slope of the curve $y = x^2$ at the point (3, 9). Then find an equation of the tangent line at (3, 9).

From Example 2, $y' = 2x$. Thus, $y'(3) = 2(3) = 6$. That is, the tangent line to the curve $y = x^2$ at (3, 9) has a slope of 6. A point-slope form of the tangent line is $y - 9 = 6(x-3)$, from which $y = 6x-9$.



b. Find the slope of the curve $y = 2x+3$ at the point where $x = 6$.

Letting $y = f(x) = 2x + 3$, we have

$$y' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[2(x+h)+3] - (2x+3)}{h}$$

$$= \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2 = 2.$$

Since $y' = 2$, the slope when $x = 6$, or at any point, is 2. Note that the curve is a straight line and thus has the same slope at each point.

Example 5:

Find $D_x(\sqrt{x})$.

If $f(x) = \sqrt{x}$, then

$$D_x(\sqrt{x}) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}.$$

As $h \longrightarrow 0$, both the numerator and denominator approach zero. This can be avoided by rationalizing the numerator.

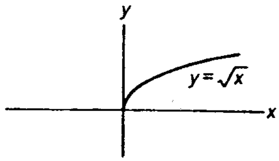
$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{\sqrt{x+h}+\sqrt{x}}.$$

Thus,

$$D_x(\sqrt{x}) = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Note that the original function, $\sqrt{x}$, is defined for $x \geqslant 0$. But the derivative, $1/(2\sqrt{x})$, is defined only when $x > 0$. From the graph of $y = \sqrt{x}$, it is clear that when $x = 0$, the tangent is a vertical line and hence does not have a slope.



If a variable, say q, is a function of some variable, say p, then we would speak of the derivative of q with respect to p and could write dq/dp.

Example 6:

If $q = f(p) = \dfrac{1}{2p}$, *find* dq/dp.

$$\frac{dq}{dp} = \lim_{h\to 0} \frac{f(p+h)-f(p)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1}{2(p+h)} - \dfrac{1}{2p}}{h} = \lim_{h\to 0} \frac{p-(p+h)}{h[2p(p+h)]}$$

$$= \lim_{h\to 0} \frac{-h}{h[2p(p+h)]} = \lim_{h\to 0} \frac{-1}{2p(p+h)} = -\frac{1}{2p^2}.$$

Note that when $p = 0$, neither the function nor its derivative exists.

As a final note we point out that the derivative of y = f(x) at x, namely f'(x), is nothing more than the limit.

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

However, we can interpret f' as a function that gives the slope of the tangent line to the curve y = f(x) at the point (x, f(x)). This interpretation is simply a geometric convenience that assists our understanding. The above limit may exist aside from any geometric consideration at all. As you will see later, there are other useful interpretations.

EXERCISE: 8 - 1

In Problems 1–16, use the definition of the derivative to find each of the following.

1. $\frac{d}{dx}[f(x)]$ if $f(x) = x$.
2. $\frac{d}{dx}[f(x)]$ if $f(x) = 4x - 1$.
3. $\frac{dy}{dx}$ if $y = 2x + 4$.
4. $\frac{dy}{dx}$ if $y = -3x$.
5. $\frac{d}{dx}(3 - 2x)$.
6. $\frac{d}{dx}\left(4 - \frac{x}{2}\right)$.
7. $f'(x)$ if $f(x) = 3$.
8. $f'(x)$ if $f(x) = 7.01$.
9. $D_x\,(x^2 + 4x - 8)$.
10. $D_x y$ if $y = x^2 + 5$.
11. $\frac{dq}{dp}$ if $q = 2p^2 + 5p - 1$.
12. $D_x\,(x^2 - x - 3)$.
13. $D_x y$ if $y = \frac{1}{x}$.
14. $\frac{dC}{dq}$ if $C = 7 + 2q - 3q^2$.
15. $f'(x)$ if $f(x) = \sqrt{x + 2}$.
16. $g'(x)$ if $g(x) = \frac{2}{x - 3}$.
17. Find the slope of the curve $y = x^2 + 4$ at the point $(-2, 8)$.
18. Find the slope of the curve $y = 2 - 3x^2$ at the point $(1, -1)$.
19. Find the slope of the curve $y = 4x^2 - 5$ when $x = 0$.
20. Find the slope of the curve $y = \sqrt{x}$ when $x = 1$.

In Problems 21–26, find an equation of the tangent line to the curve at the given point.

21. $y = x + 4$; $(3, 7)$.
22. $y = 2x^2 - 5$; $(-2, 3)$.
23. $y = 3x^2 + 3x - 4$; $(-1, -4)$.
24. $y = (x - 1)^2$; $(0, 1)$.
25. $y = \frac{3}{x + 1}$; $(2, 1)$.
26. $y = \frac{5}{1 - 3x}$; $(2, -1)$.

8.2 Rules for Differentiation:

Differentiation of a function by direct use of the definition of a derivative can be tedious. Rules for differentiation are used instead.

Rule 1: If $f(x) = c$, where c is a constant, then $f'(x) = 0$. That is, the derivative of a constant function is zero.

$$\boxed{\text{If } f(x) = c, \text{ then } f'(x) = 0.}$$

Examples:

a. If $f(x) = 3$, then $f'(x) = 0$.

b. If $g(x) = \sqrt{5}$, then $g'(x) = 0$.
$g'(4) = 0$.

c. If $s(t) = (1{,}938{,}623)^{807.4}$, then $ds/dt = 0$

Rule 2: If $f(x) = x^n$, where n is any real number, then $f'(x) = nx^{n-1}$

$$\boxed{\text{If } f(x) = x^n, \text{ then } f'(x) = nx^{n-1}.}$$

Examples:

a. If $f(x) = x^2$, then $f'(x) = 2x^{2-1} = 2x$.

b. If $g(w) = w^{9/4}$, then $g'(w) = \frac{9}{4}w^{(9/4)-1} = \frac{9}{4}w^{5/4}$.

c. If $F(x) = x = x^1$, then $\frac{d}{dx}[F(x)] = \frac{d}{dx}(x) = 1\cdot x^{1-1} = 1\cdot x^0 = 1$.

d. Suppose $y = x\sqrt{x}$.

To find $D_x y$, put $y = x^{3/2}$

$$D_x y = \tfrac{3}{2}x^{(3/2)-1} = \tfrac{3}{2}x^{1/2}$$

$$= \tfrac{3}{2}\sqrt{x}.$$

e. Suppose $h(x) = \frac{1}{x^{3/2}}$.

$$D_x\left(\frac{1}{x^{3/2}}\right) = D_x\,(x^{-3/2})$$
$$= -\frac{3}{2}x^{(-3/2)-1}$$
$$= -\frac{3}{2}x^{-5/2}.$$

Rule 3: If g(x) = c f(x) and f (x) exists, then g (x) = c f (x)

$$\boxed{\text{If } g(x) = c\,f(x), \text{ then } g'(x) = c\,f'(x).}$$

Examples:

Find the derivative of each of the following functions.

a) $g(x) = 5x^3$

$g'(x) = 5(3x^{3-1}) = 15x^2$

b) $g(p) = \frac{13}{2}p$

$g'(p) = \frac{13}{2}(1.p^{1-1}) = \frac{13}{2}$

c) $y = \frac{0.702}{x^2\sqrt{x}}$

$= 0.702x^{-\frac{5}{2}}$

$D_x y = 0.702\left(-\frac{5}{2}x^{-\frac{5}{2}-1}\right)$

$= -1.755x^{-\frac{7}{2}}$

Rule 4:

If F(x) = f(x) ± g(x) and f′(x) and g′(x) exist, then F′(x) = f′(x) ± g′(x).

That is, the derivative of a sum or difference of two functions is the sum of difference of the derivatives of the functions.

$$\boxed{\begin{array}{c} \text{If } F(x) = f(x) \pm g(x), \text{ then} \\ F'(x) = f'(x) \pm g'(x) \end{array}}$$

Examples:

Differentiate each of the following functions.

a) $F(x) = 3x^5 + x^{\frac{1}{2}}$

$F'(x) = 15x^4 + \frac{1}{2}x^{-\frac{1}{2}}$

$= 15x^4 + \frac{1}{2\sqrt{x}}$ Ans

b) $f(x) = x^5 - \sqrt[3]{x^2}$

$= x^5 - x^{\frac{2}{3}}$

$f'(x) = 5x^4 - \frac{2}{3}x^{-\frac{1}{3}}$

$= 5x^4 - \frac{2}{3\sqrt[3]{x}}$ Ans

c) $f(z) = \frac{z^4}{4} - \frac{5}{z^{\frac{1}{3}}}$

$= \frac{1z^4}{4} - 5z^{-\frac{1}{3}}$

$\frac{d}{dz}[f(z)] = \frac{1}{4}(4z^3) - 5\left(-\frac{1z^{-\frac{4}{3}}}{3}\right)$

$= z^3 + \frac{5z^{-\frac{4}{3}}}{3}$ Ans

d) $y = 6x^3 - 2x^2 + 7x - 8$

$D_xY = D_x(6x^3) - D_x(2x^2) + D_x(7x) - D_x(8)$

$= 6D_x(x^3) - 2D_x(x^2) + 7D_x(x) - D_x(8)$

$= 6(3x^2) - 2(2x) + 7(1) - 0$

$= 18x^2 - 4x + 7$ Ans

e) Find the derivative of

$f(x) = 2x(x^2 - 5x + 2)$ when $x = 2$

$f(x) = 2x^3 - 10x^2 + 4x$

$f'(x) = 6x^2 - 20x + 4$

$f'(2) = 6(2)^2 - 20(2) + 4$

$= 24 - 40 + 4$

$= -12$ Ans

f) Find an equation of the tangent line to the curve

$$y = \frac{3x^2 - 2}{x} \quad \text{when } x = 1$$

$$y = \frac{3x^2}{x} - \frac{2}{x}$$

$$= 3x - 2x^{-1}$$

$$\frac{dy}{dx} = 3 - 2(-1x^{-2})$$

$$= 3 + 2x^{-2}$$

$$= 3 + \frac{2}{x^2}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = 3 + \frac{2}{(1)^2}$$

$$= 5 = \text{the slope}(m)$$

At $x = 1, \quad y = \frac{3(1)^2 - 2}{1} = 1$

Hence the point (1, 1) lies on both the curve and the tangent line. An equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$\therefore \quad y - 1 = 5(x - 1)$$

$$y = 5x - 4$$ Ans

Rule 5: (Product Rule)

Let $F(x) = f(x).\ g(x)$. If $f'(x)$ and $g'(x)$ exist, then
$F'(x) = f(x).\ g'(x) + g(x)\ f'(x)$

That is, the derivative of the product of two functions is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\boxed{\begin{aligned} \text{If } F(x) &= f(x).\ g(x), \text{ then} \\ F'(x) &= f(x).\ g'(x) + g(x).f'(x) \end{aligned}}$$

Examples:

a) If $F(x) = (x^2 + 3x)(4x + 5)$, find $F'(x)$

$$F'(x) = (x^2 + 3x)\ D_x(4x + 5) + (4x + 5)\ D_x(x^2 + 3x)$$

$$= (x^2 + 3x)(4) + (4x + 5)(2x + 3)$$

$$= 12x^2 + 34x + 15 \qquad \underline{\text{Ans}}$$

b) Find the slope of the graph of $F(x) = (7x^3 - 5x + 2)(2x^4 + x + 7)$ When $x = 1$

$$F'(x) = (7x^3 - 5x + 2)\ D_x(2x^4 + x + 7) + (2x^4 + x + 7)\ D_x(7x^3 - 5x + 2)$$

$$= (7x^3 - 5x + 2)(8x^3 + 1) + (2x^4 + x + 7)(21x^2 - 5)$$

at $x = 1$

$$\therefore \quad F'(1) = (7 - 5 + 2)(8 + 1) + (2 + 1 + 7)(21 - 5)$$

$$= (4)(9) + (10)(16)$$

$$= 196 \qquad \underline{\text{Ans}}$$

c) If $y = (x^{\frac{2}{3}} + 3)(x^{-\frac{1}{3}} + 5x)$, find $D_x y$

$$D_x y = (x^{\frac{2}{3}} + 3)\ D_x(x^{-\frac{1}{3}} + 5x) + (x^{-\frac{1}{3}} + 5x)\ D_x(x^{\frac{2}{3}} + 3)$$

$$= (x^{\frac{2}{3}} + 3)(-\frac{1}{3}x^{-\frac{4}{3}} + 5) + (x^{-\frac{1}{3}} + 5x)\left(\frac{2}{3}x^{-\frac{1}{3}}\right)$$

$$= \frac{25x^{\frac{2}{3}}}{3} + \frac{1x^{-\frac{2}{3}}}{3} - x^{-\frac{4}{3}} + 15 \qquad \underline{\text{Ans}}$$

d) If $y = (x + 2)(x + 3)(x + 4)$, find y'

$$y = [(x + 2)(x + 3)]\ (x + 4)$$

$$y' = [(x + 2)(x + 3)]\ D_x\ (x + 4) + (x + 4)\ D_x[(x + 2)(x + 3)]$$

$$= [(x + 2)(x + 3)]\ (1) + (x + 4)\ D_x[(x + 2)(x + 3)]$$

$$= (x^2 + 5x + 6) + (x + 4)\ [(x + 2)\ D_x(x + 3) + (x + 3)\ D_x(x + 2)]$$

$$= (x^2 + 5x + 6) + (x + 4)[(x + 2)(1) + (x + 3)(1)]$$

$$= (x^2 + 5x + 6) + (x + 4)(2x + 5)$$

$$= x^2 + 5x + 6 + 2x^2 + 13x + 20$$

$$= 3x^2 + 18x + 26 \qquad \underline{\text{Ans}}$$

Note: Usually we do not use the product rule when simpler ways are obvious. For example, if $f(x) = 2x(x + 3)$

$$\therefore\ f(x) = 2x^2 + 6x$$
$$f\ (x) = 4x + 6$$

Similarly, we do not use the product rule to differentiate $y = 4(x^2 - 3)$. Since the 4 is a constant multiplier.

Rule 6: (Quotient Rule)

Let $F(x) = \dfrac{f(x)}{g(x)}$ such that $g(x) = 0$. If $f'(x)$ and $g'(x)$ exist,

Then

$$\boxed{F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.}$$

Examples:

a. *If* $F(x) = \dfrac{4x^2 - 2x + 3}{2x - 1}$, *find* $F'(x)$.

$$F'(x) = \frac{(2x - 1)\ D_x(4x^2 - 2x + 3) - (4x^2 - 2x + 3)\ D_x(2x - 1)}{(2x - 1)^2}$$

$$= \frac{(2x - 1)(8x - 2) - (4x^2 - 2x + 3)(2)}{(2x - 1)^2}$$

$$= \frac{8x^2 - 8x - 4}{(2x - 1)^2} = \frac{4(2x^2 - 2x - 1)}{(2x - 1)^2}.$$

<u>Ans</u>

b. *If* $y = \dfrac{1}{x^2}$, *find* y'.

$$y' = \frac{(x^2)\ D_x(1) - (1)\ D_x(x^2)}{(x^2)^2}$$

$$= \frac{x^2(0) - 1(2x)}{x^4}$$

$$= \frac{-2x}{x^4} = -\frac{2}{x^3}.$$

<u>Ans</u>

c. *Find an equation of the tangent line to the curve*

$$y = \frac{(x + 1)(x^2 + 2x + 5)}{1 - x} \quad \textit{at } (0, 5).$$

$$y' = \frac{(1 - x)\, D_x[(x + 1)(x^2 + 2x + 5)] - [(x + 1)(x^2 + 2x + 5)]\, D_x(1 - x)}{(1 - x)^2}.$$

$$y' = \frac{(1 - x)[(x + 1)(2x + 2) + (x^2 + 2x + 5)(1)] - [(x + 1)(x^2 + 2x + 5)](-1)}{(1 - x)^2}$$

The slope of the curve at (0, 5) is

$$y'(0) = 12.$$

An equation of the tangent line is

$$y - 5 = 12(x - 0),$$
$$y = 12x + 5.$$

<u>Ans</u>

EXERCISE: 8 - 2

In Problems **1–50**, *differentiate the functions.*

1. $f(x) = (4x + 1)(6x + 3)$.
2. $f(x) = (3x - 1)(7x + 2)$.
3. $s(t) = (8 - 7t)(t^2 - 2)$.
4. $Q(x) = (5 - 2x)(x^2 + 1)$.
5. $f(r) = (3r^2 - 4)(r^2 - 5r + 1)$.
6. $C(I) = (2I^2 - 3)(3I^2 - 4I + 1)$.
7. $y = (x^2 + 3x - 2)(2x^2 - x - 3)$.
8. $y = (2 - 3x + 4x^2)(1 + 2x - 3x^2)$.
9. $f(w) = (8w^2 + 2w - 3)(5w^3 + 2)$.
10. $f(x) = (3x - x^2)(3 - x - x^2)$.
11. $g(x) = 3(x^3 - 2x^2 + 5x - 4)(x^4 - 2x^3 + 7x + 1)$.
12. $y = -\frac{3}{2}(2x^4 - 3x + 1)(3x^3 - 6x^2 +$ [illegible] $- 4)$.
13. $y = (x^2 - 1)(3x^3 - 6x + 5) - (x + 4$ [illegible] $2x + 1)$.
14. $h(x) = 4(x^5 - 3)(2x^3 + 4) + 3(8x^2 - 5)(3x + 2)$.
15. $f(p) = \frac{3}{2}(\sqrt{p} - 4)(4p - 5)$.
16. $g(x) = (\sqrt{x} - 3x + 1)(\sqrt[4]{x} - 2\sqrt{x})$.
17. $y = (2x^{.45} - 3)(x^{1.3} - 7x)$.
18. $y = (x - 1)(x - 2)(x - 3)$.
19. $y = (2x - 1)(3x + 4)(x + 7)$.
20. $y = \dfrac{x}{x - 3}$.
21. $y = 7 \cdot \frac{2}{3}$.
22. $y = \dfrac{2x - 3}{4x + 1}$.

23. $f(x) = \dfrac{x}{x-1}$.

24. $f(x) = \dfrac{-2x}{1-x}$.

25. $y = \dfrac{x+2}{x-1}$.

26. $h(w) = \dfrac{3w^2+5w-1}{w-3}$.

27. $h(z) = \dfrac{5-2z}{z^2-4}$.

28. $y = \dfrac{x^2-4x+2}{x^2+x+1}$.

29. $y = \dfrac{8x^2-2x+1}{x^2-5x}$.

30. $f(x) = \dfrac{x^3-x^2+1}{x^2+1}$.

31. $y = \dfrac{x^2-4x+3}{2x^2-3x+2}$.

32. $F(z) = \dfrac{z^4+4}{3z}$.

33. $g(x) = \dfrac{1}{x^{100}+1}$.

34. $y = \dfrac{3}{7x^3}$.

35. $u(v) = \dfrac{v^5-8}{v}$.

36. $y = \dfrac{x-5}{2\sqrt{x}}$.

37. $y = \dfrac{3x^2-x-1}{\sqrt[3]{x}}$.

38. $y = \dfrac{x^3-2}{2x^{2.1}+1}$.

39. $y = 7 - \dfrac{4}{x-8} + \dfrac{2x}{3x+1}$.

40. $q(x) = 13x^2 + \dfrac{x-1}{2x+3} - \dfrac{4}{x}$.

41. $H(s) = \dfrac{(s+2)(s-4)}{s-5}$.

42. $y = \dfrac{(2x-1)(3x+2)}{4-5x}$.

43. $y = \dfrac{x-5}{(x+2)(x-4)}$.

44. $y = \dfrac{4-5x}{(2x-1)(3x+2)}$.

45. $s(t) = \dfrac{t^2+3t}{(t^2-1)(t^3+7)}$.

46. $y = \dfrac{(2x-3)(x^2-4x+1)}{3x^3+1}$.

47. $y = \dfrac{(x-1)(x-2)}{(x-3)(x-4)}$.

48. $f(s) = \dfrac{17}{s(5s^2-10s+4)}$.

49. $y = 3x - \dfrac{\dfrac{2}{x} - \dfrac{3}{x-1}}{x-2}$.

50. $y = 7 - 10x^2 + \dfrac{1 - \dfrac{7}{x^2+3}}{x+2}$.

51. Find the slope of the curve $y = (4x^2+2x-5)(x^3+7x+4)$ at $(-1, 12)$.

52. Find the slope of the curve $y = \dfrac{x^3}{x^4+1}$ at $(1, \frac{1}{2})$.

In Problems **53–56**, *find an equation of the tangent line to the curve at the given point*

53. $y = 6/(x-1)$; $(3, 3)$.

54. $y = \dfrac{4x+5}{x^2}$; $(-1, 1)$.

55. $y = (2x + 3)[2(x^4 - 5x^2 + 4)]$; $(0, 24)$.

56. $y = \dfrac{x + 1}{x^2(x - 4)}$; $(2, -\frac{3}{8})$.

Rule 7: (Chain Rule)

If y = f(u) is a differentiable function of u and u is a differentiable function of x, then y is a differentiable function of x and

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Examples:

a) If $y = 2u^2 - 3u - 2$ and $u = x^2 + x$, find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{du}(2u^2 - 3u - 2) \cdot \frac{d}{dx}(x^2 + x)$$

$$= (4u - 3)(2x + 1)$$

But $u = x^2 + x$. Replace u by $x^2 + x$

$$\therefore \frac{dy}{dx} = \left[4(x^2 + x) - 3\right](2x + 1)$$

$$= (4x^2 + 4x - 3)(2x + 1)$$

$$= 8x^3 + 12x^2 - 2x - 3$$ Ans

b) If $y = \sqrt{u}$ and $u = 7 - x^3$, find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{du}(\sqrt{u}) \cdot \frac{d}{dx}(7 - x^3)$$

$$= \left(\frac{1}{2}u^{-\frac{1}{2}}\right) \cdot (-3x^2)$$

$$= -\frac{3x^2}{2\sqrt{7 - x^3}}$$ Ans

c) If $y = u^{10}$ and $u = 8-t^2+t^5$, find $\frac{dy}{dt}$

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}$$

$$= \frac{d}{du}(u^{10}) \cdot \frac{d}{dt}(8-t^2+t^5)$$

$$= (10u^9) \cdot (-2t+5t^4)$$

$$= 10(8-t^2+t^5)^9(-2t+5t^4)$$ Ans

d) If $y = 4u^3+10u^2-3u-7$ and $u = \frac{4}{3x-5}$, find $\frac{dy}{dx}$ when $x=1$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{du}(4u^3+10u^2-3u-7) \cdot \frac{d}{dx}\left(\frac{4}{3x-5}\right)$$

$$= (12u^2+20u-3) \frac{[(3x-5)(0)-4(3)]}{(3x-5)^2}$$

$$= (12u^2+20u-3) \frac{(-12)}{(3x-5)^2}$$

When $x=1$, then $u = \frac{4}{3(1)-5} = -2$. Thus,

$$\left.\frac{dy}{dx}\right|_{x=1} = [12(-2)^2+20(-2)-3] \cdot \frac{-12}{[3(1)-5]^2}$$

$$= 5.(-3) = -15$$ Ans

e) If $y = \sqrt[5]{8x^2-7x}$, find y'

Let $u = 8x^2-7x$. Then $y = \sqrt[5]{u} = u^{\frac{1}{5}}$

$$y' = \frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{du}{dx}$$

$$= \frac{d}{du}(u^{\frac{1}{5}}) \cdot \frac{d}{dx}(8x^2-7x)$$

$$= (\frac{1}{5}u^{-\frac{4}{5}}).(16x-7)$$

$$= \frac{1}{5}(8x^2-7x)^{-\frac{4}{5}}(16x-7)$$ Ans

f) If $y = \frac{1}{(x^2-2)^4}$, find y'

Let $u = x^2-2$, Then $y = \frac{1}{u^4}$. Thus

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{du}\left(\frac{1}{u^4}\right) \cdot \frac{d}{dx}(x^2-2)$$

$$= \frac{-4u^3}{u^8} \cdot (2x)$$

$$= \frac{-8x}{u^5} = \frac{-8x}{(x^2-2)^5}$$ <u>**Ans**</u>

Rule 8: (Power Rule)

If $y = u^n$, where n is any real number and u is differentiable function of x, then

$$\boxed{\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}} \quad \text{or} \quad \boxed{\frac{d}{dx}\left([u(x)]^n\right) = n[u(x)]^{n-1} \cdot u'(x)}$$

Examples:

a) If $y = (x^3-1)^7$, find y'

$$u(x) = x^3-1$$

$$y' = n[u(x)]^{n-1} \cdot u'(x)$$

$$= 7(x^3-1)^{7-1} \cdot \frac{d}{dx}(x^3-1)$$

$$= 7(x^3-1)^6 \cdot (3x^2)$$

$$= 21x^2(x^3-1)^6$$ <u>Ans</u>

b) If $y = \sqrt{4x^2+3x-1}$, find $\frac{dy}{dx}$ when $x = -2$

$$y = (4x^2+3x-1)^{\frac{1}{2}} / \; u(x) = 4x^2+3x-1$$

$$\frac{dy}{dx} = \frac{1}{2}(4x^2+3x-1)^{-\frac{1}{2}} \cdot \frac{d}{dx}(4x^2+3x-1)$$

$$= \frac{1}{2}(4x^2+3x-1)^{-\frac{1}{2}}(8x+3)$$

$$= \frac{8x+3}{2\sqrt{4x^2+3x-1}}$$

$$\left.\frac{dy}{dx}\right|_{x=-2} = \frac{-13}{2\sqrt{9}} = \frac{-13}{6}$$ <u>Ans</u>

c) If $Z = \left(\frac{2s+5}{s^2+1}\right)^4$, find $\frac{dZ}{ds}$

First use the power rule

$$\frac{dZ}{ds} = 4\left(\frac{2s+5}{s^2+1}\right)^{4-1} \cdot \frac{d}{ds}\left(\frac{2s+5}{s^2+1}\right)$$

Then, use the quotient rule

$$\frac{dZ}{ds} = 4\left(\frac{2s+5}{s^2+1}\right)^3 \cdot \frac{(s^2+1)(2) - (2s+5)(2s)}{(s^2+1)^2}$$

Simplifying, we have

$$\frac{dZ}{ds} = 4 \cdot \frac{(2s+5)^3}{(s^2+1)^3} \cdot \frac{(-2s^2-10s+2)}{(s^2+1)^2}$$

$$= -\frac{8(s^2+5s-1)(2s+5)^3}{(s^2+1)^5} \qquad \underline{\text{Ans}}$$

d) If $y = (x^2-4)^5(3x+5)^4$, find y'

Since y is a product, we first apply the product rule.

$$y' = (x^2-4)^5 D_x(3x+5)^4 + (3x+5)^4 D_x(x^2-4)^5$$

Now we can use the power rule.

$$y' = (x^2-4)^5[4(3x+5)^3(3)] + (3x+5)^4[5(x^2-4)^4(2x)]$$

Simplifying, we have

$$y' = 12(x^2-4)^5(3x+5)^3 + 10x(3x+5)^4(x^2-4)^4$$

$$= 2(x^2-4)^4(3x+5)^3[6(x^2-4)+5x(3x+5)]$$

$$= 2(x^2-4)^4(3x+5)^3(21x^2+25x-24) \qquad \underline{\text{Ans}}$$

EXERCISE: 8 - 3

In Problems 1-8, use the chain rule.

1. If $y = u^2 - 2u$ and $u = x^2 - x$, find dy/dx.
2. If $y = 2u^3 - 8u$ and $u = 7x - x^3$, find dy/dx.
3. If $y = \frac{1}{w^2}$ and $w = 2 - x$, find dy/dx.
4. If $y = \sqrt[3]{z}$ and $z = x^6 - x^2 + 1$, find dy/dx.

5. If $w = u^2$ and $u = \frac{t+1}{t-1}$, find dw/dt when $t = 3$.

6. If $z = u^2 + \sqrt{u} + 9$ and $u = 2s^2 - 1$, find dz/ds when $s = -1$.

7. If $y = 3w^2 - 8w + 4$ and $w = 3x^2 + 1$, find dy/dx when $x = 0$.

8. If $y = 3u^3 - u^2 + 7u - 2$ and $u = 3x - 2$, find dy/dx when $x = 1$.

In Problems 9-44, find y'.

9. $y = (7x + 4)^8$.

10. $y = (4 - 3x)^{25}$.

11. $y = (3 - 2p^2)^{14}$.

12. $y = (x^2 - 8x)^{40}$.

13. $y = \frac{(4x^3 - 8x + 2)^{10}}{3}$.

14. $y = \frac{(7 - q^2 + q)^{12}}{9}$.

15. $y = (4r^2 - 10r + 3)^{-15}$.

16. $y = (t^2 - 5)^{-4}$.

17. $y = \frac{7}{(x^3 - x^2 + 2)^7}$.

18. $y = \frac{6}{(2 - x^2 + x)^4}$.

19. $y = 15(4z^3 - z^2 + 2)^{1/5}$.

20. $y = 2(8x - 2)^{2/3}$.

21. $y = \sqrt{2x^2 - x + 3}$.

22. $y = \sqrt[3]{8s^2 - 1}$.

23. $y = \sqrt[5]{(x^2 + 1)^3}$.

24. $y = \frac{1}{(3x^2 - x)^{2/3}}$.

25. $y = \left(\frac{x - 7}{x + 4}\right)^{10}$.

26. $y = \left(\frac{2w}{w + 2}\right)^4$.

27. $y = 2\left(\frac{q^3 - 2q + 4}{5q^2 + 1}\right)^5$.

28. $y = 3\left(\frac{x^2 + 2x - 2}{x^3 + x}\right)^8$.

29. $y = \sqrt{\frac{x - 2}{x + 3}}$.

30. $y = \sqrt[3]{\frac{8x^2 - 3}{x^2 + 2}}$.

31. $y = (x^2 + 2x - 1)^3(5x + 7)$.

32. $y = (8x^3 - 1)^3(2x^2 + 1)^2$.

33. $y = [(4x + 3)(6x^2 + x + 8)]^8$.

34. $y = \frac{2t - 5}{(t^2 + 4)^3}$.

35. $y = \frac{(2w + 3)^3}{w^2 + 4}$.

36. $y = \sqrt{(x - 1)(x + 2)^3}$.

37. $y = 6(5x^2 + 2)\sqrt{x^4 + 5}$.

38. $y = \sqrt[3]{\frac{8x - 7}{5x^2 + 6}}$.

39. $y = (4 - 3x^2)^2(2 - 3x)^3$.

40. $y = 6 + 3x - 4x(7x + 1)^2$.

41. $y = 8t + \frac{t - 1}{t + 4} - \left(\frac{8t - 7}{4}\right)^2$.

42. $y = 4[(3p - 8)(3p^2 - 2p + 1)^3]^4$.

43. $y = \frac{(8x - 1)^5}{(3x - 1)^3}$.

44. $y = \frac{(4x^2 - 2)(8x - 1)}{(3x - 1)^2}$.

In Problems 45 and 46, use the quotient rule and power rule of find y′. Do not simplify your answer.

45. $y = \dfrac{(2x+1)(3x-5)^2}{(x^2-7)^4}$.

46. $y = \dfrac{\sqrt{x+2}\,(4x^2-1)^2}{9x-3}$.

47. If $y = (5u+6)^3$ and $u = (x^2+1)^4$, find dy/dx when $x = 0$.

48. If $z = 2y^2 - 4y + 5$, $y = 6x - 5$, and $x = 2t$, find dz/dt when $t = 1$.

49. Find the slope of the curve $y = (x^2 - 7x - 8)^3$ at the point (8, 0).

50. Find the slope of the curve $y = \sqrt{x+1}$ at the point (8, 3).

In Problems 51-54, find an equation of the tangent line to the curve at the given point.

51. $y = \sqrt[3]{(x^2-8)^2}$; (3, 1).

52. $y = (2x+3)^2$; (−2, 1).

53. $y = \dfrac{\sqrt{7x+2}}{x+1}$; $(1, \frac{3}{2})$.

54. $y = \dfrac{-3}{(3x^2+1)^3}$; (0, −3).

In Problems 55 and 56, determine the percentage rate of change of y with respect to x for the given value of x.

55. $y = (x^2+9)^3$; $x = 4$.

56. $y = \dfrac{1}{(x^2+1)^2}$; $x = -3$.

Rule 9: (Logarithmic functions)

The derivative of the logarithm of a function to the base 'a' is one over (the reciprocal of) the original function multiplied by the derivative of the function, all multiplied by the logarithm of 'e' to the base 'a'.

$$\boxed{\frac{d}{dx}\left[\log_a g(x)\right] = \frac{g'(x)}{g(x)} \log_a e}$$

Examples:

Differentiate each of the following:

a). $y = f(x) = \ln x$

$$y' = \frac{\frac{d(x)}{dx} \cdot \log_e e}{x}$$

$$= \frac{1}{x} \qquad \underline{\text{Ans}}$$

b) $y = x \ln x$

Since y is a product, we first apply the product rule

$$y' = x \frac{d}{dx}(\ln x) + [\ln x] \cdot \frac{d}{dx}(x)$$

$$= x. \frac{1}{x} \log_e e + \ln x$$

$$= 1 + \ln x \qquad \underline{\text{Ans}}$$

c) $y = \ln (x^2+1)$

$$\frac{dy}{dx} = \frac{2x}{x^2+1} \cdot \log_e e$$

$$= \frac{2x}{x^2+1} \qquad \underline{\text{Ans}}$$

d) $y = x^2 . \ln(4x + 2)$

$$D_x y = x^2 D_x \ln(4x + 2) + [\ln(4x + 2)]. D_x (x^2)$$

$$= x^2 . \frac{4}{4x+2} \cdot \log_e e + [\ln(4x + 2)](2x)$$

$$= \frac{4x^2}{4x+2} + 2x.\ln(4x + 2) \qquad \underline{\text{Ans}}$$

e) $y = \ln (\ln x)$

$$y' = \frac{\frac{d(\ln x)}{dx}}{\ln x} \cdot \log_e e$$

$$= \frac{\frac{1}{x}}{\ln x} \cdot \log_e e$$

$$= \frac{1}{x \ln x} \qquad \underline{\text{Ans}}$$

f) $y = \ln(2x + 5)^3$

Use the properties of logarithms to simplify the right side

$y = 3 \ln(2x + 5)$

$$\frac{dy}{dx} = 3\left(\frac{2}{2x+5}\right) \cdot \log_e e$$

$$= \frac{6}{2x+5} \qquad \underline{\text{Ans}}$$

g) $f(p) = \ln[(p + 1)^2(p + 2)^3(p + 3)^4]$

Simplify the right side

$$f(p) = \ln (p + 1)^2 + \ln(p + 2)^3 + \ln(p + 3)^4$$

$$f(p) = 2 \ln(p + 1) + 3 \ln(p + 2) + 4 \ln(p + 3)$$

$$f'(p) = 2\left(\frac{1}{p+1}\right) . \log_e e + 3\left(\frac{1}{p+2}\right) . \log_e e + 4\left(\frac{1}{p+3}\right) . \log_e e$$

$$= \frac{2}{p+1} + \frac{3}{p+2} + \frac{4}{p+3}$$ Ans

h) $f(w) = \ln\sqrt{\dfrac{1 + w^2}{w^2 - 1}}$

Again, using properties of logarithms will simplify the work.

$$f(w) = \ln\left(\frac{1 + w^2}{w^2 - 1}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2} \ln\left(\frac{1 + w^2}{w^2 - 1}\right) = \frac{1}{2} [\ln(1 + w^2) - \ln(w^2 - 1)]$$

$$f'(w) = \frac{1}{2}\left[\frac{2w}{1+w^2} . \log_e e - \frac{2w}{w^2-1} . \log_e e\right]$$

$$= \frac{1}{2}\left[\frac{2w}{1+w^2} - \frac{2w}{w^2-1}\right] = \frac{1}{2}\ \frac{2w^3 - 2w - 2w - 2w^3}{w^4 - 1}$$

$$= -\frac{2w}{w^4 - 1}$$ Ans

i) $f(w) = \ln^3[(2x + 1)^4]$

$f(w) = [\ln(2x + 1)^4]^3 = [4 \ln(2x + 1)]^3$

By the power rule,

$$f'(w) = 3[4 \ln(2x + 1)]^2\ D_x\ [4 \ln(2x + 1)]$$

$$= 3[4 \ln(2x + 1)]^2\left[4\ \frac{2}{2x+1} \cdot \log_e e\right]$$

$$= 3.4^2[\ln(2x + 1)]^2\ \frac{8}{2x+1}$$

$$= \frac{384}{2x+1}\ [\ln(2x + 1)]^2$$

$$= \frac{384}{2x+1} . \ln^2(2x + 1)$$ Ans

EXERCISE: 8 - 4

In Problems 1-34, differentiate the functions.

1. $y = \ln (3x - 4)$.
2. $y = \ln (5x - 6)$.
3. $y = \ln x^2$.
4. $y = \ln (ax^2 + b)$.
5. $y = \ln (1 - x^2)$.
6. $y = \ln (-x^2 + 6x)$.
7. $f(p) = \ln (2p^3 + 3p)$.
8. $f(r) = \ln (2r^4 - 3r^2 + 2r + 1)$.
9. $y = \ln^4 (ax)$.
10. $y = \ln^2 (2x + 3)$.
11. $y = \ln (x^2 + 4x + 5)$.
12. $y = \ln x^{100}$.
13. $f(t) = t \ln t$.
14. $y = x^2 \ln x$.
15. $y = \log_3 (2x - 1)$.
16. $f(w) = \log (w^2 + w)$.
17. $y = (x^2 + 1) \ln (2x + 1)$.
18. $y = (ax + b) \ln (ax)$.
19. $y = \ln [(x^2 + 2)^2(x^3 + x - 1)]$.
20. $y = \ln [(5x + 2)^4(8x - 3)^6]$.
21. $f(l) = \ln \left(\dfrac{1 + l}{1 - l}\right)$.
22. $y = \ln \left(\dfrac{2x + 3}{3x - 4}\right)$.
23. $y = \ln \sqrt{1 + x^2}$.
24. $f(s) = \ln \left(\dfrac{s^2}{1 + s^2}\right)$.
25. $y = \ln [(x + 1)^3 + (x + 2)^4 + x^8]$.
26. $y = \ln x^3 + \ln^3 x$.
27. $y = \ln \sqrt[4]{\dfrac{1 + x^2}{1 - x^2}}$.
28. $y = \ln \sqrt{\dfrac{x^4 - 1}{x^4 + 1}}$.
29. $f(z) = \dfrac{\ln z}{z}$.
30. $y = \dfrac{x^2 - 1}{\ln x}$.
31. $y = x \ln \sqrt{x - 1}$.
32. $y = \ln (x^2\sqrt{3x - 2})$.
33. $y = \sqrt{4 + \ln x}$.
34. $y = \ln (x + \sqrt{1 + x^2})$.

Rule 10: (Exponential Functions)

The derivative of a constant to a variable power is the product of the original expression, multiplied by the natural logarithm of the constant, multiplied by the derivative of the power

$$\boxed{\frac{d}{dx}[a^{g(x)}] = a^{g(x)}.(\ln a)\,[g'(x)]}$$

Examples:

a) Find $\frac{d}{dx}\left(e^{x^3+3x}\right)$

$$\frac{d}{dx}\left(e^{x^3+3x}\right) = e^{x^3+3x}.(\ln e)\ D_x\ (x^3+3x)$$

$$= e^{x^3+3x}\ (1)(3x^2+3)$$

$$= 3(x^2+1)\left(e^{x^3+3x}\right) \qquad \underline{\text{Ans}}$$

b) If $y = \frac{x}{e^x}$, find y'

First use the quotient rule.

$$\frac{dy}{dx} = \frac{e^x\ D_x(x) - x\ D_x(e^x)}{(e^x)^2}$$

$$= \frac{e^x(1) - x.e^x.\ln e(1)}{e^{2x}}$$

$$= \frac{e^x - xe^x}{e^{2x}} = \frac{e^x(1-x).}{e^{2x}}$$

$$= \frac{1-x}{e^x} \qquad \underline{\text{Ans}}$$

c) If $f(w) = w^4.\ e^{2w}$, find $f'(w)$

First, use the product rule

$$f'(w) = w^4 D_w\left(e^{2w}\right) + e^{2w} D_w(w^4)$$

$$= w^4.\left[e^{2w}(2)\right] + e^{2w}(4w^3)$$

$$= 2e^{2w}.w^4 + 4e^{2w}w^3$$

$$= 2e^{2w}w^3(w+2) \qquad \underline{\text{Ans}}$$

d) Find $D_x\ [e^{x+1} . \ln(x^2 + 1)]$

By the product rule,

$$D_x\ [e^{x+1} . \ln(x^2 + 1)] = e^{x+1} . D_x\ [\ln(x^2 + 1)] + \ln(x^2 + 1)\ D_x . e^{x+1}$$

$$= e^{x+1} . \frac{2x}{x^2+1} . \ln e + \ln(x^2+1) . e^{x+1}(\ln e)(1)$$

$$= e^{x+1} . \frac{2x}{x^2+1} . \ln e + \ln(x^2+1) . e^{x+1}(\ln e)(1)$$

$$= e^{x+1} . \frac{2x}{x^2+1} + \ln(x^2 + 1) . e^{x+1}$$

$$= e^{x+1} \left[\frac{2x}{x^2+1} + \ln(x^2 + 1)\right]$$ Ans

e) If $y = e^2 + e^x + \ln 3$, find y'

(Note: e^2 and $\ln 3$ are constants)

$\therefore\ y' = 0 + e^x(\ln e)(1) + 0$

$= e^x$ Ans

f) Find $\frac{dy}{dx}$ if $y = 4^{2x^3+5x}$

$$y = 4^{2x^3+5x}$$

$$\frac{dy}{dx} = 4^{2x^3+5x}\ (\ln 4)\ \frac{d}{dx}(2x^3 + 5x)$$

$$= 4^{2x^3+5x}\ (\ln 4)(6x^2+5)$$

$$= (\ln 4)(6x^2 + 5)\ 4^{2x^3+5x}$$ Ans

g) If $y = x^{100} + 100^x$, find $D_x y$

$$D_x y = 100x^{99} + 100^x(\ln 100)(1)$$

$$= 100x^{99} + 100^x \ln 100$$ Ans

Note: This function involves **a variable to a constant power** and **a constant raised to a variable power.**

EXERCISE: 8 - 5

In Problems 1-30, differentiate the functions.

1. $y = e^{x^2+1}$.
2. $y = e^{2x^2+5}$.
3. $y = e^{3-5x}$.
4. $f(q) = e^{-q^3+6q-1}$.
5. $f(r) = e^{3r^2+4r+4}$.
6. $y = e^{9x^2+5x^3-6}$.
7. $y = xe^{x}$.
8. $y = x^2e^{-x}$.
9. $y = x^2e^{-x^2}$.
10. $y = xe^{2x}$.
11. $y = \dfrac{e^x + e^{-x}}{2}$.
12. $y = \dfrac{e^x - e^{-x}}{2}$.
13. $y = 4^{3x^2}$.
14. $y = 4^{3x+1}$.
15. $f(w) = \dfrac{e^{2w}}{w^2}$.
16. $y = 2^x x^2$.
17. $y = e^{1+\sqrt{x}}$.
18. $y = e^{x-\sqrt{x}}$.
19. $y = x^3 - 3^x$.
20. $y = (e^{3x} + 1)^4$.
21. $y = \dfrac{e^x - 1}{e^x + 1}$.
22. $f(z) = e^{1/z}$.
23. $y = e^{e^x}$.
24. $y = e^{2x}(x + 1)$.
25. $y = e^{\ln x}$.
26. $y = e^{\ln(x^2+1)}$.
27. $y = e^{x \ln x}$.
28. $y = e^{-x} \ln x$.
29. $y = (\log 2)^x$.
30. $y = \ln e^{4x+1}$.
31. Find an equation of the tangent line to the graph of $y = e^x$ when $x = 2$.
32. Find the slope of the tangent line to the graph of $y = 2e^{-4x^2}$ when $x = 0$.

Differentiation Formulas

$$\frac{d}{dx}(c) = 0, \text{ where } c \text{ is any constant.}$$

$$\frac{d}{dx}(x^n) = nx^{n-1}, \text{ where } n \text{ is any real number.}$$

$$\frac{d}{dx}[cf(x)] = cf'(x).$$

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x).$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x).$$

$$\frac{d}{dx}[f(x)\,g(x)] = f(x)g'(x) + g(x)f'(x).$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}, \text{ where } y \text{ is a function of } u \text{ and } u \text{ is a function of } x.$$

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

$$\frac{d}{dx}(\log_b u) = \frac{1}{u}(\log_b e)\frac{du}{dx}.$$

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}.$$

$$\frac{d}{dx}(a^u) = a^u(\ln a)\frac{du}{dx}.$$

$$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}.$$

8.3 Implicit Differentiation:

Until now we have always had y given explicitly (directly) in terms of x before determining y'; that is in the form $y = f(x)$. Now consider the equation $x^2 + y^2 = 4$

If we have to find y in terms of x, then

$$y^2 = 4 - x^2$$

or $$y = \pm\sqrt{4 - x^2}$$

Given a value of x, two values of y can be obtained. In fact, depending on the equation given, it may be very complicated or even impossible to find an explicit expression for y. For example, it would be difficult to solve $y\,e^x + \ln(x + y) = 0$ for y. Now consider a method to avoid such difficulties.

An equation of the form $f(x, y) = 0$, is said to express y *implicitly* as a function of x. The word *'implicitly'* is used since y is not given explicitly as a function of x. However, it is *assumed or implied* that the equation defined y as at least one differentiable function of x. Thus, for example, $x^2 + y^2 - 4 = 0$ defines at least one function of x, say $y = f(x)$. Hence to find $\frac{dy}{dx}$ we treat y as a function of x and differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}(x^2 + y^2 - 4) = \frac{d}{dx}(0)$$

$$\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} - \frac{d}{dx}(4) = \frac{d}{dx}(0)$$

$$\frac{d(x^2)}{dx} = 2x; \quad \frac{d(4)}{dx} \text{ and } \frac{d(0)}{dx} = 0$$

But $\frac{d(y^2)}{dx}$ is not 2y because we are differentiating with respect to x, not y. That is, y is not the independent variable. Since y is assumed to be a function of x, the y^2 – term has the form u^n, where y plays the role of u. Just as the power rule says that $\frac{d}{dx}(u^2) = 2u\frac{du}{dx}$, we have $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx} = 2yy'$

Hence the above equation becomes

$$2x + 2yy' = 0$$

$$2yy' = -2x$$

$$y' = \frac{-2x}{2y} = \frac{-x}{y}$$

This method of finding $\frac{dy}{dx}$ is called *implicit differentiation.*

Notice that the expression for y' involves the variable y as well as x. This means that to find y involves the variable y as well as x. This means that to find y' at a point, both coordinates of the point must be substitued into y'.

Example 1:

For each of the following, find y' by implicit differentiation

a) $y + y^3 = x$

Solution:

Treat y as a function of x and differentiate both sides with respect to x.

$$\frac{d}{dx}(y) + \frac{d}{dx}(y^3) = \frac{d}{dx}(x)$$
$$y' + 3y^2y' = 1$$
$$y'(1 + 3y^2) = 1$$
$$y' = \frac{1}{1 + 3y^2} \qquad \underline{\text{Ans}}$$

b) $x^3 + 4xy^2 - y^4 - 27 = 0$

Solution:

Assume that y is a function of x and differentiate both sides with respect to x.

$$\frac{d}{dx}(x^3) + 4\frac{d}{dx}(xy^2) - \frac{d}{dx}(y^4) - \frac{d}{dx}(27) = 0$$
$$3x^2 + 4[x(2y\,y') + y^2(1)] - 4y^3y' - 0 = 0$$
$$3x^2 + 8xy\,y' + 4y^2 - 4y^3y' = 0$$
$$8xy\,y' - 4y^3y' = -3x^2 - 4y^2$$
$$y'(8xy - 4y^3) = -3x^2 - 4y^2$$
$$y' = \frac{-3x^2 - 4y^2}{8xy - 4y^3}$$
$$\text{or} \quad = \frac{3x^2 + 4y^2}{4y^3 - 8xy} \qquad \underline{\text{Ans}}$$

Example 2:

For each of the following, find y' by implicit differentiation.

a) $e^{xy} = x + y$

Solution:

$$\frac{d}{dx}(e^{xy}) = \frac{d}{dx}(x) + \frac{d}{dx}(y)$$

$$e^{xy}\left[\frac{d}{dx}(xy)\right] = 1 + y'$$

$$e^{xy}[x.y' + y(1)] = 1 + y'$$

$$x.e^{xy}.y' + ye^{xy} = 1 + y'$$

$$x.e^{xy}.y' - y' = 1 - ye^{xy}$$

$$y' = \frac{1 - ye^{xy}}{x.e^{xy} - 1} \qquad \underline{\text{Ans}}$$

b) $x^3 = (y - x^2)^2$

Solution:

$$\frac{d}{dx}(x^3) = \frac{d}{dx}[(y - x^2)^2]$$

$$3x^2 = 2(y - x^2)(y' - 2x)$$

$$3x^2 = 2(yy' - 2xy - x^2y' + 2x^3)$$

$$3x^2 + 4xy - 4x^3 = 2y'(y - x^2)$$

$$y' = \frac{3x^2 + 4xy - 4x^3}{2(y - x^2)}$$

Then find the slope of the curve $x^3 = (y - x^2)^2$ at (1,2)

$$y'\Big|_{(1,2)} = \frac{3(1)^2 + 4(1)(2) - 4(1)^3}{2[2 - (1)^2]} = \frac{7}{2} \qquad \underline{\text{Ans}}$$

Example 3:

If $q - p = \ln q + \ln p$, find $\frac{dq}{dp}$, the rate of change of q with respect to p.

Solution:

$$\frac{d}{dp}(q) - \frac{d}{dp}(p) = \frac{d}{dp}(\ln q) + \frac{d}{dp}(\ln p)$$

$$q' - 1 = \frac{1}{q}q' + \frac{1}{p}$$

$$q' - \frac{1}{q}q' = 1 + \frac{1}{p}$$

$$q'(1 - \frac{1}{q}) = 1 + \frac{1}{p}$$

$$q' = \frac{1 + \frac{1}{p}}{1 - \frac{1}{q}} \quad \text{or} \quad \frac{P+1}{p} \cdot \frac{q}{q-1}$$

$$= \frac{q(p+1)}{p(q-1)}$$

Ans

EXERCISE: 8 - 6

In Problems 1-22 find dy/dx by implicit differentiation.

1. $x^2 \pm 4y^2 = 4.$
2. $3x^2 + 6y^2 = 1.$
3. $xy = 4.$
4. $x + xy - 2 = 0.$
5. $xy - y - 4x = 5.$
6. $x^2 + y^2 = 2xy + 3.$
7. $x^3 + y^3 - 12xy = 0.$
8. $2x^2 - 3y^2 = 4.$
9. $x^{3/4} + y^{3/4} = 7.$
10. $y^3 = 4x.$
11. $3y^4 - 5x = 0.$
12. $x^{1/5} + y^{1/5} = 4.$
13. $\sqrt{x} + \sqrt{y} = 3.$
14. $2x^3 + 3xy + y^3 = 0.$
15. $x = \sqrt{y} + \sqrt[3]{y}.$
16. $x^3y^3 + x = 9.$
17. $3x^2y^3 - x + y = 25.$
18. $y^2 + y = \ln x.$
19. $y \ln x = xe^y.$
20. $\ln(xy) + x = 4.$
21. $xe^y + y = 4.$
22. $ax^2 - by^2 = c.$
23. If $x + xy + y^2 = 7$, find y' at $(1, 2)$.
24. Find the slope of the curve $4x^2 + 9y^2 = 1$ at the point $(0, \frac{1}{3})$; at the point (x_0, y_0).
25. Find an equation of the tangent line to the curve $x^3 + y^2 = 3$ at the point $(-1, 2)$.

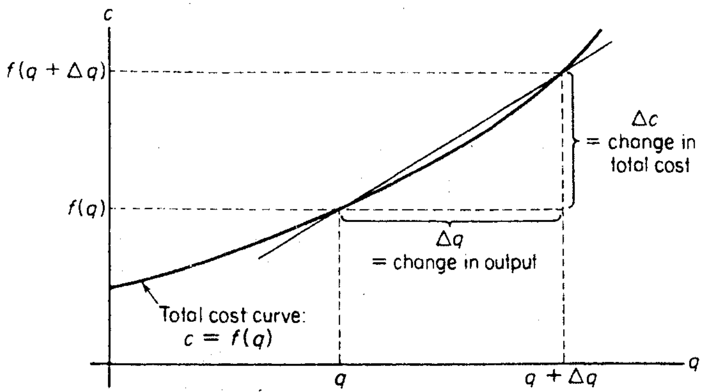
8.4 The Derivative as a Rate of Change:

To denote the change in a variable such as x, the symbol Δx (read 'delta x') is commonly used.

For example, if x changes from $x = 1$ to $x = 3$, then the change is $x = 3 - 1 = 2$. The new value of x can be written as $1 + \Delta x$. Similarly, if q increased by Δq, the new value is $q + \Delta q$. Δ – notation will be used in the following discussion.

Let $c = f(q)$ be the total cost function.
q = units of product.

The manufacturer produces q units at a total cost of f(q)



If the production level is increased by Δq units, then the total cost is $= f(q + \Delta q)$

The average cost per unit for these Δq additional units is:-

$$\frac{\textit{change in total cost c}}{\textit{change in output q}} = \frac{f(q + \Delta q) - f(q)}{\Delta q}$$

Let c = change in total cost

$$\therefore \frac{\Delta c}{\Delta q} = \frac{f(q + \Delta q) - f(q)}{\Delta q}$$

which is called the average rate of change of cost c with respect to output q over the interval $[q, q + \Delta q]$

Example:

If $c = f(q) = 0.1q^2 + 3$, where c is in dollars and q is in pounds the cost of 4 lb. is

$$f(4) = 0.1\,(4)^2 + 3 = 4.6$$

If output increases by 2 lb($\Delta q = 2$), then the new level of production is $= q + \Delta q = 2 + 4 = 6$

$$f(q) = 0.1q^2 + 3$$

$$f(q+\Delta q) = f(4+2) = f(6)$$

$$f(6) = 0.1(6)^2 + 3 = 6.6$$

$$\therefore \quad \frac{\Delta c}{\Delta q} = \frac{f(q + \Delta q) - f(q)}{\Delta q}$$

$$= \frac{f(6) - f(4)}{\Delta q} = \frac{6.6 - 4.6}{2} = \$1 \text{ per lb.}$$

This means that the average cost per lb of the additional output on the interval [4,6] is \$1 per lb.

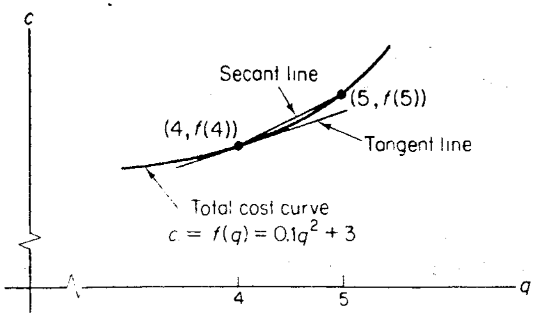
CHANGE IN OUTPUT Δq	INTERVAL $[4, 4 + \Delta q]$	AVERAGE COST PER LB OF ADDITIONAL OUTPUT $\Delta c/\Delta q$
2	[4, 6]	1
1	[4, 5]	.90
.1	[4, 4.1]	.81
.01	[4, 4.01]	.801

Similarly, as shown in the table above, for changes in output of 1, 0.1, 0.01, the corresponding results are obtained. As $\Delta q \to 0$, then $\Delta c/\Delta q \to 0.80$. This would indicate that for a small increase in output above 4 lb, the cost per lb. of that additional output is approxomately \$0.80.

$\therefore \lim_{\Delta q \to 0} \frac{\Delta c}{\Delta q}$ is the instantaneous rate of change of cost c with respect to output q when q = 4. In general, for any cost function $c = f(q)$, $\lim_{\Delta q \to 0} \frac{\Delta c}{\Delta q}$ is the *instantaneous rate of change of c with respect to q.* This limit is also called *marginal cost.*

now $\frac{\Delta c}{\Delta q}$ is the slope of a secant line

This if $\Delta q \to 0$, then $\frac{\Delta c}{\Delta q}$ approaches the slope of a tangent line; that is, $\lim_{\Delta q \to 0} \frac{\Delta c}{\Delta q} = \frac{dc}{dq}$



$$\therefore \frac{dc}{dq} = \left\{ \begin{array}{c} \text{instantaneous rate} \\ \text{of change of } c \\ \text{with respect to } q \end{array} \right\} = \text{marginal cost.}$$

Marginal cost: is the approximate change in cost resulting from one additional unit of output.

Example 1:

If the cost function $c = f(q) = 0.1q^2 + 3$, find the marginal cost when 4 1b are produced.

Solution:

$$c = 0.1q^2 + 3$$

$$\frac{dc}{dq} = 0.2q$$

To find marginal cost when 4 lb are produced, we evaluate $\frac{dc}{dq}$ when $q = 4$

$$\left.\frac{dc}{dq}\right|_{q=4} = 0.2q = 0.2(4)$$

$$= \$0.80 \qquad \underline{\text{Ans}}$$

In general, the rate of change applies not only to cost functions, but also to any function $y = f(x)$

$$\frac{dy}{dx} = \begin{cases} \text{Instantaneous rate of} \\ \text{change of y with respect to x} \end{cases}$$

Example 2.

Find the (instantaneous) rate of change of $y = x^4$ with respect to x. Evaluate when $x = 2$ and when $x = -1$.

The rate of change is given by dy/dx:

$$\frac{dy}{dx} = 4x^3$$

Thus the rate at which x^4 changes with respect to x is $4x^3$. When $x = 2$, the $dy/dx = 4(2)^3 = 32$. This means that y is increasing 32 times as fast as x does. When $x = -1$, then $dy/dx = 4(-1)^3 = -4$. The significance of the minus sign on -4 is that y is decreasing 4 times as fast as x increases.

Example 3:

Let $p = 100 - q^2$ be the demand function for a manufacturer's product. Find the rate of change of price p per unit with respect to quantity q. How fast is the price changing with respect to q when $q = 5$?

The rate of change of p with respect to q is dp/dq.

$$\frac{dp}{dq} = \frac{d}{dq}(100 - q^2) = -2q$$

Thus,

$$\left.\frac{dp}{dq}\right|_{q=5} = -2(5) = -10$$

This means that when units are demanded, an increase of one extra unit demanded will decrease the price per unit that consumers are willing to pay be approximately \$10.

If c is the total cost of producing q units of a product, then the average cost per unit, $\bar{c}$, for producing q units is

$$\bar{c} = \frac{c}{q}$$

For example, if the total cost of 20 units is \$100, then the average cost per unit is $\bar{c} = 100/20 = \$5$. By multiplying both sides of Eq. (2) by q, we have

$$c = q\bar{c}$$

That is, total cost is the product of the number of unit produced and the average cost per unit.

Example 4:

$$\textit{If} \qquad \bar{c} = .0001q^2 - .02q + 5 + \frac{5000}{q}$$

is a manufacturer's average cost equation, find the marginal cost function. What is the marginal cost when 50 units are produced?

We first find total cost c. Since $c = q\bar{c}$, then

$$c = q\bar{c}$$

$$= q\left(.0001q^2 - .02q + 5 + \frac{5000}{q}\right).$$

$$c = .0001q^3 - .02q^2 + 5q + 5000$$

Differentiating c, we have the marginal cost function:

$$\frac{dc}{dq} = .001(3q^2) - .02(2q) + 5(1) + 0$$

$$= .0003q^2 - .04q + 5$$

The marginal cost when 50 units are produced is

$$\left.\frac{dc}{dq}\right|_{q=50} = .0003(50)^2 - .04(50) + 5 = 3.75$$

Let us interpret this result. If c is in dollars and production is increased by one unit from q = 50 to q = 51, then the cost of the additional unit is approximately \$3.75. If production is increased by $\frac{1}{3}$ unit from q = 50, then the cost of the additional output is approximately $(\frac{1}{3})$ (3.75) = \$1.25.

Suppose r = f(q) is the total revenue function of a manufacturer. The equation r = f(q) states that the total dollar value received for selling q units of the product is r. The *marginal revenue* is defined as the rate of change of the total dollar value received with respect to the total number of units sold. Hence, marginal revenue is merely the derivative of r with respect to q.

$$\text{marginal revenue} = \frac{dr}{dq}$$

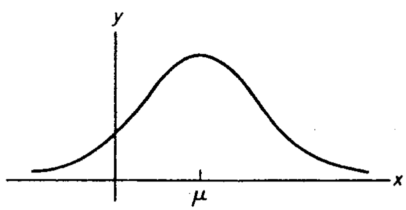
Marginal revenue indicates the rate at which revenue changes with respect to units sold. We interpret it as the approximate change in revenue that results from selling one additional unit of output.

Example 5:

An inportant function used on economic and business decisions is the normal distribution density function

$$y = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(x-\mu)/\sigma]^2}$$

where σ (a Greek letter read 'sigma') and μ (a Greek letter read 'mu') are constants. Its graph, called the normal curve, is 'bell-shaped'. Determine the rate of



change of y with respect to x when x = μ

$$\frac{dy}{dx} = \frac{1}{\sigma\sqrt{2\pi}}[e^{-(1/2)[(x-\mu)/\sigma]^2}]\left[-\frac{1}{2}(2)\left(\frac{x-\mu}{\sigma}\right)\left(\frac{1}{\sigma}\right)\right].$$

Evaluating dy/dx when $x = \mu$, we obtain

$$\left.\frac{dy}{dx}\right|_{x=\mu} = 0.$$

Marginal Revenue: is defined as the rate of change of the total amount received with respect to the total number of units sold.

It $r = f(q)$ is the total revenue function

$$\text{marginal revenue} = \frac{dr}{dq}$$

Marginal revenue indicates the rate at which revenue changes with respect to units sold. It is the approximate change in revenue that results from selling one additional unit of output.

Example 1:

Suppose a manufacturer sells a product at \$5 per unit. If q units are sold, the total revenue is given by

$$r = 5q.$$

Thus the marginal revenue function is

$$\frac{dr}{dq} = \frac{d}{dq}(5q) = 5.$$

The marginal revenue when q = 10 is

$$\left.\frac{dr}{dq}\right|_{q=10} = 5.$$

Suppose $r = f(q) = 2q$ gives the total revenue r (in dollars) that a manufacturer receives for selling q units of his product. The rate of change of revenue with respect to number of units sold is

$$\frac{dr}{dq} = 2.$$

This means that revenue is changing at the rate of \$2 per unit, regardless of the number of units sold. Although this is valuable information, it may be more significant when compared to r itself. For example, if q = 50 then r = 2(50) = \$100. Thus the rate of change of revenue is 2/100 = .02 of r. On the other hand, if q = 5000 then

$r = 2(5000) = \$10,000$, so the rate of change of r is $2/10,000 = .0002$ of r. Although r changes at the same rate at each level, when compared to r itself this rate is relatively smaller when $r = 10,000$ than when $r = 100$. By considering the ratio

$$\frac{dr/dq}{r},$$

we have a means of comparing the rate of change of r with r itself. This ratio is called the relative rate of change of the revenue function $r = f(q)$. We have shown above that the relative rate of change when $q = 50$ is

$$\frac{dr/dq}{r} = \frac{2}{100} = .02,$$

and when $q = 5000$, it is

$$\frac{dr/dq}{r} = \frac{2}{10,000} = .0002.$$

By multiplying these relative rates by 100, we obtain so-called *percentage rates of change.* Thus the percentage rate of change when $q = 50$ is $(.02)(100) = 2$ percent; when $q = 500$ it is $(.0002)(100) = .02$ percent.

In general, for any function f we have the following definition.

DEFINITION:

The relative rate of change of $f(x)$ is

$$\frac{f'(x)}{f(x)}.$$

The percentage rate of change of $f(x)$ is

$$\frac{f'(x)}{f(x)} \cdot 100.$$

Example 2:

If the demand equation for a manufacturer's product is $p = 1000/(0 + 5)$, find the marginal revenue function and evaluate it when $q = 45$.

The revenue r received for selling q units is

$$\text{revenue} = (\text{price})(\text{quantity}),$$
$$r = pq.$$

Thus the revenue function is

$$r = \left(\frac{1000}{q+5}\right)q$$

or

$$r = \frac{1000q}{q+5}.$$

To find the marginal revenue function, all we must determine is dr/dq.

$$\frac{dr}{dq} = \frac{(q+5)\,D_q(1000q) - (1000q)\,D_q(q+5)}{(q+5)^2}$$

$$= \frac{(q+5)1000 - (1000q)(1)}{(q+5)^2} = \frac{5000}{(q+5)^2}.$$

When $q = 45$,

$$\frac{dr}{dq} = \frac{5000}{(45+5)^2} = \frac{5000}{2500} = 2.$$

This means that selling one additional unit beyond 45 results in approximately $2 more in revenue.

EXERCISE: 8 - 7

In Problems 1-6, find (a) the rate of change of y with respect to x, and (b) the relative rate of change of y. At the given value of x, find (c) the rate of change of y, (d) the relative rate of change of y, and (e) the percentage rate of change of y.

1. $y = f(x) = x + 4; \quad x = 5.$
2. $y = f(x) = 4 - 2x; \quad x = 3.$
3. $y = 3x^2 + 6; \quad x = 2.$
4. $y = 2 - x^2; \quad x = 0.$
5. $y = 8 - x^3; \quad x = 1.$
6. $y = x^2 + 3x - 4; \quad x = -1.$

In Problems 7-12, cost functions are given where c is the cost of producting q units of a product. In each case find the marginal cost function. What is the marginal cost at the given value(s) of q?

7. $c = 500 + 10q; \quad q = 100.$

8. $c = 5000 + 6q; \quad q = 36$

9. $c = .3q^2 + 2q + 850; \quad q = 3.$

10. $c = .1q^2 + 3q + 2; \quad q = 3$

11. $c = q^2 + 50q + 1000; \quad q = 15, q = 16, q = 17.$

12. $c = .03q^3 - .6q^2 + 4.5q + 7700; \quad q = 10, q = 20, q = 100.$

In Problems 13-16, c represents average cost, which is a function of the number q of units produced. Find the marginal cost function and the marginal cost for the indicated values of q.

13. $\bar{c} = .01q + 5 + \dfrac{500}{q}; \quad q = 50, q = 100.$

14. $\bar{c} = 2 + \dfrac{1000}{q}; \quad q = 25, q = 235.$

15. $\bar{c} = .00002q^2 - .01q + 6 + \dfrac{20{,}000}{q}; \quad q = 100, q = 500.$

16. $\bar{c} = .001q^2 - .3q + 40 + \dfrac{7000}{q}; \quad q = 10, q = 20.$

In Problems 17-20, r represents total revenue and is a function of the number q of units sold. Find the marginal revenue function and the marginal revenue for the indicated values of q.

17. $r = .7q; \quad q = 8, q = 100, q = 200.$

18. $r = q(15 - \frac{1}{30}q); \quad q = 5, q = 15, q = 150.$

19. $r = 250q + 45q^2 - q^3; \quad q = 5, q = 10, q = 25.$

20. $r = 2q(30 - .1q); \quad q = 10, q = 20.$

21. For the cost function $c = .2q^2 + 1.2q + 4$, how fast does c change with respect to q when $q = 5$? Determine the percentage rate of change of c with respect to q when $q = 5$.

22. For the cost function $c = .4q^2 + 4q + 5$, find the rate of change of c with respect to q when $q = 2$. Also, what is $\Delta c/\Delta q$ over the interval [2, 3]?

23. The total cost function for a hosiery mill is estimated by Dean:

$$c = -10{,}484.69 + 6.750q - .000328q^2,$$

where q is output in dozens of pairs and c is total cost in dollars. Find the marginal cost function and evaluate it when $q = 5000$.

24. The total cost function for an electric light and power plant is estimated by Nordin:

$$c = 32.07 - .79q + .02142q^2 - .0001q^3, \qquad 20 < q < 90$$

where q is eight-hour total output (as percentage of capacity) and c is total fuel cost in dollars. Find the marginal cost function and evaluate it when $q = 70$.

25. Find the marginal revenue function if the demand function is $p = 25/\ln(q+2)$.

26. A total cost function is given by $c = 25 \ln(q+1) + 12$. Find the marginal cost when $q = 6$.

For each of the demand equations in Problems 27 and 28, find the rate of change of price p with respect to quantity q. What is the rate of change for the indicated value of q?

27. $p = 15e^{-.001q}$; $q = 500$.

28. $p = 8e^{-3q/800}$; $q = 400$.

29. The population P of a city t years from now is given by $P = 20{,}000e^{.03t}$. Show that $dP/dt = kP$ where k is a constant. This means that the rate of change of population at any time is proportional to the population at that time.

In Problems 36 and 37, $\bar{c}$ is the average cost of producing q units of a product. Find the marginal cost function and the marginal cost for the given values of q.

30. $\bar{c} = (7000e^{q/700})/q$; $q = 350, q = 700$.

31. $\bar{c} = \dfrac{850}{q} + 4000\dfrac{e^{(2q+6)/800}}{q}$; $q = 97, q = 197$.

32. For a firm the daily output q on the t-th day of a production run is given by $q = 500(1 - e^{-.2t})$. Find the instantaneous rate of change of output q with respect to t on the tenth day.

33. For the normal density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

find $f'(0)$.

Marginal propensity to consume: is defined as the rate of change of consumption (c) with respect to income (I)

If C = f(I) is the consumption function

$\therefore$ Marginal propensity to consume $= \dfrac{dC}{dI}$

Marginal propensity to save: is defined as the rate of change of savings (S) with respect to income (I). (how fast savings change with respect to income)

Assume that savings (S) is the difference between income (I) and Consumption (C).

$$S = I - C.$$

$\therefore$ Marginal propensity to save $= \frac{dS}{dI}$

$$\frac{dS}{dI} = \frac{d}{dI}(I) - \frac{d}{dI}(C) = 1 - \frac{dC}{dI}.$$

Example:

If the consumption function is given by

$$C = \frac{5(2\sqrt{I^3} + 3)}{I + 10},$$

determine the marginal propensity to consume and the marginal propensity to save when I = 100.

$$\frac{dC}{dI} = \frac{(I + 10)\, D_I[5(2I^{3/2} + 3)] - 5(2\sqrt{I^3} + 3)\, D_I[I + 10]}{(I + 10)^2}$$

$$= \frac{(I + 10)[5(3I^{1/2})] - 5(2\sqrt{I^3} + 3)[1]}{(I + 10)^2}.$$

When $I = 100$ the marginal propensity to consume is

$$\left.\frac{dC}{dI}\right|_{I=100} = \frac{6485}{12{,}100} \approx .536.$$

The marginal propensity to save when I = 100 is 1 − .536 = .464. This means that if a current income of \$100 billion increases by \$1 billion, then the nation will consume approximately 53.6 percent (536/1000) and save 46.4 percent (464/1000) of that increase.

EXERCISE: 8 - 8

1. For the U.S. (1922–42), the consumption function is estimated by

$$C = .672I + 113.1.$$

Find the marginal propensity to consume.

2. Repeat Problem 59 if $C = .712I + 95.05$ for the U.S. for 1929–41.†

In Problems 3 - 6, each equation represents a consumption function. Find the marginal propensity to consume and the marginal propensity to save for the given value of I.

3. $C = 2 + 2\sqrt{I}$; $I = 9$.

4. $C = 6 + \frac{3I}{4} - \frac{\sqrt{I}}{3}$; $I = 25$.

5. $C = \dfrac{16\sqrt{I} + .8\sqrt{I^3} - .2I}{\sqrt{I} + 4}$; $I = 36$.

6. $C = \dfrac{20\sqrt{I} + .5\sqrt{I^3} - .4I}{\sqrt{I} + 5}$; $I = 100$.

Marginal revenue product:

It is approximately the change in revenue that results when a manufacturer hires an extra employee.

Suppose a manufacturer hires m employees who produce a total of q units of a product per day. We can think of q as a function of m. If r is the total revenue the manufacturer receives for selling the q units produced by the m employees, then r can be considered a function of m.

Total revenue is given by r = pq.

$\therefore \frac{dr}{dm} =$ marginal revenue product.

where p is the price per unit. Here p is function of q and is determined by the product's demand equation. By the product rule,

$$\frac{dr}{dm} = p\frac{d}{dm}(q) + q\frac{d}{dm}(p) = p\frac{dq}{dm} + q\frac{dp}{dm}.$$

But by the chain rule,

$$\frac{dp}{dm} = \frac{dp}{dq} \cdot \frac{dq}{dm}.$$

Therefore,

$$\frac{dr}{dm} = p\frac{dq}{dm} + q\frac{dp}{dq} \cdot \frac{dq}{dm}$$

or

$$\frac{dr}{dm} = \frac{dq}{dm}\left(p + q\frac{dp}{dq}\right).$$

The derivative dr/dm is called the **marginal revenue product.** It is approcimately the change in revenue that results when a manufacturer hires an extra employee.

Example:

A manufacturer determines that m employees will produce a total of q units of a product per day where $q = 10m^2/\sqrt{m^2 + 19}$. *If the demand equation for the product is* $p = 900/(q + 9)$, *determine the marginal revenue product when* $m = 9$.

Solution:

$$\frac{dq}{dm} = \frac{(m^2 + 19)^{1/2}\, D_m(10m^2) - (10m^2)\, D_m\left[(m^2 + 19)^{1/2}\right]}{\left[(m^2 + 19)^{1/2}\right]^2}$$

$$= \frac{(m^2 + 19)^{1/2}(20m) - 10m^2\left[(1/2)(m^2 + 19)^{-1/2}(2m)\right]}{m^2 + 19}$$

$$= \frac{10m(m^2 + 19)^{-1/2}[(m^2 + 19)(2) - m(m)]}{m^2 + 19}$$

$$= \frac{10m(m^2 + 38)}{(m^2 + 19)^{3/2}}.$$

Since $p = 900(q + 9)^{-1}$, then by the power rule,

$$\frac{dp}{dq} = 900\left[(-1)(q + 9)^{-2}(1)\right] = -\frac{900}{(q + 9)^2}.$$

Substituting into $\frac{dr}{dm}$, we have the marginal revenue product:

$$\frac{dr}{dm} = \frac{10m(m^2 + 38)}{(m^2 + 19)^{3/2}} \left[p + q\left(-\frac{900}{[q + 9]^2} \right) \right].$$

When m = 9, then q = 81 and p = 10. Thus

$$\left. \frac{dr}{dm} \right|_{m=9} = 10.71.$$

EXERCISE: 8 - 9

In Problems 1 - 4, q is the total number of units produced per day by m employees of a manufacturer, and p is the price per unit at which the q units are sold. In each case find the marginal revenue product for the given value of m.

1. $q = 2m, p = -.5q + 20; \quad m = 5.$

2. $q = (200m - m^2)/20, p = -.1q + 70; \quad m = 40.$

3. $q = 10m^2/\sqrt{m^2 + 9}\,, p = 525/(q + 3); \quad m = 4.$

4. $q = 100m/\sqrt{m^2 + 19}\,, p = 4500/(q + 10); \quad m = 9.$

5. Suppose $p = 100 - \sqrt{q^2 + 20}$ is a demand equation for a manufacturer's product.

   a) Find the rate of change of p with respect to q.
   b) Find the relative rate of change of p with respect to q.
   c) Find the marginal revenue function.

6. If p = c/q, where c is a constant, is the demand equation for a manufacturer's product, and q = f(m) defines a function that gives the total number of units produced per day by m employees, show that the marginal revenue product is always zero.

7. Suppose the cost c of producing q units of a product is given by $c = 4000 + 10q + .1q^2$ If the price per unit p is given by the equation q = 800 – 2.5q, use the chain rule to find the rate of change of cost with respect to price per unit when p = 80.

8. Suppose that for a certain group of 20,000 births, the number l_x of people surviving to age x years is

$$l_x = 2000\sqrt{100 - x}, \qquad 0 < x < 100.$$

a) Find the rate of change of l_x with respect to x and, evaluate your answer for x = 36.
b) Find the relative rate of change of l_x when x = 36.

In Problems 9 and 10, each equation represents a consumption function. Find the marginal propensity to consume and the marginal propensity to save for the given value of I.

9. $C = \dfrac{20\sqrt{I} + .5\sqrt{I^3} - .4I}{\sqrt{I + 100}}; \quad I = 100.$ 10. $C = \dfrac{5(2I + \sqrt{I + 9})}{\sqrt{I + 9}}; \quad I = 135.$

11. If the total cost function for a manufacturer is given by

$$c = \frac{5q^2}{\sqrt{q^2 + 3}} + 5000.$$

find the marginal cost function.

8.5 Higher-Order Derivatives:

Since the derivative of a function is itself a function, it too may be differentiated. When this is done, the result may also be differentiated. Continuing in this way, we obtain higher-order derivatives.

If y = f(x), then f'(x) is called the *first derivative* of f with respect to x. The derivative of f'(x), denoted f″(x), is called the *second derivative* of f with respect to x, etc. Some of the various ways in which higher-order derivatives may be written are given in table below.

first derivative:	y',	$f'(x)$,	$\dfrac{dy}{dx}$,	$\dfrac{d}{dx}[f(x)]$,	$D_x y$.
second derivative:	y'',	$f''(x)$,	$\dfrac{d^2y}{dx^2}$,	$\dfrac{d^2}{dx^2}[f(x)]$,	$D_x^2 y$.
third derivative:	y''',	$f'''(x)$,	$\dfrac{d^3y}{dx^3}$,	$\dfrac{d^3}{dx^3}[f(x)]$,	$D_x^3 y$.
fourth derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\dfrac{d^4y}{dx^4}$,	$\dfrac{d^4}{dx^4}[f(x)]$,	$D_x^4 y$.

Note: The symbol $D_x^2 y$ represents the second derivative of y. It is not the same as $[D_x y]^2$, the square of the first derivative of y.

$$D_x^2 y \neq [D_x y]^2.$$

Example 1:

a. *If* $y = 2x^4 + 6x^3 - 12x^2 + 6x - 2$, *find* y'''.

Differentiating y, we obtain

$$y' = 8x^3 + 18x^2 - 24x + 6.$$

Differentiating y', we obtain

$$y'' = 24x^2 + 36x - 24.$$

Differentiating y'', we obtain

$$y''' = 48x + 36.$$

b. *If* $f(x) = 7$, *find* $f''(x)$.

$$f'(x) = 0.$$

$$f''(x) = 0.$$

c. *If* $y = e^{x^2}$, *find* $\dfrac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = e^{x^2}(2x) = 2xe^{x^2}.$$

$$\frac{d^2y}{dx^2} = 2\left[x(e^{x^2})(2x) + e^{x^2}(1)\right]$$

$$= 2e^{x^2}(2x^2 + 1).$$

d. *If* $y = f(x) = \dfrac{x^2}{x + 4}$, *find* $\dfrac{d^2y}{dx^2}$ *and evaluate it when* $x = 4$.

$$\frac{dy}{dx} = \frac{(x + 4)(2x) - (x^2)(1)}{(x + 4)^2} = \frac{x^2 + 8x}{(x + 4)^2}.$$

$$\frac{d^2y}{dx^2} = \frac{(x + 4)^2(2x + 8) - (x^2 + 8x)(2)(x + 4)}{(x + 4)^4}$$

$$= \frac{(x + 4)[(x + 4)(2x + 8) - (x^2 + 8x)(2)]}{(x + 4)^4}$$

$$= \frac{32}{(x+4)^3}.$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=4} = \frac{1}{16}.$$

The second derivative evaluated at x = 4 can also denoted $f''(4)$ or $y''(4)$.

e. *If $f(x) = x \ln x$, find the rate of change of $f''(x)$.*

To find the rate of change of any function, we must find its derivative. Thus we want $D_x[f''(x)]$ which is $f'''(x)$.

$$f'(x) = x\left(\frac{1}{x}\right) + (\ln x)(1) = 1 + \ln x.$$

$$f''(x) = 0 + \frac{1}{x} = \frac{1}{x}.$$

$$f'''(x) = -\frac{1}{x^2}.$$

We shall now find a higher-order derivative by means of implicit differentiation. Keep in mind that we shall assume y to be a function of x.

Example 2:

a. *Find y'' if $x^2 + 4y^2 = 4$.*

$$x^2 + 4y^2 = 4.$$

Differentiating both sides with respect to x, we obtain

$$2x + 8yy' = 0,$$

$$y' = \frac{-x}{4y} \quad \text{.......................} (1)$$

$$y'' = \frac{4y\,D_x(-x) - (-x)\,D_x(4y)}{(4y)^2}$$

$$= \frac{4y(-1) - (-x)(4y')}{16y^2}.$$

$$= \frac{-4y + 4xy'}{16y^2}.$$

$$y'' = \frac{-y + xy'}{4y^2} \quad \text{.......................} (2)$$

Since $y' = \frac{-x}{4y}$ from Eq. (1), by substituting into Eq. (2) we have

$$y'' = \frac{-y + x\left(\frac{-x}{4y}\right)}{4y^2} = \frac{-4y^2 - x^2}{16y^3}$$

$$= -\frac{4y^2 + x^2}{16y^3}.$$

Since $x^2 + 4y^2 = 4$

$$y'' = -\frac{4}{16y^3} = -\frac{1}{4y^3}$$

b) *Find y'' if $y^2 = e^{x+y}$*

$$y^2 = e^{x+y}$$

$$2yy' = e^{x+y}(1+y')^*$$

* Explanation:

$$2yy' = e^{x+y}(1+y')$$

$$2yy' = y^2 + y^2 y'$$

$$2yy' - y^2 y' = y^2$$

$$y' = \frac{y}{2-y}$$

Solving for y', we obtain

$$y' = \frac{y}{2-y}$$

$$y'' = \frac{(2-y)\ y' - y(-y')}{(2-y)^2}$$

$$= \frac{2y'}{(2-y)^2}$$

Since $y' = \frac{y}{2-y}$,

$$y'' = \frac{2y}{(2-y)^3}$$

EXERCISE: 8 - 10

In Problems **1–20**, *find the indicated derivatives.*

1. $y = 4x^3 - 12x^2 + 6x + 2, y'''$.

2. $y = 2x^4 - 6x^2 + 7x - 2, y'''$.

3. $y = 7 - x, \dfrac{d^2y}{dx^2}$.

4. $y = -x - x^2, \dfrac{d^2y}{dx^2}$.

5. $y = x^3 + e^x, y^{(4)}$.

6. $f(q) = \ln q, f'''(q)$.

7. $f(x) = x^2 \ln x, D_x^2[f(x)]$.

8. $y = \dfrac{1}{x}, y'''$.

9. $f(p) = \dfrac{1}{6p^3}, f'''(p)$.

10. $f(x) = \sqrt{x}, D_x^2[f(x)]$.

11. $f(r) = \sqrt{1 - r}, f''(r)$.

12. $y = e^{-4x^2}, y''$.

13. $y = \dfrac{1}{5x - 6}, \dfrac{d^2y}{dx^2}$.

14. $y = (2x + 1)^4, y''$.

15. $y = \dfrac{x + 1}{x - 1}, y''$.

16. $y = 2x^{1/2} + (2x)^{1/2}, y''$.

17. $y = \ln [x(x + 1)], y''$.

18. $y = \ln \dfrac{(2x - 3)(4x - 5)}{x + 3}, y''$.

19. $f(z) = z^2 e^z, f''(z)$.

20. $y = \dfrac{x}{e^x}, \dfrac{d^2y}{dx^2}$.

In Problems **21–30**, *find* y''.

21. $x^2 + 4y^2 - 16 = 0$.

22. $x^2 - y^2 = 16$.

23. $y^2 = 4x$.

24. $4x^2 + 3y^2 = 4$.

25. $\sqrt{x} + 4\sqrt{y} = 4$.

26. $y^2 - 6xy = 4$.

27. $xy + y - x = 4$.

28. $xy + y^2 = 1$.

29. $y^2 = e^{x+y}$.

30. $e^x - e^y = x^2 + y^2$.

31. Find the rate of change of $f'(x)$ if $f(x) = (5x - 3)^4$.

32. Find the rate of change of $f''(x)$ if $f(x) = 6\sqrt{x} + \dfrac{1}{6\sqrt{x}}$.

33. If $c = .3q^2 + 2q + 850$ is a cost function, how fast is marginal cost changing when $q = 100$?

34. If $p = 1000 - 45q - q^2$ is a demand equation, how fast is marginal revenue changing when $q = 10$?

REVIEW PROBLEMS: 8 - 11

In Problems 1–54, *differentiate.*

1. $y = 6^3$.

2. $y = x$.

3. $y = 7x^4 - 6x^3 + 5x^2 + 1$.

4. $y = \sqrt{x+3}$.

5. $y = 2e^x + e^2 + e^{x^2}$.

6. $y = 1/x^3$.

7. $y = \dfrac{x^2+3}{5}$.

8. $y = \dfrac{1}{2x+1}$.

9. $f(r) = \ln(r^2 + 5r)$.

10. $y = e^{\ln x}$.

11. $y = (x^2 + 6x)(x^3 - 6x^2 + 4)$.

12. $y = (x^2+1)^{100}(x-6)$.

13. $f(x) = (2x^2 + 4x)^{100}$.

14. $y = 2^{7x^2}$.

15. $y = (8 + 2x)(x^2+1)^4$.

16. $f(t) = \log_6 \sqrt{t^2+1}$.

17. $y = \sqrt[3]{4x-1}$.

18. $y = \sqrt[3]{(1-3x^2)^2}$.

19. $y = e^x(x^2+2)$.

20. $f(w) = we^w + w^2$.

21. $f(z) = \dfrac{z^2-1}{z^2+1}$.

22. $y = \dfrac{x-5}{(x+2)^2}$.

23. $y = \dfrac{\ln x}{e^x}$.

24. $y = \dfrac{e^x + e^{-x}}{x^2}$.

25. $y = e^{x^2+4x+5}$.

26. $y = (2x)^{3/5} + e$.

27. $y = \dfrac{x(x+1)}{2x^2+3}$.

28. $g(z) = \dfrac{-7z}{(z-1)^{-1}}$.

29. $2xy + y^2 = 6$ (find y').

30. $y = (x-6)^4(x+4)^3(6-x)^2$.

31. $y = \sqrt{(x-6)(x+5)(9-x)}$.

32. $4x^2 - 9y^2 = 4$ (find y').

33. $f(q) = \ln[(q+1)^2(q+2)^3]$.

34. $y = x^{x^3}$.

35. $y = \dfrac{1}{\sqrt{1-x}}$.

36. $y = \sqrt{\dfrac{(x-2)(x+3)}{\sqrt{x-1}}}$.

37. $y = \log_2 (8x+5)^2$.

38. $y + xy + y^2 = 1$ (find y').

39. $y = (x+1)^{x+1}$.

40. $y = (x+2)^{\ln x}$.

41. $y = \dfrac{x^2+6}{\sqrt{x^2+5}}$.

42. $y = \dfrac{(x+3)^5}{x}$.

43. $x^2y^2 = 1$ (find y').

44. $f(x) = 5x\sqrt{1-2x}$.

45. $y = 2x^{-3/8} + (2x)^{-3/8}$.

46. $f(t) = e^{\sqrt{t}}$.

47. $f(l) = \ln(1 + l + l^2 + l^3)$.

48. $y = \sqrt{\frac{x}{2}} + \sqrt{\frac{2}{x}}$.

49. $y = (x^3 + 6x^2 + 9)^{3/5}$.

50. $y = (e + e^2)^0$.

51. $f(u) = \ln(u^2\sqrt{1-u})$.

52. $y = \frac{1+e^x}{1-e^x}$.

53. $y = \frac{(x^2+2)^{3/2}(x^2+9)^{4/9}}{(x^3+6x)^{4/11}}$.

54. $y = \frac{\ln x}{\sqrt{x}}$.

In Problems 55–62, find the indicated derivative at the given point. It is not necessary to simplify the derivative before substituting the coordinates.

55. $y = x^4 - 2x^3 + 6x, y''', (1, 5)$.

56. $y = x^2e^x, y''', (1, e)$.

57. $y = \frac{x}{\sqrt{x-1}}, y'', \left(5, \frac{5}{2}\right)$.

58. $y = \frac{2}{1-x}, y'', \left(-2, \frac{2}{3}\right)$.

59. $x + xy + y = 5, y'', (2, 1)$.

60. $xy + y^2 = 2, y'', (1, 1)$.

61. $y = \frac{4x}{x^2+4}, y'', (2, 1)$.

62. $y = (x+1)^3(x-1), y'', (-1, 0)$.

In Problems 63–68, find an equation of the tangent line to the curve at the point corresponding to the given value of x.

63. $y = x^2 - 6x + 4, x = 1$.

64. $y = -2x^3 + 6x + 1, x = 2$.

65. $y = e^x, x = \ln 2$.

66. $y = \frac{x}{1-x}, x = 3$.

67. $x^2 - y^2 = 9, x = 7, y > 0$.

68. $xy = 6, x = 1$.

69. If $r = q(20 - .1q)$ is a total revenue function, find the marginal revenue function

70. If $c = .0001q^3 - .02q^2 + 3q + 6000$ is a total cost function, find the marginal cost when $q = 100$.

71. If $C = 7 + .6I - .25\sqrt{I}$ is a consumption function, find the marginal propensity to consume and the marginal propensity to save when $I = 16$.

72. If $p = (q + 14)/(q + 4)$ is a demand equation, find the rate of change of price p with respect to quantity q.

73. If $p = -.5q + 450$ is a demand equation, find the marginal revenue function.

74. If $\bar{c} = (500/q)e^{q/300}$ is an average cost function, find the marginal cost function.

75. The total cost function for an electric light and power plant is estimated by

$$c = 16.68 + .125q + .00439q^2, \qquad 20 < q < 90$$

where q is eight-hour total output (as percentage of capacity) and c is total fuel cost in dollars. Find the marginal cost function and evaluate it when $q = 70$.

76. A manufacturer has determined that m employees will produce a total of q units per day where $q = m(50 - m)$. If his demand function is given by $p = -.01q + 9$, find the marginal revenue product when $m = 10$.

CHAPTER 9

APPLICATIONS OF DIFFERENTIATION

9.1 Intercepts and Symmetry:

In this section, we shall examine two features of the graph of an equation, namely, **intercepts and symmetry.**

A point where a graph intersects the x-axis is called an x-intercept of the graph and has the form (x, 0). A y-intercept is a point (0,y) where the graph intersects the y-axis.

Example 1:

Find the x – and y – intercepts of the graph of the equation $x^2 + y^2 = 25$.

Solution:

To determine the x-intercepts, put y = 0

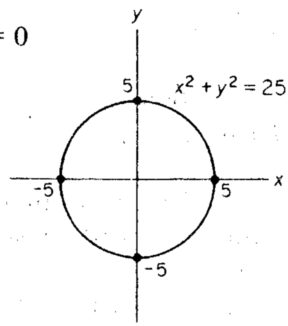
$$x^2 + 0^2 = 25.$$
$$x = \pm 5.$$

The x-intercepts are thus (5, 0) and (–5, 0) <u>Ans</u>

Similarly to find the y-intercepts, set x = 0

$$0^2 + y^2 = 25.$$
$$y = \pm 5.$$

Thus the y-intercepts are (0, 5) and (0, –5) <u>Ans</u>

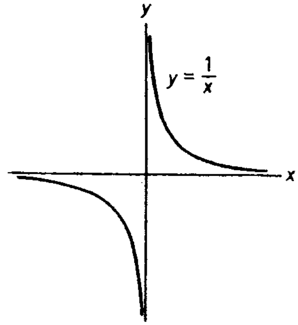


Example 2:

Find the x – and y – intercepts of $y = \frac{1}{x}$

Solution:

Since x cannot be 0, the graph has no y-intercept. If $y = 0$, then $0 = \frac{1}{x}$ and this equation has no solution. Thus no x-intercepts exist either <u>Ans</u>



Remark:

At times it may be quite difficult or even impossible to find intercepts. For example, the x-intercepts of the graph of $y - \sqrt{2}\, x^5 - 4x^2 - 7 = 0$ would be difficult to find, although the y-intercept is easily found to be (0, 7). In cases such as this, we settle for those intercepts that we can find conveiently.

Symmetry:

1. A graph is **symmetric with respect to the y-axis** if and only if (–x0, y0)

Example:

Show that the graph $y = x^2$ is symmetric with respect to the y-axis.

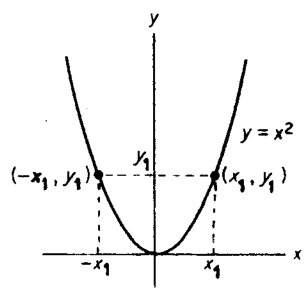
Solution:

Suppose (x_1, y_1) is any point on the graph of $y = x^2$ Then $y_1 = x_1^2$

To show that $y = x^2$ is symmetric with respect to the y-axis, we must show that $(-x_1, y_1)$ satisfies $y = x^2$

$$y_1 = (-x_1)^2$$

$$y_1 = x_1^2$$



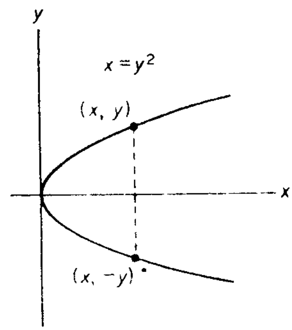
Thus the graph is symmetric with respect to the y-axis. Ans

Note: (x_1, y_1) can be any point on the graph.

2. A graph is **symmetric with respect to the x-axis** if and only if $(x, -y)$ lies on the graph when (x, y) does

Example:

In the case $x = y^2$



If (x, y) lies on the graph, then $(x, -y)$ also lies on it. We know that any point lies on the graph by substitute the values of the point into the equation of the graph. If the point satifies the equation, then it lies on it.

3. A graph is **symmetric with respect to the origin** if and only if $(-x, -y)$ lies on the graph when (x, y) does.

Example:

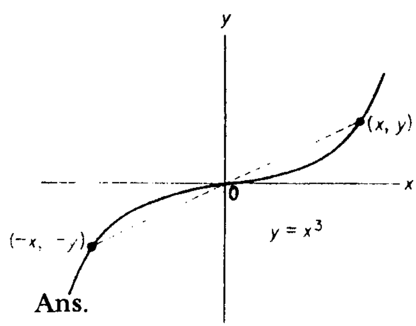
In case of $y = x^3$

If (x, y) is any point on the graph, then test for the point $(-x, -y)$

$$-y = (-x)^3$$

$$-y = -x^3$$

$$y = x^3$$



The point $(-x, -y)$ satisfies the condition. It, therefore, lies on the graph. Hence there is a symmetry with respect to the origin. Ans.

Example:

Determine whether or not the graph of $y = f(x) = 1 - x^4$ is symmetric with respect to the x-axis, the y-axis, or the origin. Then find the intercepts and sketch the graph.

Solution:

a) With respect to the x-axis, replace y by -y

$$y = 1 - x^4$$
$$-y = 1 - x^4$$
$$y = -1 + x^4$$

which is not equivalent to the given equation. Thus the graph is not symmetric with respect to the x-axis.

b) With respect to the y-axis, replace x by -x

$$y = 1 - x^4$$
$$y = 1 - (-x)^4$$
$$y = 1 - x^4$$

which is equivalent to the given equation. Thus the graph is symmetric with respect to the y-axis,

c) With respect to the origin, replace (x,y) by (-x, -y)

$$y = 1 - x^4$$
$$-y = 1 - (-x)^4$$
$$-y = 1 - x^4$$
$$y = -1 + x^4$$

which is not equivalent to the given equation. Thus the graph is not symmetric with respect to the origin.

d) To find the intercepts:

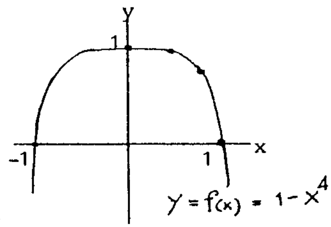
For x-intercepts, put $y = 0$

$$0 = 1 - x^4$$
$$(1 - x^2)(1 + x^2) = 0$$
$$x = \pm 1$$

The x-intercepts are $(1,0)$ and $(-1, 0)$. For y-intercepts, put $x = 0$

$$y = 1 - 0^4$$
$$y = 1$$

The y-intercept is $(0,1)$



EXERCISE: 9 - 1

In Problems 1 - 16 find the x-and y-intercepts of the graphs of the equations. Also determine whether or not the graphs are symmetric with respect to the x-axis, the y-axis, or the origin. Do not sketch the grophs.

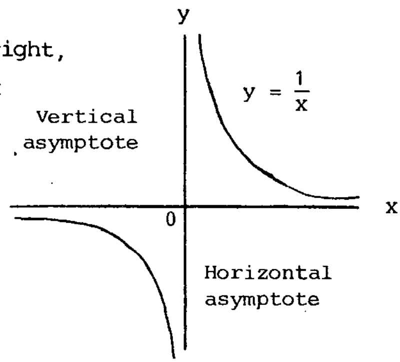
1. $y = 5x$.
2. $y = f(x) = x^2 - 4$.
3. $2x^2 + y^2 x^4 = 8 - y$.
4. $x = y^3$.
5. $4x^2 - 9y^2 = 36$.
6. $y = 7$.
7. $x = -2$.
8. $y = |2x| - 2$.
9. $x = -y^{-4}$.
10. $y = \sqrt{x^2 - 4}$.
11. $x - 4y - y^2 + 21 = 0$.
12. $x^3 - xy + y^2 = 0$.
13. $y = f(x) = x^3/(x^2 + 5)$.
14. $x^2 + xy + y^2 = 0$.
15. $e^{x^2 + y^2} - 5 = 0$.
16. $y = e^{x^2}$.

In Problems 17 - 24, find the x-and y-intercepts of the graphs of the equations. Also determine whether or not the graphs are symmetric with respect to the x-axis, the y-axis, or the origin. Then sketch the graphs.

17. $|x| - |y| = 0$.
18. $x = y^4$.
19. $2x + y^2 = 4$.
20. $y = x - x^3$.
21. $y = f(x) = x^3 - 4x$.
22. $x^2 + y^2 = 16$.
23. $4x^2 + y^2 = 16$.
24. $x^2 - y^2 = 1$.

9.2 Asymptotes:

Consider $y = \frac{1}{x}$



As x approaches zero from the right, $\frac{1}{x}$ becomes positively infinite; as x approaches zero from the left, $\frac{1}{x}$ becomes negatively infinite.

In terms of limits,

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \quad \text{and}$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

The line x = 0 (the y-axis) is *vertical asymptote* of the graph $y=\frac{1}{x}$.
This means it is a vertical line near which the graph rises or fall without bound.

On the other hand, as x approaches ∞, as well as $-\infty$, $\frac{1}{x}$ approaches 0.

That is,

$$\lim_{x\to\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0$$

The line y = 0 (the x-axis) is a *horizontal asymptote* of the graph $y = \frac{1}{x}$.

Definition:

a) *The line $x = a$ is a* <u>*vertical asymptote*</u> *of the graph of $f(x)$ if and only if*

$$\lim_{x \to a^+} f(x) = \infty \text{ (or } -\infty)$$

$$\text{or} \quad \lim_{x \to a^-} f(x) = \infty \text{ (or } -\infty)$$

b) *The line $y = a$ is a* <u>*horizontal asymptote*</u> *of the graph of $f(x)$ if and only if*

$$\lim_{x \to \infty} f(x) = a$$

$$\text{or} \quad \lim_{x \to -\infty} f(x) = a$$

** *Note that if $x = a$ is a vertical asymptote, the function cannot be continuous at a — in fact, it has an infinite discontinuity at a.*

Example 1:

Determine the horizontal and vertical asymptotes for the graph of $y = \dfrac{1}{x - 2} + 3$

Solution:

To test for horizontal asymptotes,

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left(\frac{1}{x - 2}\right) + 3$$

$$= \lim_{x \to \infty} 3 = 3$$

Thus the line $y = 3$ is a horizontal asymptote.

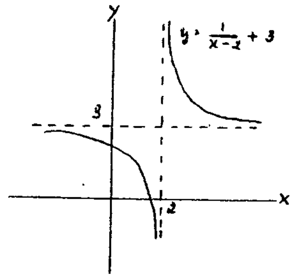
Also,

$$\lim_{x \to -\infty} \left(\frac{1}{x - 2} + 3\right) = 0 + 3 = 3$$

Hence the graph will move near the line $y = 3$ both as $x \to \infty$ and $x \to -\infty$

To determine vertical asymptotes, make $\frac{1}{x - 2} = \infty$ and this can be obtained by putting the denominator $(x - 2) = 0$

Thus $x = 2$



$$\text{Therefore} \quad \lim_{x \to 2^+} \left(\frac{1}{x - 2} + 3\right) = \infty$$

Hence, the line $x = 2$ is a vertical asymptote.
If x is slightly less than 2, then $x - 2$ is very close to zero but negative.

Thus $\frac{1}{x - 2}$ is "very negative" and so

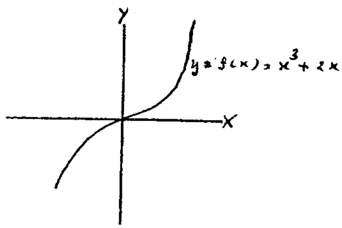
$$\lim_{x \to 2^-} \left(\frac{1}{x - 2} + 3\right) = -\infty$$

Thus the function increases without bound as $x \to 2^+$ and decreases without bound ad $x \to 2^-$

Example 2:

Determine the horizontal and vertical asymptotes for $y = f(x) = x^3 + 2x$.

Solution:



Testing for horizontal asymptotes, we have

$$\lim_{x \to \infty} (x^3 + 2x) = \infty \quad \text{and}$$

$$\lim_{x \to -\infty} (x^3 + 2x) = -\infty$$

Thus the graph does not approach any value as $x \to \infty$ or $x \to -\infty$. Hence, there are no horizontal asymptotes. <u>Ans</u>

Testing for vertical asymptotes, since $y = x^3 + 2x$ is a polynomial function which is continous everywhere. Thus the graph has no vertical asymptotes. <u>Ans</u>

Example 3:

Find the horizontal and vertical asymptotes of

$$y = e^x - 1$$

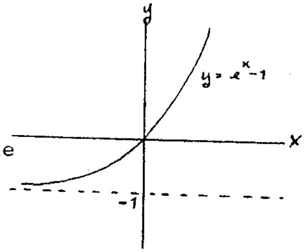
Solution:

For horizontal asymptotes, let $x \to \infty$

Therefore, $\lim_{x \to \infty} (e^x - 1) = \infty$

Thus the graph does not approach any value
However, as $x \to -\infty$

$$\lim_{x \to -\infty} (e^x - 1) = 0 - 1 = -1$$



Therefore the line $y = -1$ is a horizontal asymptote <u>Ans</u>

For vertical asymptotes, there is none since $e^x - 1$ cannot increase without bound around any fixed value of x. <u>Ans</u>

Example 4:

Sketch the graph of $y = \dfrac{x^2}{x^2 - 1}$ *with the aid of intercepts, symmetry, and asymptotes.*

Solution: **Intercepts**

To find x-intercepts, put $y = 0$

Therefore, $0 = \dfrac{x^2}{x^2 - 1}$

$\therefore x = 0$

To find y-intercepts, set $x = 0$. Then,

$$y = \frac{0}{0 - 1} = 0$$

The x-intercept, as well as the y-intercept, is = (0,0)

Symetry:

a) Testing for y-axis symmetry, replace x by $-x$

$$\therefore y = \frac{(-x)^2}{(-x)^2 - 1} = \frac{x^2}{x^2 - 1}$$

The graph is symmetric with respect to the y-axis

b) With respect to the x-axis, since y is a function of x and is not the zero function, then the graph is not symmetric with respect to the x-axis

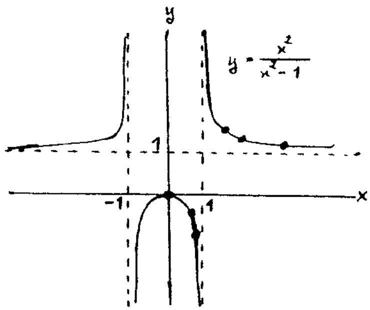
c) With regard to the origin:- Symmetry with respect to exactly one axis implies that the graph cannot be symmetric with respect to the origin.

Asymptotes:

Testing for horizontal asymptote,

$$\lim_{x \to \infty} \left(\frac{x^2}{x^2 - 1} \right) = 1$$

and $$\lim_{x \to -\infty} \frac{x^2}{x^2 - 1} = 1$$



As $x \to \infty$ and $x \to -\infty$, the graph approaches the line $y = 1$, a horizontal asymptote.

Testing for vertical asymptote, set the denominator equal to zero

$$\therefore \quad x^2 - 1 = 0$$

$$(x + 1)(x - 1) = 0$$

$$\text{or} \quad x = 1, -1$$

Therefore, $\lim_{x \to 1^-} \frac{x^2}{x^2 - 1} = -\infty$ and $\lim_{x \to 1^+} \frac{x^2}{x^2 - 1} = \infty$

By Symmetry, $\lim_{x \to -1^+} \frac{x^2}{x^2 - 1} = -\infty$ and $\lim_{x \to -1^-} \frac{x^2}{x^2 - 1} = \infty$

Thus the lines $x = 1$ and $x = -1$ are vertical asymptotes.

In general,

1. For vertical asymptotes of a function $y = f(x)$, they are at those values of x for which a denominator of $f(x) = 0$.
2. Any polynomial function that is not a constant has neither horizontal nor vertical esymptotes.
3. If y is a function of x and it is not the zero function, then the graph is not symmetric with respect to the x-axis

EXERCISE: 9 - 2

In Problems 1 - 14, determine the horizontal and vertical asymptotes of the graphs of the functions. Do not sketch the graphs.

1. $y = \frac{4}{x}$.

2. $y = -\frac{4}{x^2}$.

3. $y = \frac{4}{x-6} + 4$.

4. $y = \frac{2x+1}{2x-1}$.

5. $y = x^3 - 5x + 8$.

6. $y = \frac{x^3}{x^2 - 9}$.

7. $f(x) = \frac{x-1}{2x+3}$.

8. $f(x) = \frac{x^2}{5}$.

9. $f(x) = \frac{2x^2}{x^2 + x - 6}$.

10. $f(x) = \frac{5}{2x^2 - 9x + 4}$.

11. $f(x) = \sqrt[3]{x^2}$.

12. $y = \frac{x^2(x^2-9)}{x^2}$.

13. $y = 2e^{x+2} + 4$.

14. $f(x) = e^{x^3}$.

In Problems 15 - 26, find the x - and y-intercepts of the graphs of the function; determine whether the graphs are symmetric with respect to the x-axis, y-axis, or origin; determine horizontal and vertical asymptotes; sketch the graphs.

15. $y = \frac{3}{x-1}$.

16. $y = \frac{x}{4-x}$.

17. $f(x) = \frac{8}{x^3}$.

18. $f(x) = \frac{1}{x^4}$.

19. $f(x) = \frac{1}{x^2-1}$.

20. $f(x) = \frac{x^2}{x^2-4}$.

21. $y = \frac{x^2-1}{x^2-4}$.

22. $y = \frac{x^3-x}{x}$.

23. $y = \frac{x^2(x^2-9)}{x^2}$.

24. $f(x) = \begin{cases} 1/x, & \text{if } x > 0. \\ (x+1)/x, & \text{if } x < 0. \end{cases}$

25. $y = 3 - e^{2x}$.

26. $y = e^{-x} - 1$.

27. In discussing the time pattern of purchasing, Mantell and Sing use the curve

$$y = \frac{x}{a + bx}$$

as a mathematical model. They claim that $y = \frac{1}{b}$ is an asymptote. Verify this.

28. Sketch the graphs of $y = 6 - 3e^{-x}$ and $y = 6 + 3e^{-x}$. Show that they are both asymptotic to the same line. What is the equation of this line?

29. For a new product the yearly number of thousand packages sold, y, after t years from its introduction is predicted to be

$$y = f(t) = 150 - 76e^{-t}.$$

Show that $y = 150$ is a horizontal asymptote of the graph. This shows that after the product is established with consumers, the market tends to be constant.

9.3 Maxima and Minima:

In curve sketching, plotting points at random usually is not good enough to properly determine a curve's shape. For example, the points (-1, 0), (0, -1), and (1, 0) satisfy the equation $y = (x+1)^3(x-1)$. You might hastily conchede that its graph should appear as in Fig. 1, but in fact the actual shape is given in Fig.2.

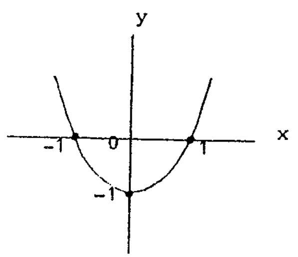


Fig. 1.

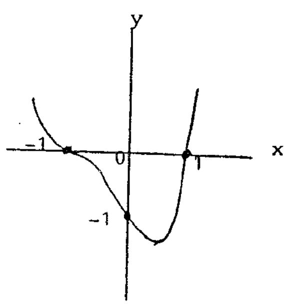
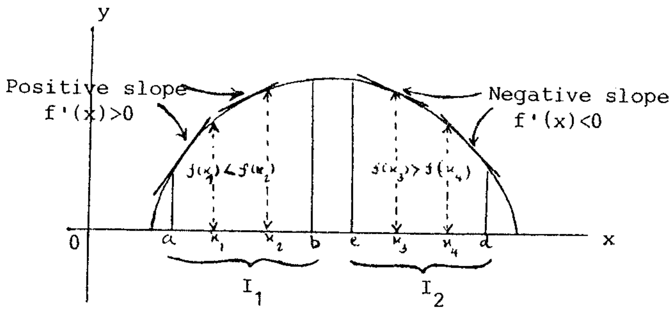


Fig. 2.

In this section, we shall explore the role that differentiation plays in analyzing a function so that we may determine the true shape and behavior of its graph.

Consider the graph $y = f(x)$ as shown in the figure below. As x increases (goes from left to right) on the interval I_1 determined by a and b, the corresponding values of y



increase and the curve is rising. Symbolically, if x_1 and x_2 are any two points in I_1 such that $x_1 < x_2$, then $f(x_1) < f(x_2)$ and is said to be an increasing function on I_1. Moreover, for this portion of the curve, any tangent line will have a positive slope, and thus $f'(x)$ must be positive for all x in I_1. On the other hand, as x increases on the interval I_2, determined by c and d, the curve is falling. Here $x_3 < x_4$ implies $f(x_3) > f(x_4)$ and f is said to be a decreasing function on I_2. In this case any tangent line has a negative slope, and thus $f'(x) < 0$ for all x in I_2.

- In summary, a function f is an increasing [or decreasing] function on the interval I if and only if for any two points x_1, x_2 in I such that $x_1 < x_2$, then $f(x_1) < f(x_2)$ [or $f(x_1) > f(x_2)$]
- And If $f'(x) > 0$ on an interval I, then f is an increasing function on I. If $f'(x) < 0$ on I, then f is a decreasing function on I.

Example:

Find the intervals on which $y = 18x - \frac{2x^3}{3}$ *is increasing or decreasing.*

SOLUTION:

$$y = 18x - \frac{2x^3}{3}$$

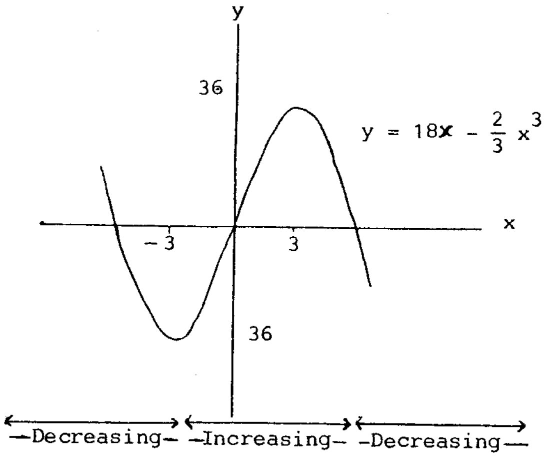
$$y' = 18 - 2x^2 = 2(9 - x^2)$$

$$= 2(3 + x)(3 - x)$$

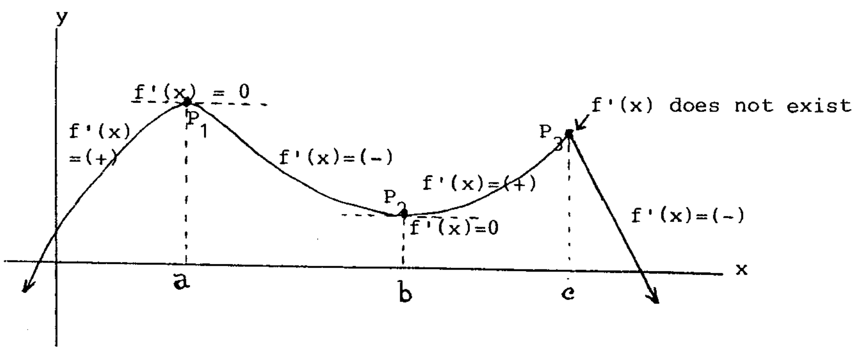
Now we can find the sign of f'(x) by considering the intervals determined by the roots of $2(3 + x)(3 - X) = 0$, namely 3 and -3. In each interval the sign of f'(x) is determined by the signs of its factors:

if $x < -3$, then $f'(x) = 2(-)(+) = (-)$ and f is decreasing;
if $-3 < x < 3$, then $f'(x) = 2(+)(+) = (+)$ and f is increasing;
if $x > 3$, then $f'(x) = 2(+)(-) = (-)$ and f is decreasing.

Thus f is decreasing on $(-\infty, -3)$ and $(3, \infty)$, and is increasing on $(-3, 3)$ as shown in the figure below.



Consider the figure below:



P_1 is higher than any other "nearby" points on the curve-likewise for P_3. The point P_2 is lower than any other "nearly" points on the curve. Since P_1, P_2, and P_3 may not necessarily be the highest or lowest points on the entire curve, we simply say that f has a *relative maximum* at x = a and x = c, and a relative minimum at x =b.

Actually, there is an *absolute maximum* (highest point on the entire curve) at x = a but there is no *absolute minimum* (lowest point on the entire curve), since the curve is assumed to extend downward indefinitely. We define these new terms as follows:

Definition: "A function given by f(x) has a relative maxinum (minimum) at a point, x = a, if there is a small interval containing x = a such that f(a) is larger (smaller) than the value of the function at any other point in the interval."

"A function f has an absolute maximum (absolute minimum) at x = a if f(a) takes on its largest (smallest) value for all x in the domain f . (a relative maximum or a relative minimum can be termed as relative extremum).

9.3.1 To Find a Relative Extremum:

The following rule can be followed:

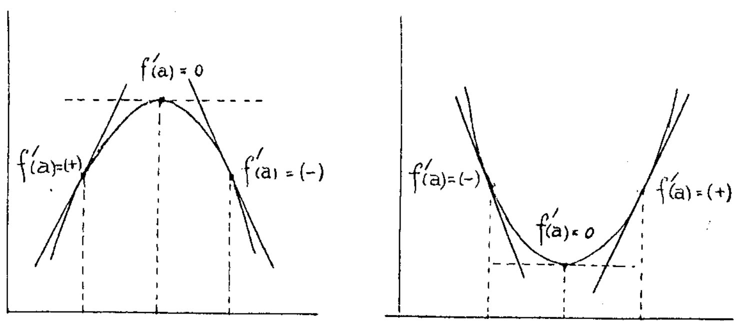
If f has a relative extremum at x = a,

then a) f'(a) = 0

or b) f'(a) is not defined.

Consider Case a) f'(a) = 0:

If x = a is in the domain of f and f'(x) changes from positive to negaive as x increases through a, then f has a relative maximum when x = a. If f'(x) changes from negative to positive as x increases through a, then f has a relative minimum when x = a.

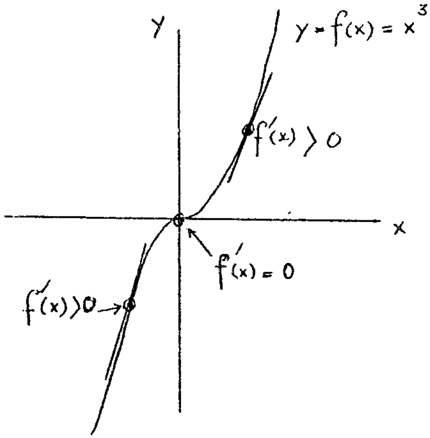


Case b) f′(a) is not defined:

If f'(a) 0 or f'(a) is not defined, x = a is called a *critical value* of f. If "a" is a critical value and is in the domain of f, then "a" is call a *critical point*.

Example:

$$y = f(x) = x^3$$
$$f'(x) = x^3$$
$$f'(x) = 0$$
$$\therefore x = 0.$$



Take note that not every critical value corresponds to a relative extremum.

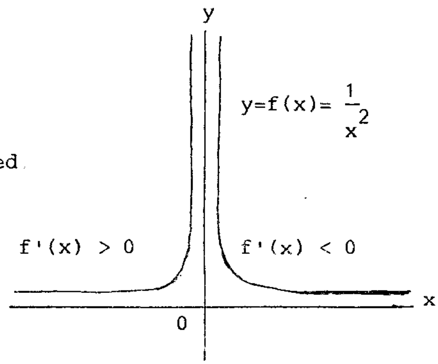
Consider $y = f(x) = \dfrac{1}{x^2}$

$$f'(x) = -\frac{2}{x^3}$$

When x = 0, f'(x) is not defined.

$\therefore$ 0 is a critical value.

If $x < 0$, then $y' > 0$
If $x > 0$, then $y' < 0$



Although a change in sign of y' occurs around x = 0, no relative maximum exists there since 0 is not in the domain of f. Nevertheless, this critical value is important in determining the intervals over which f is increasing or decreasing. Here f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. Therefore, critical value is only a candidate for a relative extremum. It may correspond to a relative maximum, a relative minimum, or neither.

Summary:

> *The necessary condition, also called the* <u>*first-order condition,*</u> *for a function, given by $f(x)$, to have a relative maximum or minimum at a point a, is that $f'(a) = 0$*

To find the relative extremum, follow the following steps:-

Step 1. Find $f'(x)$

2. Determine critical values

3. On the intervals suggested by the critical values, determine whether f is increasing $[f'(x) > 0]$ or decreasing $[f'(x) < 0]$.

4. For each critical value a in the domain of f, determine whether $f'(x)$ changes sign as x increases through $x = a$

   - There is a relative maximum when $x = a$, if $f'(x)$ changes from (+) to (−), and a relative minimum if $f(a)$ changes from (−) to (+).

   - If $f'(x)$ does not change sign, there is no relative maximum or minimum when $x = a$.

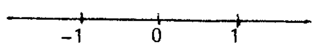
Example 1:

Find the relative maximum or relative minimum point of $y = f(x) = 3x - x^3 - 1$.

Solution:

$$f(x) = 3x - x^3 - 1$$

$$f'(x) = 3 - 3x^2$$



The first-order condition to have a relative maximum or relative minimum at a point "a", is that $f'(a) = 0$

$$\therefore \quad f'(x) = 3 - 3x^2 = 0$$

$$x = \pm 1$$

- Picking out a point in $f(x) = 3x - x^3 - 1$ to the left of $x = 1$, say $x = 0$ and test in $f'(x)$
  $f'(0) = 3 = +$, this indicates that the original function is increasing to the left of the point $x = 1$.

- Picking out a point to the right of x = 1, say x = 2,
  $\therefore$ $f'(2) = 3 - 3(4) = -9$ indicating that the original function is decreasing
  Therefore we have a relative maximum at x = 1.
  Check out x = -1 as an exercise.

Example 2:

If $y = f(x) = x + \dfrac{4}{x+1}$, use the first-derivative test to determine intervals on which f is increasing or decreasing and locate all relative extrema.

Solution:

1. $f'(x) = 1 - \dfrac{4}{(x+1)^2} = \dfrac{(x+1)^2 - 4}{(x+1)^2} = \dfrac{x^2 + 2x - 3}{(x+1)^2} = \dfrac{(x+3)(x-1)}{(x+1)^2}$

2. Setting $f'(x) = 0$ gives the critical values $x = -3, 1$. Since $f'(-1)$ does not exist, $x = -1$ is also a critical value.

3. There are four intervals to consider (Fig. 1).

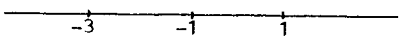


FIG. 1

if $x < -3$, then $f'(x) = \dfrac{(-)(-)}{(+)} = (+)$ and f is increasing;

if $-3 < x < -1$, then $f'(x) = \dfrac{(+)(-)}{(+)} = (-)$ and f is decreasing;

if $-1 < x < 1$, then $f'(x) = \dfrac{(+)(-)}{(+)} = (-)$ and f is decreasing;

if $x > 1$, then $f'(x) = \dfrac{(+)(+)}{(+)} = (+)$ and f is increasing (Fig. 2).

Thus f is increasing on $(-\infty, -3)$ and $(1, \infty)$ and is decreasing on $(-3, -1)$ and $(-1, 1)$.

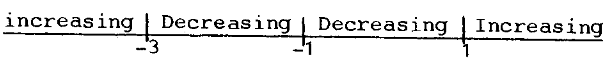


FIG. 2

4. When $x = -3$, there is a relative maximum since $f'(x)$ changes from (+) to (-). When $x = 1$, there is a relative minimum since $f'(x)$ changes from (-) to (+). We ignore $x = -1$ since -1 is not in the domain of f (Fig. 3).

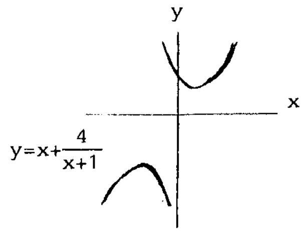


FIG. 3

Example 3:

Test $y = f(x) = x^{2/3}$ for relative extrema.

We have $f'(x) = \frac{2}{3}x^{-1/3} = 2/(3\sqrt[3]{x})$. When $x = 0$, then $f'(x)$ is not defined and thus $x = 0$ is a critical value. If $x < 0$, then $f'(x) < 0$. If $x > 0$, then $f'(x) > 0$. Since 0 is also in the domain of f, there is a relative (as well as an absolute) minimum when $x = 0$ (see Fig. 4).

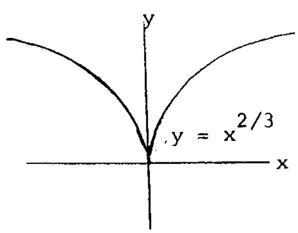


FIG. 4

Example 4:

Test $y = f(x) = x^2e^x$ for relative extrema.

By the product rule,

$$f'(x) = x^2e^x + e^x(2x) = xe^x(x + 2).$$

Since e^x is always positive, the critical values are $x = 0$, -2. From the signs of $f'(x)$ given in Fig. 5, we conclude that there is a relative maximum when $x = -2$ and a relative minimum when $x = 0$.

$f'(x)=(-)(+)(-) = (+)$	-2	$f'(x)=(-)(+)(+) = (-)$	0	$f'(x)=(+)(+)(+) = (+)$

FIG. 5

Example 5:

Sketch the graph of $y = f(x) = 2x^2 - x^4$.

Intercepts. If $x = 0$, then $y = 0$. If $y = 0$, then

$$0 = 2x^2 - x^4 = x^2(2 - x^2) = x^2(\sqrt{2}+x)(\sqrt{2}-x),$$

and thus $x = 0, \pm\sqrt{2}$. The intercepts are $(0, 0)(\sqrt{2}, 0)$, and $(-\sqrt{2}, 0)$.

Symmetry. Tesing for y-axis symmetry, we have

$$y = 2(-x)^2-(-x)^4,$$
$$y = 2x^2 - x^4.$$

Since this is the original equation, there is y-axis symmetry. It can be shown that there is no x-axis symmetry and hence no symmetry with respect to the origin.

Asymptotes: No horizontal or vertical asymptotes exist, since f is a nonconstant polynomial function.

First-Derivative Test.

1. $y' = 4x - 4x^3 = 4x(1 - x^2) = 4x(1 + x)(1 - x)$.

2. Setting $y' = 0$ gives the critical values $x = 0. \pm 1$. The critical points are $(-1, 1)$, $(0, 0)$, and $(1, 1)$. The y-coordinates of these points were found by substituting $x = 0, \pm 1$ into the **original** equation, $y = 2x^2 - x^4$.

3. There are four intervals to consider in Fig. 6.

	-1		0		1	

FIG. 6

if $x < -1$, then $y' = 4(-)(-)(+) = (+)$ and f is increasing;
if $-1 < x < 0$, then $y' = 4(-)(+)(+) = (-)$ and f is decreasing;
if $0 < x < 1$, then $y' = 4(+)(+)(+) = (+)$ and f is increasing;
if $x > 1$, then $y' = 4(+)(+)(-) = (-)$ and f is decreasing (Fig. 7).

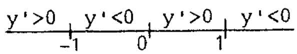


FIG. 7

4. Relative maxima occur at (-1, 1) and 1, 1); a relative minimum occurs at (0, 0).

Discussion. In Fig. 8 (a) we have plotted the intercepts and the horizontal tangents at the relative maximum and minimum points. We know the curve rises from the left, has a

(a) (b)

FIG. 8

relative maximum, then falls, has a relative minimum, then rises to a relative maximum, and falls thereafter. A sketch is shown in Fig. 8 (b)

In Example 4 relative maxima, as well as absolute maxima, occur at $x = \pm 1$ [see Fig 8(b)]. Although there is a relative minimum, there is no absolute minimum.

If the domain of a function is an interval that contains an endpoint, to determine absolute extrema we must not only examine the function for relative extrema, but we must also take into consideration the values of f(x) at the endpoints. Although endpoints are not considered when we look for relative maxima or minima, they may yield **absolute** maxima or minima.

Example 6:

Find all extrema (relative and absolute) for $y = f(x) = x^2 - 4x + 5$ on the closed interval [1, 4].

1. $f'(x) = 2x - 4 = 2(x - 2)$.

2. Setting $f'(x) = 0$ gives the critical value $x = 2$, which is in the domain of f.

3. The intervals to consider are when $x < 2$ and when $x > 2$.

4. If $x < 2$, then $f'(x) < 0$ and f is decreasing; if $x > 2$, then $f'(x) > 0$ and f is increasing. Thus there is a relative minimum when $x = 2$. It occurs on the graph at the point (2, 1) [see Fig 9].

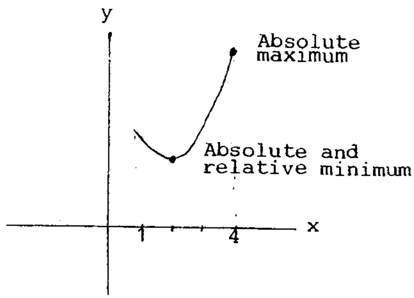


FIG. 9

5. Since f is decreasing for $x < 2$, then an absolute maximum may *possibly* occur at the left-hand endpoint of the domain of f, that is, when $x = 1$. Similarly, since f is increasing for $x + 2$, an absolute maximum may *possibly* occur at the right-hand endpoint, that is, when $x = 4$. Testing the endpoints, we have $f(1) = 2$ and $f(4) = 5$. Noting that $f(4) > f(1)$, we conclude that an absolute maximum occurs when $x = 4$. When $x = 2$ there is an absolute, as well as a relative, minimum.

EXERCISE: 9 - 3a

In Problems 1–28 determine when the function is increasing or decreasing and locate all relative maxima and minima. Do not sketch the graph.

1. $y = x^2 + 2$.
2. $y = x^2 + 4x + 3$.
3. $y = x - x^2 + 2$.
4. $y = 4x - x^2$.
5. $y = -\dfrac{x^3}{3} - 2x^2 + 5x - 2$.
6. $y = 4x^3 - 3x^4$.
7. $y = x^4 - 2x^2$.
8. $y = -2 + 12x - x^3$.
9. $y = x^3 - 6x^2 + 9x$.
10. $y = x^3 - 6x^2 + 12x - 6$.
11. $y = 3x^5 - 5x^3$.
12. $y = 5x - x^5$.
13. $y = -x^5 - 5x^4 + 200$.
14. $y = 3x^4 - 4x^3 + 1$.
15. $y = \dfrac{1}{x-1}$.
16. $y = \dfrac{3}{x}$.
17. $y = \dfrac{10}{\sqrt{x}}$.
18. $y = \dfrac{x}{x+1}$.
19. $y = \dfrac{x^2}{1-x}$.
20. $y = x + \dfrac{4}{x}$.
21. $y = (x+2)^3(x-5)^2$.
22. $y = x^2(x+3)^4$.
23. $y = e^{-2x}$.
24. $y = x \ln x$.
25. $y = x^2 - 2 \ln x$.
26. $y = xe^{x}$.
27. $y = e^x + e^{-x}$.
28. $y = e^{-x^2}$.

In Problems 29–40, determine: intervals on which the functions are increasing or decreasing; relative maxima and minima; symmetry; horizontal and vertical asymptotes; those intercepts that can be obtained conveniently. Then sketch the graphs.

29. $y = x^2 - 6x - 7$.
30. $y = 2x^2 - 5x - 12$.
31. $y = 3x - x^3$.
32. $y = x^4 - 16$.
33. $y = 2x^3 - 9x^2 + 12x$.
34. $y = x^3 - 9x^2 + 24x - 19$.
35. $y = x^4 + 4x^3 + 4x^2$.
36. $y = x^5 - \frac{5}{4}x^4$.
37. $y = \dfrac{x+1}{x-1}$.
38. $y = \dfrac{x^2}{x^2+1}$.
39. $y = \dfrac{x^2}{x+3}$.
40. $y = x + \dfrac{1}{x}$.

In Problems 41–46, find when absolute maxima and minima occur for the given function on the given interval.

41. $f(x) = x^2 - 2x + 3$, $[-1, 2]$.
42. $f(x) = -2x^2 - 6x + 5$, $[-2, 3]$.
43. $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 1$, $[0, 2]$.
44. $f(x) = \frac{1}{4}x^4 - \frac{3}{2}x^2$, $[0, 1]$.
45. $f(x) = 4x^3 + 3x^2 - 18x + 3$, $[\frac{1}{2}, 3]$.
46. $f(x) = x^{4/3}$, $[-8, 8]$.
47. If $c_f = 25{,}000$ is a fixed cost function, show that the average fixed cost function $\bar{c}_f = c_f/q$ is a decreasing function for $q > 0$. Thus, as output q increases, each unit's portion of fixed cost declines.
48. If $c = 4q - q^2 + 2q^3$ is a cost function, when is marginal cost increasing?

49. Given the demand function $p = 400 - 2q$, find when marginal revenue is increasing.

50. For the cost function $c = \sqrt{q}$, show that marginal and average costs are always decreasing for $q > 0$.

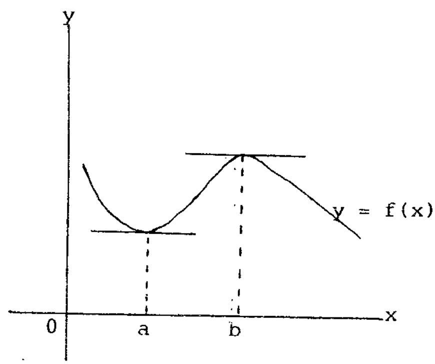
51. For a manufacturer's product, the revenue function is given by $r = 240q + 57q^2 - q^3$. Determine the output for maximum revenue.

52. In his model for storage and shipping costs of materials for a manufacturing process, Lancaster[†] derives the following cost function:

$$C(k) = 100\left[100 + 9k + \frac{144}{k}\right], \qquad 1 < k < 100,$$

where $C(k)$ is the total cost (in dollars) of storage and transportation for 100 days of operation if a load of k tons of material is moved every k days. (a) Find $C(1)$. (b) For what value of k does $C(k)$ have a minimum? (c) What is the minimum value?

9.3.2 The Second-Derivative Test:



The second derivative may be used to test certain critical values for relative extrema. From the figure, when x = a there is a horizontal tangent; that is $f'(a) = 0$. This suggests a relative maximum or minimum. However, we see also that the curve is bending upward there (that is, $f''(a) > 0$). This leads us to conclude that there is a relative minimum at "a". On the other hand, $f'(a) = 0$ but the curve is bending downward at "a" (that is, $f''(a) < 0$). From this we conclude that a relative maximum exists there. This technique of examining the second derivative at points where $f'(x) = 0$ is called the *second derivative test* for relative maxima and minima.

Second-Derivative Test:

Suppose $f'(a) = 0$

If $f''(a) < 0$, *then f has a relative maximum at "a"*

If $f''(a) > 0$, *then f has a relative minimum at "a"*

If $f''(a) = 0$, *the test gives no information, that is, at "a" there may be a relative maximum, a relative minimum or neither (use the first derivative test).*

Example 1:

Use the second-derivative test to examime the following for relative maxima and minima.

a) $y = 18x - \frac{2}{3}x^3$

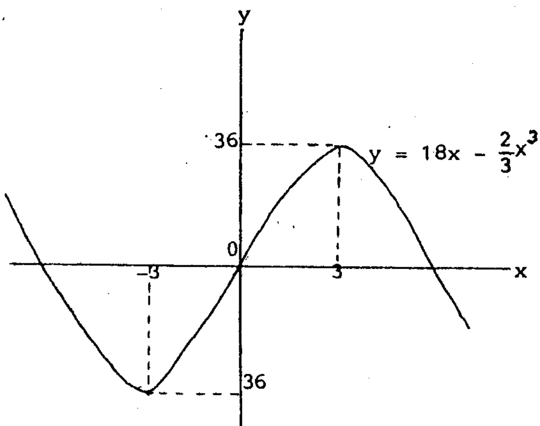
$$y' = 18 - 2x^2 = 2(9 - x^2) = 2(3 + x)(3 - x)$$

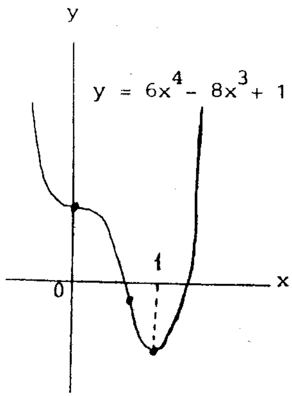
$$y = -4x$$

Solving $y' = 0$ gives the critical valves $x = \pm 3$.

If $x = 3$, then $y'' = -4(3) = -12$ so there is a relative maximum.

If $x = -3$, then $y'' = -4(-3) = 12$ so there is a relative minimum.





b) $y = 6x^4 - 8x^3 + 1$

$$y' = 24x^3 - 24x^2 = 24x^2(x - 1)$$
$$y'' = 72x^2 - 48x$$

Solving $y' = 0$ gives the critical values $x = 0, 1$

At $x = 1, y'' > 0$ giving a relative minimum at that point.

At $x = 0, y'' = 0$ so the second-derivative test gives no information. Use the first-derivative test.

If $x < 0$, then $y' < 0$; and

if $0 < x < 1$, then $y' < 0$.

Thus no relative maximum or minimum exists when $x = 0$.

Example 2:

Identify any relative max - min. points using the first and second-order conditions, and evaluate f(x) at any relative maximum or minimum point located.

a) $f(x) = 4x^3 + x + 1$

$f'(x) = 12x^2 + 1$

$f''(x) = 24x$

Solve for x when $f'(x) = 12x^2 + 1 = 0$

$x = \pm\sqrt{-\frac{1}{12}}$ which is imaginary.

This function is never zero. There are no relative maximum or minimum points. Ans

b) $f(x) = x^3 - 12x + 7$

$f'(x) = 3x^2 - 12$

$f''(x) = 6x$

Solve for x when $f'(x) = 3x^2 - 12 = 0$

$x = \pm 2$

At $x = 2$, $f''(2) = 6(2) = 6(2) = 12$ which is a relative minimum. Ans.

At $x = -2$, $f''(-2) = 6(-2) = -12$ which is a relative maximum. Ans.

c) $f(x) = 4x^2 - 2x + 3$

$f'(x) = 8x - 2$

$f''(x) = 8$

Solve for x when $f'(x) = 8x - 2 = 0$

$\therefore x = \frac{1}{4}$

Evaluate $f''(x) = 8$, at $x = \frac{1}{4}$

At $x = \frac{1}{4}$, $f''(\frac{1}{4}) = 8 > 0$

$\therefore$ f(x) has a relative minimum at $x = \frac{1}{4}$

and $f(\frac{1}{4}) = 2\frac{3}{4}$ Ans.

d) $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

$f''(x) = 2a$

Solve for x when $f'(x) = 2ax + b = 0$

$\therefore x = -\frac{b}{2a}$

Evaluate $f''(x) = 2a$, at $x = -\frac{b}{2a}$

At $x = -\frac{b}{2a}$, $f''\left(-\frac{b}{2a}\right) = 2a$ (2a may either be positive or negative depending on the value f "a").

$\therefore$ This function has a relative maximum at $x = -\frac{b}{2a}$ if $a < 0$ and relative minimum if $a > 0$. Ans

EXERCISE: 9 - 3b

In Problems 1-10, test for relative maxima and minimum by using the second-derivative test. In Problem 1-4, state whether the relative extrema are also absolute extrema.

1. $y = x^2 - 5x + 6.$
2. $y = -2x^2 + 6x + 12.$
3. $y = -4x^2 + 2x - 8.$
4. $y = 3x^2 - 5x + 6.$
5. $y = x^3 - 27x + 1.$
6. $y = x^3 - 12x + 1.$
7. $y = -x^3 + 3x^2 + 1.$
8. $y = x^4 - 2x^2 + 4.$
9. $y = 2x^4 + 2.$
10. $y = -x^7.$

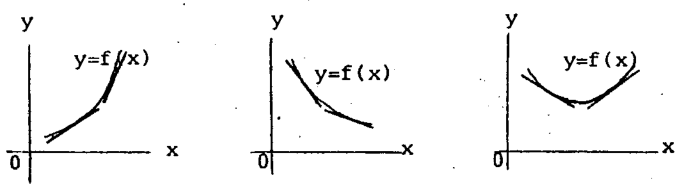
9.4 Concavity:

describes the nature of the curve of the function whether the curve bends upward or downward.

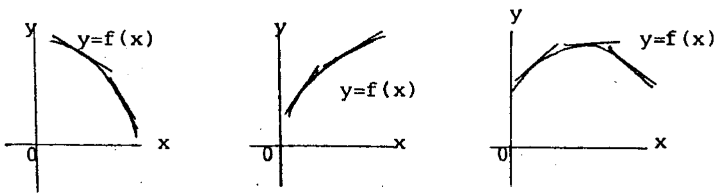
Definition: A function f is said to be concave up (concave down) on an interval I if f' is an increasing [decreasing] function on I.

- note that concavity relates to whether f' is increasing or decreasing.

Rule: If $f''(x) > 0$ on an interval I, then f is concave up on I. If $f''(x) < 0$ on I, then f is concave down on I.



In each case the curve y = f(x) "bends" upward. If tangent lines are drawn their slopes increase in value as x increases. Since f'(x) gives the slope at a point, f' is an increasing function. The curve is **concave up.**



Example 3:

Sketch the graph of $y = 2x^3 - 9x^2 + 12x$

Solution:

$y = 2x^3 - 9x^2 + 12x$

To find x - y intercepts: set $y = 0$

$y = 2x^3 - 9x^2 + 12x = x(2x^2 - 9x + 12) = 0$

$\therefore x = 0$

Note: $2x^2 - 9x + 12 = 0$ gives no real roots.

Thus the only intercept is (0,0)

Maxima and minima: set $y' = 0$

$$y' = 6x^2 - 18x + 12$$
$$= 6(x^2 - 3x + 2) = 6(x - 1)(x - 2) = 0$$
$$\therefore x = 1, 2$$

\Using the second-derivative test.

If $x = 1$, $f''(1) = 12(1) - 18 = (-)$, giving a relative maximum.

If $x = 2$, $f''(2) = 12(2) - 18 = (+)$, giving a relative minimum.

$\therefore$ There is a relative maximum when $x = 1$ and a relative minimum when $x = 2$.

Inflection Point: set $y'' = 0$

$$y'' = 12x - 18 = 0$$
$$6(2x - 3) = 0$$
$$\therefore x = \frac{3}{2}$$

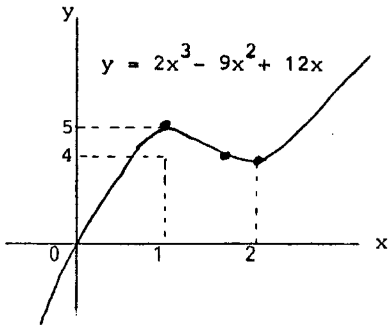
The inflection point is at $x = \frac{3}{2}$

Concavity:

When $x < \frac{3}{2}$, then $y'' = 12x - 18 < 0$ and f is concave down.

When $x > \frac{3}{2}$, then $y'' = 12x - 18 > 0$, and f is concave up.

Then find the coordinates of the important points on the graph.



Example 4:

Sketch the graph of $y = \dfrac{4x}{x^2 + 1}$

SOLUTION:

Intercepts: set y = 0

$$y = \frac{4x}{x^2 + 1} = 0$$

$$\therefore x = 0$$

The only intercept is at (0,0)

Maxima and Minima: set y' = 0

$$y' = \frac{(x^2 + 1)(4) - 4x(2x)}{(x^2 + 1)^2}$$

$$= \frac{4 - 4x^2}{(x^2 + 1)^2} = \frac{4(1 + x)(1 - x)}{(x^1 + 1)^2} = 0$$

$$\therefore x = 1, -1$$

The critical values are $x = \pm 1$

To test for maximum and minimum points, use the first derivative test.

If $x < -1$, then $f'(x) = \dfrac{4(-1)(+)}{(+)} = (-)$ and f is decreasing;

if $-1 < x < 1$, then $f'(x) = \dfrac{4(+)(+)}{(+)} = (+)$ and f is increasing;

if $x > 1$, then $f'(x) = \dfrac{4(+)(-)}{(+)} = (-)$ and f is decreasing.

$\therefore$ There is a relative minimum when $x = -1$ and a relative maximum when $x = 1$

Inflection: set $y'' = 0$

$$y'' = \frac{(x^2+1)^2(-8x) - (4 - 4x^2)(2)(x^2+1)(2x)}{(x^2+1)^4}$$

$$= \frac{8x(x^2+1)(x^2-3)}{(x^2+1)^4} = \frac{8x(x+\sqrt{3})(x-\sqrt{3})}{(x^2+1)^3} = 0$$

$$\therefore \quad x = 0, \pm\sqrt{3}$$

Possible points of inflection are when $x = \pm\sqrt{3}$, 0

Concavity:

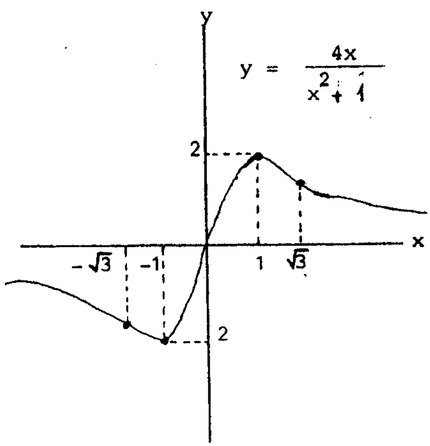
If $x > -\sqrt{3}$, then $f''(x) = \dfrac{8(-)(-)(-)}{(+)} = (-)$ and f is concave down;

if $-\sqrt{3} < x < 0$, then $f''(x) = \dfrac{8(-)(+)(-)}{(+)} = (+)$ and f is concave up;

if $0 < x < \sqrt{3}$, then $f''(x) = \dfrac{8(+)(+)(-)}{(+)} = (-)$ and f is concave down;

if $x > \sqrt{3}$, then $f''(x) = \dfrac{8(+)(+)(+)}{(+)} = (+)$ and f is concave up.

Then find coordinates of the important points and draw the graph.



EXERCISE: 9 - 4

In Problems 1-14, determine concavity and where points of inflection occur. Do not sketch the graphs.

1. $y = -2x^2 + 4x$.

2. $y = 3x^2 - 6x + 5$.

3. $y = 4x^3 + 12x^2 - 12x$.

4. $y = x^3 - 6x^2 + 9x + 1$.

5. $y = x^4 - 6x^2 + 5x - 6$.

6. $y = \frac{x^4}{4} + \frac{9x^2}{2} + 2x$.

7. $y = \frac{x + 1}{x - 1}$.

8. $y = x + \frac{1}{x}$.

9. $y = \frac{x^2}{x^2 + 1}$.

10. $y = \frac{x^2}{x + 3}$

11. $y = e^x$.

12. $y = e^x - e^{-x}$.

13. $y = xe^x$.

14. $y = xe^{-x}$.

In Problems 15-44 sketch each curve. Determine: intervals on which the function is increasing, decreasing, concave up, concave down; relative maxima and minima; inflection points; symmetry; horizontal and vertical asymptotes; those intercepts which can be obtained conveniently.

15. $y = x^2 + 4x + 3$.

16. $y = x^2 + 2$.

17. $y = 4x - x^2$.

18. $y = x - x^2 + 2$.

19. $y = 2x^2 - 5x - 12$.

20. $y = x^2 - 6x - 7$.

21. $y = x^3 - 9x^2 + 24x - 19$.

22. $y = 3x - x^3$.

23. $y = \frac{x^3}{3} - 3x$.

24. $y = x^3 - 6x^2 + 9x$.

25. $y = x^3 - 3x^2 + 3x - 3$.

26. $y = 2x^3 - 9x^2 + 12x$.

27. $y = 4x^2 - x^4$.

28. $y = -\frac{x^3}{3} - 2x^2 + 5x - 2$.

29. $y = 4x^3 - 3x^4$.

30. $y = x^4 - 2x^2$.

31. $y = -2 + 12x - x^3$.

32. $y = (3 + 2x)^3$

33. $y = x^3 - 6x^2 + 12x - 6$.

34. $y = 3x^5 - 5x^3$.

35. $y = 5x - x^5$.

36. $y = \frac{x^5}{100} - \frac{x^4}{20}$.

37. $y = 3x^4 - 4x^3 + 1$.

38. $y = x(1 - x)^3$.

39. $y = \frac{3}{x}$.

40. $y = \frac{1}{x - 1}$.

In each case, as x increases, the slopes of the tangent lines are decreasing and the curves are bending downward. Thus f' is a decreasing function and f is **concave up.**

9.5 Inflection Point:

A point of inflection occurs when a curve changes from "facing down (up)" to "facing up (down)".

** A necessary condition for an inflection point at x = a is that $f''(a) = 0$

Example 1:

Test $y = 6x^4 - 8x^3 + 1$ for points of inflection.

Solution:

$$y = 6x^4 - 8x^3 + 1$$
$$y' = 24x^3 - 24x^2$$
$$y'' = 72x^2 - 48x = 24x(3x-2) = 72x\left(x - \frac{2}{3}\right)$$

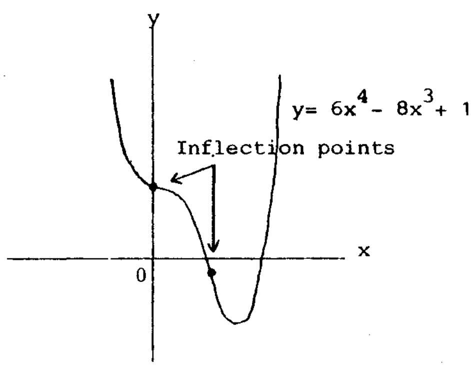
Setting $y'' = 0$ gives $x = 0, \frac{2}{3}$ as possible points of inflection.

if $x < 0$, then $y'' = 72(-)(-) = (+)$ the curve is concave up.

if $0 < x < \frac{2}{3}$, then $y'' = 72(+)(-) = (-)$ and the curve is concave down;

if $x > \frac{2}{3}$, then $y'' = 72(+)(+) = (+)$ and the curve is concave up.

Since concavity changes when $x = 0$ and $x = \frac{2}{3}$, inflection points occur for these values of x. <u>Ans</u>.



Example 2:

Test $y = f(x) = (x - 1)^3 + 1$ for concavity

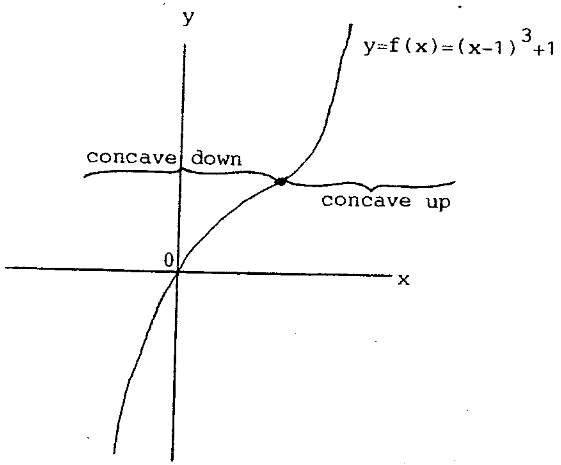
Solution:

$$y = (x-1)^3 + 1$$
$$y' = 3(x-1)^2$$
$$y'' = 6(x-1)$$

Solve for x, when $y' = 3(x-1)^2 = 0$

$$x = 1$$

$$\left.\begin{array}{l}\text{when } x > 1,\ y'' = 6(x-1) > 0, \text{ then f is concave up.}\\ \text{when } x < 1,\ y'' = 6(x-1) < 0, \text{ then f is concave down.}\end{array}\right\} \underline{\text{Ans}}$$



9.6 Curve Sketching:

By using the maxima and minima thęory in curve sketching, we make use of the roots, critical points and inflection points of f. They are useful because

(i) they tell us when the curve crosses the x - axis,

(ii) they tell us when it reaches its high and low points, and

(iii) they tell us when it changes from "facing down" to "facing up".

To find the roots of a function, put $f(x) = 0$

To find the maximum and minimum points, put $f'(x) = 0$

To find the inflection points, put $f''(x) = 0$

Example 1:

Graph the function $f(x) = x^3 - 3x^2 - x + 3$

Solution:

$$f(x) = x^3 - 3x^2 - x + 3$$
$$= (x + 1)(x - 1)(x - 3)$$

To find the roots, set $f(x) = 0$

$$\therefore x = 1, -1, +3$$
$$f'(x) = 3x^2 - 6x - 1$$

To find max-min point, set $f'(x) = 0$

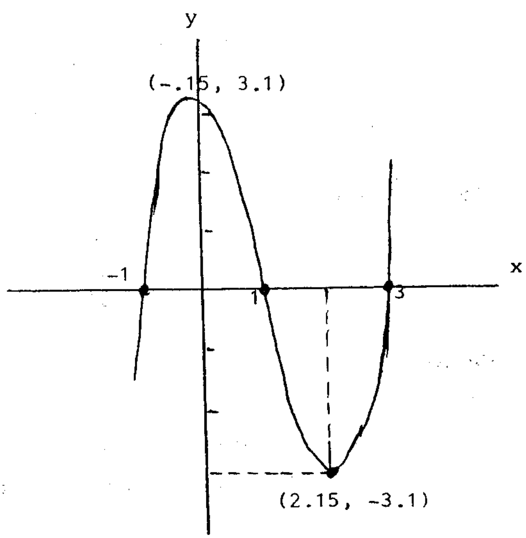
$$f'(x) = 3x^2 - 6x - 1 = 0$$
$$x = 1 \pm \frac{2}{\sqrt{3}}$$

Then test for the point of maximum and minimum

$$f''(x) = 6x - 6$$

At $x = 1 + \frac{2}{\sqrt{3}}$ or 2.15, $f''(2.15) = 6(2.15) - 6 = +$ giving a relative minimum.

At $x = 1 - \frac{2}{3}$ or -0.15, $f''(-.15) = 6(-0.15) - 6 = (-)$ giving a relative maximum.



To find inflection point; set $f''(x) = 0$

$$f''(x) = 6x - 6 = 0$$
$$\therefore x = 1$$

To test for concavity.

When $x < 1$, $f''(x) = 6x - 6 < 0$, then f is a concave down

When $x > 1$, $f''(x) = 6x - 6 > 0$, then f is a concave up

Example 2:

Using the calculus where applicable, graph the function $f(x) = x^3 - x$

Solution:

To find the x-, y- intercepts set $f(x) = 0$

$$f(x) = x^3 - x = x(x + 1)(x - 1) = 0$$
$$\therefore x = 0,\ 1,\ -1$$

To find the relative max-min points, set $f'(x) = 0$

$$f'(x) = 3x^2 - 1 = 0$$
$$\therefore x = \pm \frac{1}{\sqrt{3}}$$

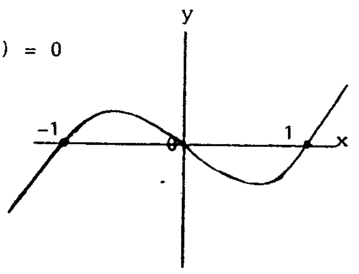
To test for the max-min point, use the second-derivative test

$f''(x) = 6x$

At $x = \frac{1}{\sqrt{3}}$, $f''\left(\frac{1}{\sqrt{3}}\right) = \frac{6}{\sqrt{3}} = +$, giving a relative minimum.

At $x = -\frac{1}{\sqrt{3}}$, $f''\left(-\frac{1}{\sqrt{3}}\right) = -\frac{6}{\sqrt{3}} = -$, giving a relative maximum.

To find the inflection point, set $f''(x) = 0$



$$f''(x) = 6x = 0$$
$$\therefore x = 0$$

Inflection point occurs when $x = 0$

To test for concavity

$$f''(x) = 6x$$

if $x < 0$, then $f''(x) = 6x < 0$, and the curve is concave down,

if $x > 0$, then $f''(x) = 6x > 0$, and the curve is concave up.

41. $y = \dfrac{x}{x+1}$.

42. $y = \dfrac{10}{\sqrt{x}}$.

43. $y = x^2 + \dfrac{1}{x^2}$.

44. $y = \dfrac{x^2}{1-x}$.

45. Show that the graph of the demand equation $p = 100/(q + 2)$ is decreasing and concave up for $q > 0$.

46. For the cost function $c = 3q^2 + 5q + 6$, show that the graph of the average cost function $\bar{c}$ is always concave up for $q > 0$.

9.7 Applied Maxima and Minima:

By using techniques of the previous setions, we can examine situations that require determining the value of a variable that will maximize or minimize a function. For example, we might want to maximize profit or minimize cost.

To do this; then:

1. Set up the function required.
2. Find its derivative and test the resulting critical values. (Use the first-derivative test or the second-derivative test)

Common equations

Revenue = *(Price)(quantity)*

Profit = *Revenue - cost*

Total Cost = *Variable Cost + fixed Cost*

Average Cost = $\dfrac{\textit{Total Cost}}{\textit{quantity}}$

Example 1:

For the total cost function $c = \dfrac{q^2}{4} + 3q + 400$, *where* q *is the number of units produced, at what level of output will average cost per unit be a minimum? What is this minimum?*

Solution:

The quantity to be minimixed is average cost $\bar{c}$
The average cost function is given by

$$\bar{c} = \frac{c}{q} = \frac{\frac{q^2}{4} + 3q + 400}{q} = \frac{q}{4} + 3 + \frac{400}{q}$$

$$\bar{c}' = \frac{1}{4} - \frac{400}{q^2} = \frac{q^2 - 1600}{4q^2}$$

$$\text{Set } \bar{c}' = \frac{q^2 - 1600}{4q^2} = 0$$

$$\therefore q = 40$$

To determine if this level of output gives a relative minimum use the second-derivative test.

$$\bar{c}'' = \frac{800}{q^3}$$

$$\bar{c}''(40) = \frac{800}{40^3} > 0, \text{ giving } c \text{ as a relative minimum}$$

when $q = 40$ <u>Ans</u>

$$\text{This minimum is } \bar{c} = \frac{q}{4} + 3 + \frac{400}{q}$$

$$= \frac{40}{4} + 3 + \frac{400}{40} = 10 + 3 + 10$$

$$= 23$$ <u>Ans</u>

Example 2:

The demand equation for a manufacturer's product is $p = \frac{(80 - q)}{4}$ *where q is the number of units and p is price per unit. At what value of q will there be maximum revenue?*

Solution:

Let r be total revenue
Then revenue = (price)(quantity)

Thus, $$r = pq = \frac{80 - q}{4} \cdot q = \frac{80q - q^2}{4}$$

$$\frac{dr}{dq} = \frac{80 - 2q}{4} = 0$$

$$q = 40 \qquad \underline{\text{Ans}}$$

When q = 40, gives the maximum revenue.

Test for maximum value:

$$\frac{d^2r}{dq^2} = -\frac{1}{2} < 0 \text{ giving q as a maximum}$$

Example 3:

A manufacturer annually produces and sells 10,000 units of a product. Sales are uniformly distributed throughout the year. He wishes to determine the number of units to be manufactured in each production run in order to minimize annual set-up costs and carrying costs. The size of such production runs is referred to as the economic lot size or economic order quantity. The production cost of each unit is $20 and carrying costs (insurance, interest, storage, etc.) are estimated to be 10 percent of the value of the average inventory. Set-up costs per production run are $40. Find the economic lot size.

Solution:

Let q be the number of units in a production run. The average inventory is taken to be $\frac{q}{2}$ units. The production costs are $20 per unit, and so the value of the average inventory is $20\left(\frac{q}{2}\right)$

Carrying costs are 10 percent of this value: $= 0.10(20)\left(\frac{q}{2}\right)$

The number of production run per year is $= \frac{10,000}{q}$

Thus the total set-up costs are $= 40\left(\frac{10,000}{q}\right)$

Hence the total annual carrying costs and set-up costs C are

$$C = 0.10(20)\left(\frac{q}{2}\right) + 40\left(\frac{10,000}{q}\right)$$

$$= q + \frac{400,000}{q}$$

$$\frac{dc}{dq} = 1 - \frac{400,000}{q^2} = \frac{q^2 - 400,000}{q^2}$$

Setting $\frac{dc}{dq} = 0$, we get

$$q^2 = 400{,}000$$

$$\text{or} \quad q = \pm 200\sqrt{10}$$

$$= \pm 632.5$$

Since $q > 0$, we have $q = 632.5$ <u>Ans</u>

To test for minimum value:

If $0 < q < \sqrt{400{,}000}$, then $\frac{dc}{dq} < 0$

If $q > \sqrt{400{,}000}$, then $\frac{dc}{dq} > 0$

Since C always decreases between $q = 0$ and $q = \sqrt{400{,}000}$, and always increases to the right of $q = \sqrt{400{,}000}$, we conclude that there is an absolute minimum at $q = 632.5$.

Example 4:

The Vista TV Cable Co. currently has 2000 subscribers who are paying a monthly rate of \$5. A survey reveals that there will be 50 more subscribers for each \$.10 decrease in the rate. At what rate will maximum revenue be obtained and how many subscribers will there be at this rate?

Solution:

Let x be the rate

Then the total decrease in the rate is $= 5 - x$

And the number of \$0.10 decreases is $= \frac{5 - x}{0.10}$

For each of these decreases there will be 50 more subscribers

Thus the total of new subscribers is $= 50\left(\frac{5 - x}{0.10}\right)$

and the total of all subscribers is $= 2000 + 50\left(\frac{5 - x}{0.10}\right)$

The revenue r is given by $r = (\text{rate})(\text{number of subscribers})$

$$r = x\left[2000 + 50\left(\frac{5 - x}{0.10}\right)\right]$$

$$= 4500x - 500x^2$$

Setting $r' = 0$, we have: $r' = 4500 - 1000x = 0$

$x = \$4.50$ <u>Ans</u>

To test for maximum:

Since $r'' = -1000 < 0$, we have a relative maximum when $x = 4.50$.

The total of all subscribers is $= 2000 + 50\left(\frac{5 - x}{0.10}\right)$

$= 2250$ subscribers <u>Ans</u>

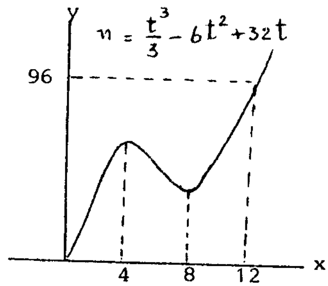
Example 5:

A health economist determined that if a particular health-care program for the elderly were initiated, then t years after its start, n thousand elderly people would receive direct benefits, where.

$$n = \frac{t^3}{3} - 6t^2 + 32t, \quad 0 \leqslant t \leqslant 12.$$

For what value of t does the maximum number receive benefits?

Solution:



Setting $\frac{dn}{dt} = 0$, we have

$$\frac{dn}{dt} = t^2 - 12t + 32 = 0$$
$$(t - 4)(t - 8) = 0$$
$$t = 4 \text{ or } t = 8$$

Now, $\frac{d^2n}{dt^2} = 2t - 12,$

If $t = 4$, $\frac{d^2n}{dt^2} = 2(4) - 12 = (-)$

If $t = 8$, $\frac{d^2n}{dt^2} = 2(8) - 12 = (+)$

Thus, there is a relative maximum when $t = 4$

Therefore, $n = \frac{(4)^3}{3} - 6(4)^2 + 32(4)$

$$= \frac{64}{3} - 96 + 128$$

$$= 53\frac{1}{3}$$ Ans

Example 6:

Suppose that the demand equation for a monopolist's product is $p = 400 - 2q$ and the average cost function is $\bar{c} = .2q + 4 + (400/q)$, where q is number of units, and both p and $\bar{c}$ are expressed in dollars.

a) Determine the level of output at which profit is maximized.
b) Determine the price at which maximum profit occurs.
c) Determine the maximum profit.
d) If, as a regulatory device, the government imposes a tax of \$22 per unit on the monopolist, what is the new price for profit maximization?

Solution:

Since revenue $r = pq = 400\,q - 2q^2$

Total cost $c = q\bar{c} = 0.2q^2 + 4q + 400$

Profit (P) is $P = r - c = 400q - 2q^2 - (0.2q^2 + 4q + 400)$

$P = 396\,q - 2.2\,q^2 - 400$

a) Setting $\frac{dP}{dq} = 0$, we have

$$\frac{dP}{dq} = 396 - 4.4q = 0$$

$$q = 90 \qquad \underline{\text{Ans}}$$

Since $\frac{d^2P}{dq^2} = -4.4 < 0$, we conclude that $q = 90$

gives a maximum.

b) From the demand equation, $p = 400 - 2(90)$
$= \$220$ $\underline{\text{Ans}}$

c) Maximum profit can be obtained by substituting $q = 90$ in the profit equation which gives $P = 17{,}420$ $\underline{\text{Ans}}$

d) The tax of \$22 per unit means that for q units the total cost increases by 22q. The new cost function is

$$c_1 = 0.2q^2 + 4q + 400 + 22q,$$

and the profit P_1 is given by

$$P_1 = 400q - 2q^2 - (0.2q^2 + 4q + 400 + 22q)$$
$$= 374\,q - 2.2q^2 - 400$$

Setting $\frac{dP_1}{dq} = 0$ gives:

$$\frac{dP_1}{dq} = 374 - 4.4q = 0$$
$$q = 85$$

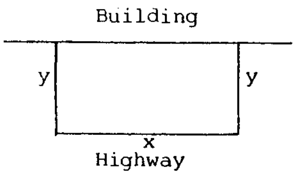
Thus to maximize profit, the monopolist restricts output to 85 units at a higher price of

$$P_1 = 400 - 2(85) = 230$$

Since this is only \$10 more than before, only part of the tax has been shifted to the consumer, and the monopolist must bear the cost of the balance. The profit now is \$15,495, which is lower than the former profit.

Example 7:

For insurance purposes a manufacturer plans to fence in a 10,800-sq-ft rectangular storage area adjacent to a building by using the building as one side of the enclosed area (see Fig.). The fencing parallel to the building faces a highway and will cost \$3 per ft. installed, while the fencing for the other two sides costs \$2 per ft. installed. Find the amount of each type of fence so that the total cost of the fence will be a minimum. What is the minimum cost?



Solution:

Let x be the length in feet of the side parallel to the building and y be the lengths of the other two sides. The cost (in dollars) of the fencing along the highway is = 3x, and along each of the other sides it is = 2y.

Thus, the total cost C of the fencing is

$$C = 3x + 2y + 2y = 3x + 4y$$

The storage area = xy must be 10,800 sq-ft.

$$\therefore y = \frac{10,800}{x}$$

Then

$$C = 3x + 4\left(\frac{10,800}{x}\right)$$

$$= 3x + \frac{43,200}{x}$$

To minimize C, set $\frac{dc}{dx} = 0$ and solve for x:

$$\frac{dc}{dx} = 3 - \frac{43200}{x^2} = 0$$

$$x^2 = \frac{43200}{3} = 14,400$$

$$x = 120 \text{ (since } x > 0)$$

Test for Minimum:

Since $\frac{d^2c}{dx^2} = \frac{86,400}{x^3} > 0$ for $x = 120$, then $x = 120$ gives the minimum value of C.

When $x = 120$, then $y = \dfrac{10{,}800}{120} = 90$

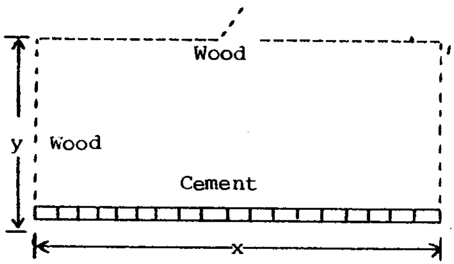
Thus 120 ft. of the $3 fencing and 180 ft. of the $2 fencing are needed. This gives a cost of $720. Ans

Example 8:

The manager of a department store wants to build a 600-square-foot rectangular enclosure on the store's parking lot in order to display some equipment. Three sides of the enclosure will be built out of redwood fencing, at a cost of $7 per running foot. The fourth side will be built out of cement blocks, at a cost of $14 per running foot. Find the dimensions of the enclosure that minimizes the total cost of the building materials.

Solution:

Let x be the length of the side built out of cement blocks and let y be the length of an adjacent side



Since the area of the enclosure must be 600 square feet, then $xy = 600$

We are asked to minimize the total cost.

Now Cost of redwood = (length of redwood fencing) x (cost per foot).

$= (x + 2y)\,7 = 7(x) + 14y$

and Cost of cement blocks = (length of cement wall) x (cost per foot).

$= (x)(14)$

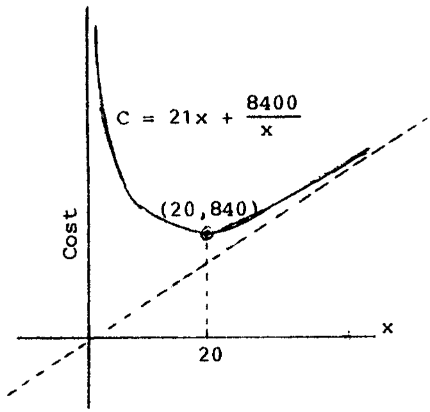
If c denotes the total cost of the materials, then

$$C = (7x + 14y) + 14x = 21x + 14y$$

Since $\quad xy = 600$

$$y = \frac{600}{x}$$

$$\therefore \quad C = 21x + 14\left(\frac{600}{x}\right) = 21x + \frac{8400}{x}$$



$$\frac{dc}{dx} = 21 - \frac{8400}{x^2} = 0$$

$$x^2 = \frac{8400}{21}$$

$$x = 20 (\text{since } x > 0)$$

(The test for the value of $x = 20$ creating a minimum cost can be done as usual)

$$\therefore \quad C = 21(20) + \frac{8400}{20}$$

$$= 420 + 420$$

$$= 840$$

The minimum cost of \$840 occurs when $x = 20$ <u>Ans</u>

Example 9:

The cost of constructing a high-rise office building is made up of fixed costs (land, permits), costs proportional to the number of floors (plumbing fixtures, paneling), and costs that vary with a power of the number of floors (elevator, main structural support, etc.). Suppose that the cost of constructing a building of n floors is $200 + 250n + 2n^2$ thousands of dollars. When the cost per floor is low, the cost per square foot of office space will be low. How many floors should the building have in order for the average cost per floor to be as small as possible?

Solution:

$$\text{Average Cost} = \frac{\text{total cost}}{\text{number of floors}}$$

$$= \frac{200 + 250n + 2n^2}{n}$$

$$= \frac{200}{n} + 250 + 2n \text{ (in thousand dollars)}$$

Setting $f(n) = \frac{200}{n} + 250 + 2n$, we must find the value of n for which $f(n)$ is minimized.

$$f'(n) = \frac{-200}{n^2} + 2$$

$$f''(n) = \frac{400}{n^3}$$

Setting $f'(n) = 0$, we have

$$-\frac{200}{n^2} + 2 = 0$$

$$n = 10 \text{ (since } n > 0)$$

Since $f''(10) = 0.4$, which is positive, the graph $f(n)$ is concave up at $n = 10$, and so $f(x)$ has a minimum at $n = 10$. Thus the building should have 10 floors. Ans

Example 10:

A farmer uses his 1000-acre farm to raise two grain crops. Grain A is sold at market and grain B is fed to his herd of 1000 cattle. Using the data below, determine how many acres of each crop should be planted in order to realize the greatest profit.

a) An acre planted with grain A yields 150 bushels, which sell for \$5 per bushel.
b) Feeding the herd the yield of x acres of grain B will result in each steer weighing $350 + x - .001x^2$ pounds. Steer sell for \$1 per pound.

Solution:

When x acres are planted with crop B, (1000 − x) acres are left to be planted with crop A.

Profit from grain A: $= (5)(150)(1000 - x)$ dollars
Profit from grain B: $= (1)(1000)(350+x-0.001x^2)$ dollars
The total profit is $= (5)(150)(1000-x) + (1)(1000)(350+x-0.001x^2)$ dollars.

$$= 750{,}000-750x+350{,}000+1000x-x^2$$

$$= 1{,}100{,}000+250x-x^2$$

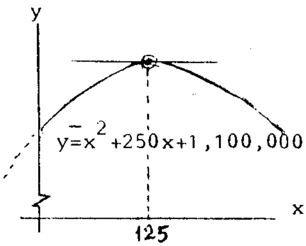
Thus the profit function $f(x) = 1{,}100{,}000+250x-x^2$

We want to find the value of x that maximizes the profit, $f(x)$

$$f'(x) = 250 - 2x$$

Setting $f'(x) = 0$, we have

$$250 - 2x = 0$$
$$x = 125$$



The profit at $x = 125$ is a maximum, since $f''(125) = -2 < 0$. Therefore the farmer should plant 125 acres of grain B for his cattle and plant the remaining $1000 - 125 = 875$ acres with grain A to be sold. Ans

Example 11:

(An Inventory Problem). A publishing company sells 400,000 copies of a certain book each year. If the managing editor orders the entire amount printed at the beginning of the year, she will tie up valuable storage space. However, running off the copies in several partial runs throughout the year results in added costs for setting up each printing run. Using the data below find the number of production runs that minimizes the publisher's costs.

1. *Each book costs \$4 for materials and labor.*
2. *Setting up each production run costs \$1000.*
3. *The inventory cost per book per year is \$.50.*

Note: Since the number of books in storage will vary throughout the year, inventory costs should be computed on the basis of the average number of books in storage. For example, if there are x runs per year, each run will produce $\frac{400,000}{x}$ books, and the number of books in storage will range from $\frac{400,000}{x}$ to 0. If we assume that the books will sell at a relatively uniform rate throughout the year, then the average number of books in storage will be $\frac{1}{2} \cdot \frac{400,000}{x}$.

Solution:

If there are x runs per year, each run will produce $\frac{400,000}{x}$ books, and the number of books in storage will range from $\frac{400,000}{x}$ to 0.

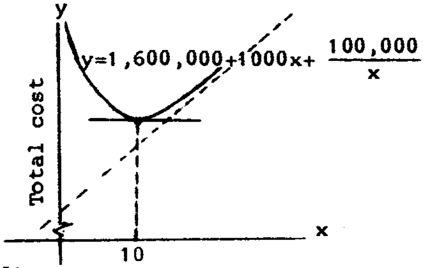
If we assume that the books will sell at a relatively uniform rate throughout the year, then the average number of books in storage will be

$$= \frac{1}{2} \cdot \frac{400,000}{x}$$

Let $f(x)$ be the total cost of making x production runs

Then $$f(x) = (400,000)(4) + 1000x + \left(\frac{1}{2} \cdot \frac{400,000}{x}\right)(0.50)$$
$$= 1,600,000 + 1000x + \frac{100,000}{x}$$

We want to find the value of x that minimizes the total cost $f(x)$.



$$f'(x) = 1000 - \frac{100,000}{x^2}$$
$$f''(x) = \frac{200,000}{x^3}$$

Setting $f'(x) = 0$, we have

$$1000 - \frac{100,000}{x^2} = 0$$
$$x = 10 \text{ (since } x > 0)$$

Since $f''(10) = \frac{200,000}{(10)^3} > 0$, the graph of $f(x)$ is concave up at $x = 10$, and so $f(x)$ has a minimum there.

Thus the total cost of publishing the 400,000 books will be minimized by scheduling ten production runs during the year. Ans

EXERCISE: 9 - 5

In each of the following, p is price per unit (in dollars) and q is output per unit of time.

1. A manufacturer finds that the total cost c of producing his product is given by the cost function $c = .05q^2 + 5q + 500$. At what level of output will average cost per unit be at a minimum?

2. The cost per hour C (in dollars) of operating an automobile is given by
$$C = .12s - .0012s^2 + .08, \quad 0 \leq s \leq 60,$$
where s is the speed in miles per hour. At what speed is the cost per hour a minimum?

3. The demand equation for a monopolist's product is $p = -5q + 30$. At what price will revenue be maximized?

4. For a monopolist's product, the demand function is $q = 10,000e^{-.02p}$. Find the value of p for which maximum revenue is obtained.

5. For a monopolist's product, the demand equation is $p = 72 - .04q$ and the cost function is $c = 500 + 30q$. At what level of output will profit be maximized? At what price does this occur and what is the profit?

6. For a monopolist, the cost per unit of producing a product is \$3 and the demand equation is $p = 10/\sqrt{q}$. What price will give the greatest profit?

7. For a monopolist, the demand equation is $p = 42 - 4q$ and the average cost function is $\bar{c} = 2 + (80/q)$. Find the profit-maximizing price.

8. For a monopolist's product, the demand function is $p = 50/\sqrt{q}$ and the average cost function is $\bar{c} = .50 + (1000/q)$. Find the profit-maximizing price and output. At this level, show marginal revenue is equal to marginal cost.

9. For XYZ Manufacturing Co., total fixed costs are \$1200, material and labor costs combined are \$2 per unit, and the demand equation is $p = 100/\sqrt{q}$. What level of output will maximize profit? Show that this occurs when marginal revenue is equal to marginal cost. What is the price at profit maximization?

10. A real estate firm owns 70 garden-type apartments. At \$125 per month each apartment can be rented. However, for each \$5 per month increase, there will be two vacancies with no possibility of filling them. What rent per apartment will maximize montly revenue?

11. A manufacturer finds that for the first 500 units of his product that are produced and sold, the profit is \$50 per unit. The profit on each of the units beyond 500 is decreased by \$.10 times the number of additional units produced. For example, the total profit when 502 units are produced and sold is 500(50) + 2(49.80). What level of output will maximize profit?

12. A TV cable company has 1000 subscribers who are paying \$5 per month. It can get 100 more subscribers for each \$.10 decrease in the monthly fee. What rate will yield maximum revenue and what will this revenue be?

13. Find two numbers whose sum is 40 and whose product is a maximum.

14. Find two nonnegative numbers whose sum is 20 and such that the product of twice one number and the square of the other number will be a maximum.

15. A company has set aside \$3000 to fence in a rectangular portion of land adjacent to a stream by using the stream for one side of the enclosed area. The cost of the fencing

parallel to the stream is $5 per foot installed, and the fancing for the remaining two sides is $3 per foot installed.

Find the dimensions of the maximum enclosed area.

16. The owner of the Laurel Nursery Garden Center wants to fence in 1000 square feet of land in a rectangular plot to be used for different types of shrubs. The plot is to be divided into four equal plots with three fences parallel to the same pair of sides as shown in Fig. 1. What is the least number of feet of fence needed?

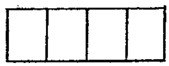
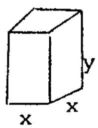


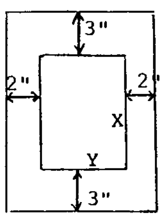
FIG. 1

17. A container manufacturer is designing a rectangular box, open at the top and with a square base, that is to have a volume of 32 cu ft. If the box is to require the least amount of material, what must be the dimensions of the box?



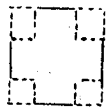
18. An open-top box with a square base is to be constructed from 192 sq ft of material. What should be the dimenstions of the box if the volume is to be a maximum? What is the maximum volume?

19. A rectangular cardboard poster is to have 150 sq in. for printed matter. It is to have a 3-in. margin at the top and bottom and a 2-in. margin on each side. Find the dimensions of the poster so that the amount of cardboard used is minimized.

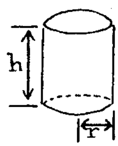


20. An open box is to be made by cutting equal squares from each corner of a 12-in. square piece of cardboard and then folding up the sides. Find the length of the side of the square that

must be cut out if the volume of the box is to be maximized. What is the maximum volume?



21. A cylindrical can, open at the top, is to have a fixed volume of K. Show that if the least amount of material is to be used, then both the radius and height are equal to $\sqrt[3]{K/\pi}$.



Volume $= \pi r^2 h$
Surface area $= 2\pi rh + \pi r^2$

Open at top

22. A cylindrical can, open at the top, is to be made from a fixed amount of material, K. If the volume is to be a maximum, show that both the radius and height are equal to $\sqrt{K/(3\pi)}$.

23. The demand equation for a monopolist's product is $p = 600 - 2q$ and the total cost function is $c = .2q^2 + 28q + 200$. Find the profit-maximizing output and price, and determine the corresponding profits. If the government were to impose a tax of \$22 per unit on the manufacturer, what would be the new profit-maximizing output and price? What is the profit now?

24. Use the *original* data in Problem 23 and assume that the government impoes a license fee of \$100 on the manufacturer. This is a lump-sum amount without regard to output. Show that marginal revenue and marginal cost do not change and, hence, the profit maximizing price and output remain the same. Show, however, that there will be less profit.

25. A manufacturer has to produce annually 1000 units of a product that is sold at a uniform rate during the year. The production cost of each unit is \$10 and carrying costs (insurance, interest, storage, etc.) are estimated to be 12.8 percent of the value of average inventory. Set-up costs per production run are \$40. Find the economic lot size.

26. For a monopolist's product, the cost function is $c = .004q^3 + 20q + 5000$ and the demand function is $p = 450 - 4q$. Find the profit-maximizing output. At this level, show that marginal cost = marginal revenue.

27. Imperial Educational Services (I.E.S.) is considering offering a workshop in resource allocation to key personnel at Acme Corp. To make the offering economically feasible, I.E.S. feels that at least thirty persons must attend at a cost of \$50 each. Moreover, I.E.S. will agree to reduce the charge for *everybody* by \$1.25 for each person over the thirty who attends. How many people should be in the group for I.E.S. to maximize revenue? Assume that the maximum allowable number in the group is forty.

28. The Kiddie Toy Company plans to lease an electric motor to be used 90,000 horsepower-hours per year in manufacturing. One horsepower-hour is the work done in one hour by a one-horsepower motor. The annual cost to lease a suitable motor is \$150 plus \$.60 per horsepower. The cost per horsepower-hour of operating the motor is \$.006/N where N is the horsepower. What size motor, in horsepower, should be leased in order to minimize cost?

29. For a manufacturer, the cost of making a part is \$3 per unit for labor and \$1 per unit for materials; overhead is fixed at \$2000 per week. If more than 5000 units are made each week, labor is \$4.50 per unit for those units in excess of 5000. At what level of production will average cost per unit be at a minimum?

30. The cost of operating a truck on a throughway (excluding the salary of the driver) is .11 + (s/600) dollars per mile, where s is the (steady) speed of the truck in miles per hour. The truck driver's salary is \$6 per hour. At what speed should the truck driver operate the truck to make a 700-mile trip most economical?

31. A company produces daily x tons of chemical A ($x \leqslant 4$) and y tons of chemical B where $y = (24 - 6x)/(5 - x)$. The profit on chemical A is \$2000 per ton and on B it is \$1000 per ton. How much of chemical A should be produced per day to maximize profit? Answer the same question if the profit on A is P per ton and that on B is P/2 per ton.

32. To erect an office building, fixed costs are \$250,000 and include land, architect's fee, basement, foundation, etc. If x floors are to be constructed, the cost (excluding fixed costs) is $c = (x/2)\,[100{,}000 + 5000(x - 1)]$. The revenue per month is \$5000 per floor. Find the number of floors that will yield a maximum rate of return on investment (rate of return = total revenue/total cost).

33. The profit from producing x units of a commodity is $-.001x^2 + x - 5000$. At what level should production be set in order to maximize profit?

34. A manufacturer produces x units of a certain commodity at a cost of $C(x) = 5x + 200$ dollars and receives a revenue of $R(x) = 10x - .01x^2$ dollars. How many units should be produced in order to maximize the profit? (Note: The profit from producing x units is $P(x) = R(x) - C(x)$.)

35. A rectangular garden of area 75 square feet is to be surrounded on three sides by a brick wall costing $10 per foot and on one side by a fence costing $5 per foot. Find the dimensions of the garden such that the cost of the materials is minimized. [Note: Letting x be the length of the fence and h be the length of the other dimension, the cost of the materials is $5x + 10(2h + x)$.].

36. A closed rectangular box with square base and a volume of 12 cubic feet is to be constructed using two different types of materials. The top is made of a metal costing $2 per square foot and the remainder of wood costing $1 per square foot. Find the dimensions of the box for which the cost of materials is minimized.

37. An apple orchard produces a profit of $40 per tree when planted with 1000 trees. Due to overcrowding, the profit per tree (for each tree in the orchard) is reduced by 2¢ for each additional tree that is planted. How many trees should be planted in order to maximize the total profit from the orchard?

38. A certain pond can support a maximum population of 20,000 fish. If there are fewer fish in the pond, the growth rate will be proportional to the product of the current population and the difference of the current population from 20,000. For what size population will the growth rate be maximal? [Note: Denote by $f(x)$ the growth rate for a population of size x; then $f(x) = kx(20,000 - x)$.].

39. A book publisher sells 100,000 copies of a certain book each year. Setting up each run costs $2000, each book costs $5 in material and labor, and the storage fees are $1 per book per year. Assume a uniform rate of sales throughout the year and find the number of runs that minimizes the publisher's cost.

40. A manufacturer can produce x units of a certain commodity at a cost of $C(x) = .01x^3 - .04x^2 + 2x + 100$ dollars and receives a revenue of $R(x) = 50x - .04x^2$ dollars. How many units should be produced in order to maximize the profit? [Note: The profit from producing x units is $P(x) = R(x) - C(x)$.].

41. A swimming club offers memberships at the rate of $200, provided that a minimum of 100 people join. For each member in excess of 100, the membership fee will be reduced by one dollar per person (for every member). At most, 160 memberships will be sold. How many memberships should the club try to sell in order to maximize its profits?

42. In the planning of a sidewalk cafe, it is estimated that if there are 12 tables, the daily profit will be \$10 per table. Due to overcrowding, for each additional table the profit per table (for every table in the cafe) will be reduced by \$50. How many table should be provided in order to maximize the profit from the cafe?

43. Suppose that the cost of constructing an office building of n floors is estimated to be

$$1000 + 500n + 1.6n^2$$

thousand dollars. How many floors should the building have in order to minimize the average cost per floor?

44. Foggy Optics, Inc. makes a laboratory microscope at a cost of \$200 for materials, labor, and similar items. Each production run costs \$2500. The expense of holding one microscope in inventory for one year is \$50. Suppose that the company expects to sell 1600 of these microscopes at a fairly uniform rate throughout the year. Determine the number of production runs that will minimize the company's expenses.

45. The demand equation for a certain product is $p = 6 - \frac{1}{2}x$. Find the level of production that results in maximum revenue $(x = 6,\ R(x) = 18)$.

46. The WMA Bus Lines offers sightseeing tours of Washington, D.C. One of the tours, priced at \$7 per person, had an average demand of about 1000 customers per week. When the price was lowered to \$6, the weekly demand jumped to about 1200 customers. Assuming that the demand equation is linear, find the tour price that should be charged per person in order to maximize the total revenue each week. $(p = \$6)$.

47. Suppose that the demand equation for a monopolist is $p = 100 - 0.01x$ and the cost function is $C(x) = 50x + 10{,}000$. Find the value of x that maximizes the profit and determine the corresponding price and total profit for this level of production. (x = 2500 units, p = \$75/unit, Profit = \$52,500).

48. Rework the problem No. 47 under the condition that the government imposes an excise tax of \$10 per unit. (x = 2000 units, p = \$80/unit, Profit = \$30,000)

49. Given the cost function $C(x) = x^3 - 6x^2 + 13x + 15$, find the minimum marginal cost.

50. Suppose that a total cost function is $C(x) = .0001x^3 - .06x^2 + 12x + 100$. Is the marginal cost increasing, decreasing, or not changing at $x = 100$? Find the minimum marginal cost.

51. The revenue function for a one-product firm is $R(x) = 200 - \frac{1600}{x + 8} - x$. Find the value of x that results in maximum revenue.

52. The revenue function for a particular product is $R(x) = x(4 - .0001x)$. Find the largest possible revenue.

53. A one-product firm estimates that its daily total cost function (in suitable units) is $C(x) = x^3 - 6x^2 + 13x + 15$ and its total revenue function is $R(x) = 28x$. Find the value of x that maximizes the daily profit.

54. A small tie shop sells men's ties for \$3.50 each. The daily cost function is estimated to be C(x) dollars, where x is the number of ties sold on a typical day and $C(x) = .0006x^3 - .03x^2 + 2x + 80$. Find the value of x that will maximize the store's daily profit.

55. The demand equation for a certain commodity is $p = \frac{1}{12}x^2 - 10x + 300$, $0 \leq x \leq 60$. Find the value of x and the corresponding price p that maximize the revenue.

56. The demand equation for a product is $p = 2 - .001x$. Find the value of x and the corresponding price p that maximize the revenue.

57. Some years ago it was estimated that the demand for steel approximately satisfied the equation $p = 256 - 50x$, and the total cost of producing x units of steel was $C(x) = 182 + 56x$. (The quantity x was measured in millions of tons and the price and total cost were measured in millions of dollars.) Determine the level of production and the corresponding price that maximize the profits.

58. Until recently hamburgers at the city sports arena cost \$1 each. The food concessionaire sold an average of 10,000 hamburgers on a game night. When the price was raised to \$1.20, hamburger sales dropped off to an average of 8000 per night.

   a) Assuming a linear demand curve, find the price of a hamburger that will maximize the nightly hamburger revenue.

   b) Suppose that the concessionaire has fixed costs of \$1000 per night and the variable cost per hamburger is \$.30. Find the price of a hamburger that will maximize the nightly hamburger profit.

59. The average ticket price for a concert at the opera house was \$9.50. The 4000-seat auditorium was filled for nearly every performance. When the ticket price was raised to \$10, attendance declined to an average of 3800 persons per performance. Salaries, electricity, and maintenance expenses

total \$40,000 per performance. Costs that vary with the size of the audience are negligible. What should the average ticket price be in order to maximize the profit for the opera house? (Assume a linear demand curve.)

60. A California distributor of sporting equipment expects to receive orders during the coming year for 100,000 cans of tennis balls. Yearly inventory costs per can are about \$.50, and the cost of placing an order with the manufacturer is \$10. Assuming a fairly constant demand for the tennis balls, determine the optimal reorder quantity for the distributor.

61. A fabric store purchases scissors for \$5 a pair and resells them for \$9. It costs \$10 to place an order with the manufacturer, and the yearly inventory costs per pair of scissors is \$.80. If the store sells about 900 pairs of scissors each year at a relatively constant rate, how many scissors should be ordered at one time?

62. The monthly demand equation for an electric utility company is estimated to be

$$p = 60 - (10^{-5})\,x$$

where p is measured in dollars and x is measured in thousands of kilowatt hours. The utility has fixed costs of \$7,000,000 per month and variable costs of \$30 per thousand kilowatt hours of electricity generated, so that the cost function is

$$C(x) = 7.10^{6} + 30x.$$

a) Find the value of x and the corresponding price for a thousand kilowatt hours that maximize the utility's profit.

b) Suppose that rising fuel costs increase the utility's variable costs from \$30 to \$40, so that its new cost function is

$$C_1(x) = 7.10^{6} + 40x.$$

Should the utility pass all this increase of \$10 per thousand kilowatt hours on to consumers? Explain your answer.

63. The demand equation for a monopolist is $p = 200 - 3x$, and the cost function is $C(x) = 75 + 80x - x^2$, $0 \leq x \leq 40$.

a) Determine the value of x and the corresponding price that maximize the profit.

b) Suppose that the government imposes a tax on the monopolist of \$4 per unit quantity produced. Determine the new price that maximizes the profit.

c) Suppose that the government imposes a tax of T dollars per unit quantity produced, so that the new cost function is

$$C(x) = 75 + (80 + T)\,x - x^2, \quad 0 \leq x \leq 40$$

Determine the new value of x that maximizes the monopolist's profit as a function of T. Assuming that the monopolist cuts back production to this level, express the tax revenues received by the government as a function of T. Finally, determine the value of T that will maximize the tax revenue received by the government.

9.8 Partial Derivatives:

In the differentiation so far, we have been dealing with functions of a single variable. However, in many cases the dependent variable is a function of several independent variables. For example, profits are a function of sales, revenues and expenses. The extent of repair or service facilities required depends on the rate of demand for such service and the length of time required to accomplish it. Demand for a product may depend on consumer's disposable income, advertising, prices, credit, and perhaps other variables as well.

What we discuss here is the rate of change of the dependent variable with respect to a single one of the independent variables. When one variable is examined at a time, other variables are treated as constants.

In order to indicate that the new derivative is taken with respect to only one of the independent variables, a new symbol is used.

Given a function $z = f(x,y)$, a partial derivative can be found with respect to each independent variable. The partial derivative taken with respect to x is denoted by

$$\frac{\partial z}{\partial x} \quad \text{or} \quad f_x$$

The partial derivative taken with respect to y is denoted by

$$\frac{\partial z}{\partial y} \quad \text{or} \quad f_y$$

Partial derivatives are found by using the same differentiation rules. The only exception is that when a partial derivative is found with respect to one independent variable, the other independent variable is assumed to be held constant.

Examples:

a) Let $f(x,y) = 5x^2 + 6y^3$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Solution:

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 10x \qquad (6y^3 \text{ is considered as a constant}) \\ \frac{\partial f}{\partial y} &= 18y^2 \qquad (5x^2 \text{ is considered as a constant}) \end{aligned}\right\} \underline{\text{Ans}}$$

b) Let $f(x,y) = 5x^3y^2$. Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Solution:

$$f(x,y) = 5x^3y^2 = (5y^2)\,x^3$$

$$\frac{\partial f}{\partial x} = (5y^2).\,3x^2 = 15x^2y^2 \qquad \underline{\text{Ans}}$$

To compute $\frac{\partial f}{\partial y}$, then $f(x,y) = (5x^3)\ y^2$

$$\therefore \quad \frac{\partial f}{\partial y} = 5x^3.2y = 10x^3y \qquad \underline{\text{Ans}}$$

c) If $y = f(x,z) = x^2 + xz + z^3$, find $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial z}$

Solution:

$$\left.\begin{aligned} \frac{\partial y}{\partial x} &= 2x + z \\ \frac{\partial y}{\partial z} &= x + 3z^2 \end{aligned}\right\} \underline{\text{Ans}}$$

d) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x,y) = e^{xy^2}$

Solution:

To compute $\frac{\partial f}{\partial x}$, consider y as a constant.

$$f(x,y) = e^{x(y^2)}$$

Thus, $\frac{\partial f}{\partial x} = y^2.e^{x(y^2)} = y^2.e^{xy^2}$ $\quad \underline{\text{Ans}}$

To compute $\frac{\partial f}{\partial y}$, put x as a constant.

$$f(x,y) = e^{(x)y^2}$$

Therefore $\frac{\partial f}{\partial y} = e^{xy^2}.2xy = 2xy\, e^{xy^2}$ Ans

e) Compute $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *for* $f(x,y) = (4x + 3y - 5)^8$

Solution:

$$f(x,y) = (4x + 3y - 5)^8$$

$$\frac{\partial f}{\partial x} = 8(4x + 3y - 5)^7.(4)$$

$$= 32(4x + 3y - 5)^7 \quad \text{Ans}$$

$$\frac{\partial f}{\partial y} = 8(4x + 3y - 5)^7.(3)$$

$$= 24(4x + 3y - 5)^7 \quad \text{Ans}$$

9.9 Mixed Partial Derivative:

Just as we formed second derivaties in the case of one variable, we can form second partial derivatives of a function f(x,y) of two variables. Since $\frac{\partial f}{\partial x}$ is a function of x and y, we can differentiate it with respect to x or y.

The partial derivative of $\frac{\partial f}{\partial x}$ with respect to x is denoted by $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}

The partial derivative of $\frac{\partial f}{\partial x}$ with respect to y is denoted by $\frac{\partial^2 f}{\partial y\, \partial x}$ or f_{yx}

The partial derivative of $\frac{\partial f}{\partial y}$ with respect to x is denoted by $\frac{\partial^2 f}{\partial x\, \partial y}$ or f_{xy}

The partial derivative of $\frac{\partial f}{\partial y}$ with respect to y is denoted by $\frac{\partial^2 f}{\partial y^2}$ or f_{yy}

One property which is note-worthy is that

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 f}{\partial x\, \partial y} = \text{Cross partial derivatives.}$$

Partial derivatives can be computed for functions of any number of variables. When taking the partial with respect to one variable, we treat the other variables as constants.

Examples:

a) Let $f(x,y) = x^2 + 3xy + 2y^2$. *Calculate* $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$.

Solution:

First compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$$f(x, y) = x^2 + 3xy + 2y^2$$

$$\frac{\partial f}{\partial x} = 2x + 3y$$

$$\frac{\partial f}{\partial y} = 3x + 4y$$

To compute $\frac{\partial^2 f}{\partial x^2}$, differentiate $\frac{\partial f}{\partial x}$ with respect to x.

$$\therefore \frac{\partial^2 f}{\partial x^2} = 2$$

Similarly, to compute $\frac{\partial^2 f}{\partial y^2}$, differentiate $\frac{\partial f}{\partial y}$ with respecte to y:

$$\frac{\partial^2 f}{\partial x^2} = 4$$

To compute $\frac{\partial^2 f}{\partial x \partial y}$, differentiate $\frac{\partial f}{\partial y}$ with respect to x:

$$\frac{\partial^2 f}{\partial x \partial y} = 3$$

Finally, to compute $\frac{\partial^2 f}{\partial y \partial x}$, differentiate $\frac{\partial f}{\partial x}$ with respect to y:

$$\frac{\partial^2 f}{\partial y \partial x} = 3$$

b) Determine all first and second order derivatives of

$$z = 3x^3 + 2x^2y - 3xy^2 + 4y^3$$

Solution:

$$\frac{\partial z}{\partial x} = 9x^2 + 4xy - 3y^2$$

$$\frac{\partial z}{\partial y} = 2x^2 - 6xy + 12y^2$$

$$\frac{\partial^2 z}{\partial x^2} = 18x + 4y \quad ; \quad \frac{\partial^2 z}{\partial x \partial y} = 4x - 6y$$

$$\frac{\partial^2 z}{\partial y^2} = -6x + 24y \quad ; \quad \frac{\partial^2 z}{\partial y \partial x} = 4x - 6y$$

c) Determine all first and second derivatives of

$$z = (x^2 + 3y^3)^4$$

Solution:

$$z = (x^2 + 3y^3)^4$$

$$\frac{\partial z}{\partial x} = 4(x^2 + 3y^3)^3(2x) = 8x(x^2 + 3y^3)^3$$

$$\frac{\partial z}{\partial y} = 4(x^2 + 3y^3)^3(9y^2) = 36y^2(x^2 + 3y^3)^3$$

$$\frac{\partial^2 z}{\partial x^2} = 8x[3(x^2 + 3y^3)^2(2x)] + 8(x^2 + 3y^3)^3(1)$$

$$= 48x^2(x^2 + 3y^3)^2 + 8(x^2 + 3y^3)^3$$

$$\frac{\partial^2 z}{\partial y^2} = 36y^2.[3(x^2 + 3y^3)^2(9y^2)] + 36(x^2 + 3y^3)^3(2y)$$

$$= 972\ y^4(x^2 + 3y^3)^2 + 72y(x^2 + 3y^3)^3$$

$$= 36y(x^2 + 3y^3)^2(27y^3 + 2x^2 + 6y^3)$$

$$= 36y(x^2 + 3y^3)^2(2x^2 + 33y^3)$$

$$\frac{\partial^2 z}{\partial x \partial y} = 8x.[3(x^2 + 3y^3)^2(9y^2)] + 8(x^2 + 3y^3)^3(0)$$

$$= 216xy^2(x^2 + 3y^3)^2$$

$$\frac{\partial^2 z}{\partial y \partial x} = 36y^2.[3(x^2 + 3y^3)^2(2x)] + 36(x^2 + 3y^3)^3(0)$$

$$= 216\ xy^2(x^2 + 3y^3)^2$$

9.10 Interpretation of Partial Derivatives:

The partial derivative of a function $Z = f(x)$ with respect to the variable x describes the change in the function for an incremental change in the value of x.

If the dependent variable is functionally related to two independent variables, as $Z = f(x,y)$, this relationship can be shown by a three - dimensional graph. The partial derivative of Z with respect to x gives the rate of change of the function for an incremental change in x, assuming that y (or other variables) is held constant. Similarly the partial derivative of Z with respect to y gives the rate of change of the function for an incremental change in y with x held constant.

Examples:

1. *Consider the production function $f(x,y) = 60x^{3/4}y^{1/4}$, which gives the number of units of goods produced when utilizing x units of labor and y units of capital.*

   a) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

   b) Evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $x = 81$, $y = 16$

   c) Interpret the numbers computed in part (b).

Solution:

$$f(x,y) = 60x^{\frac{3}{4}} y^{\frac{1}{4}}$$

a)
$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 60 . \frac{3}{4} x^{-\frac{1}{4}} y^{\frac{1}{4}} = 45x^{-\frac{1}{4}} y^{\frac{1}{4}} \\ \frac{\partial f}{\partial y} &= 60 . x^{\frac{3}{4}} . \frac{1}{4} y^{-\frac{3}{4}} = 15x^{\frac{3}{4}} y^{-\frac{3}{4}} \end{aligned}\right\} \underline{\text{Ans}}$$

At $x = 81$, $y = 16$

b)
$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 45(81)^{-\frac{1}{4}} . (16)^{\frac{1}{4}} = \frac{45(2)}{3} = 30 \\ \frac{\partial f}{\partial y} &= 15(81)^{\frac{3}{4}}(16)^{-\frac{3}{4}} = \frac{15(27)}{8} = 50\frac{5}{8} \end{aligned}\right\} \underline{\text{Ans}}$$

c) The quantity $\frac{\partial f}{\partial x}$ is referred to as the marginal productivity of labor. If the amount of capital is held fixed at $y = 16$ and the amount of labor increases by one unit, then the quantity of goods produced will increase by approximately 30 units.

Similarly, the quantity $\frac{\partial f}{\partial y}$ is referred to as the marginal productivity of capital. An increase in capital of one unit (with labor fixed at 81) results in an increase in production of approximately $50\frac{5}{8}$ units of goods. Ans

2. *Estimated sales (s) of a product are related to price p, advertising (a), and the number of salesmen (n). The functional relationship is*

$$S = (10{,}000 - 800p)\, n^{1/2}\, a^{1/2}$$

If the current number of salesmen is n = 100, find the marginal effect of an additional salesman for p = \$5, and a = \$10,000.

Solution:

$$S = (10{,}000 - 800p)\, n^{1/2} \cdot a^{1/2}$$

$$\frac{\partial s}{\partial n} = (10{,}000 - 800p)\, a^{1/2} \cdot \frac{1}{2} n^{-1/2}$$

$$= \frac{1}{2}[10{,}000 - 800(5)] \cdot 10{,}000^{1/2} \cdot \frac{1}{100^{1/2}}$$

$$= \frac{1}{2}(10{,}000 - 4000)\cdot(100)\cdot\frac{1}{10}$$

$$= \frac{1}{2}(6000)(10) = \$30{,}000 \qquad \text{Ans}$$

3. *A national manufacturer estimates that the number of units it sells each year is a function of its expenditures on ratio and TV advertising. The function specifying this relationship is*

$$Z = 50{,}000x + 40{,}000y - 10x^2 - 20y^2 - 10xy.$$

where Z equals the number of units sold annually, x equals the amount spent for TV advertising, and y equals the amount spent for radio advertising (both in thousand dollars).

Find: a) the number of units sold assuming that the firm is currently spending \$40,000 on TV advertising and \$20,000 on radio advertising.

b) the effect on annual sales if \$1,000 more is spent on TV advertising.

c) the effect if an additional \$1000 is spent on radio advertising.

Solution:

a) $z = 50{,}000x + 40{,}000y - 10x^2 - 20y^2 - 10xy$

TV advertising = \$40,000 or $x = 40$ and
Radio advertising = \$20,000 or $y = 20$

$$\therefore z = 50{,}000(40)+40{,}000(20)-10(40)^2-20(20)^2-10(40)(20)$$
$$= 2{,}000{,}000 + 800{,}000 - 16{,}000 - 8{,}000 - 8{,}000$$
$$= 2{,}768{,}000$$

Therefore, 2,768,000 units will be sold <u>Ans</u>

b) If \$1,000 more is spent on TV advertising, then

$$\frac{\partial z}{\partial x} = 50{,}000 - 20x - 10y$$
$$= 50{,}000 - 20(40) - 10(20)$$
$$= 50{,}000 - 800 - 200 = 49{,}000$$

Thus, an increase in TV expenditures of \$1,000 should result in additional sales of approximately 49,000 units <u>Ans</u>

c) If an additional \$1000 is spent on radio advertising.

$$\frac{\partial z}{\partial y} = 40{,}000 - 40y - 10x$$
$$= 40{,}000 - 40(20) - 10(40)$$
$$= 40{,}000 - 800 - 400$$
$$= 38{,}800$$

Thus an increase of \$1,000 in radio advertising expenditures will lead to an approximate increase of 38,800 units. <u>Ans</u>

EXERCISE: 9 - 6

In each of Problems 1 - 26, find all partial derivatives.

1. $f(x,y) = x - 5y + 3$.
2. $f(x,y) = 4 - 5x^2 + 6y^3$.
3. $f(x,y) = 3x - 4$.
4. $f(x,y) = \sqrt{7}$.
5. $g(x,y) = x^5y^4 - 3x^4y^3 + 7x^3 + 2y^2 - 3xy + 4$.
6. $g(x,y) = x^8 - 2x^6y^5 + 3x^5y^3 + x^3y^3 + 3x - 4$.
7. $g(p,q) = \sqrt{pq}$.
8. $g(w,z) = \sqrt[3]{w^2 + z^2}$.

9. $h(s,t) = \dfrac{s^2 + 4}{t - 3}$.

10. $h(u,v) = \dfrac{4uv^2}{u^2 + v^2}$

11. $u(q_1, q_2) = \frac{3}{4} \ln q_1 + \frac{1}{4} \ln q_2$.

12. $Q(l,k) = 31^{.41}k^{.59}$.

13. $h(x,y) = \dfrac{x^2 + 3xy + y^2}{\sqrt{x^2 + y^2}}$.

14. $h(x,y) = \dfrac{\sqrt{x + 4}}{x^2y + y^2x}$.

15. $z = e^{5xy}$.

16. $z = (x^2 + y)e^{3x+4y}$.

17. $z = 5x \ln (x^2 + y)$.

18. $z = \ln (3x^2 + 4y^4)$.

19. $f(r,s) = \sqrt{r + 2s}(r^3 - 2rs + s^2)$.

20. $f(r,s) = \sqrt{rs}\, e^{2+r}$.

21. $f(r,s) = e^{3-r} \ln (7 - s)$.

22. $f(r,s) = (5r^2 + 3s^3)(2r - 5s)$.

23. $g(x,y,z) = 3x^2y + 2xy^2z + 3z^3$.

24. $g(x,y,z) = x^2y^3z^5 - 3x^2y^4z^3 + 5xz$.

25. $g(r,s,t) = e^{s+t}(r^2 + 7s^3)$.

26. $g(r,s,t,u) = rs \ln (2t + 5u)$.

In Problems 27 - 32, evaluate the given partial derivatives.

27. $f(x,y) = x^3y + 7x^2y^2$; $f_x(1, -2)$.

28. $z = \sqrt{5x^2 + 3xy + 2y}$; $\left.\dfrac{\partial z}{\partial x}\right|_{\substack{x=0 \\ y=2}}$

29. $g(x,y,z) = e^x\sqrt{y + 2z}$; $g_z(0,1,4)$.

30. $g(x,y,z) = \dfrac{3x^2 + 2y}{xy + xz}$; $g_y(1,1,1)$.

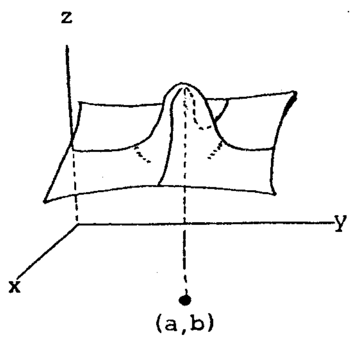
31. $h(r,s,t,u) = (s^2 + tu) \ln (2r + 7st)$; $h_s(1,0,0,1)$.

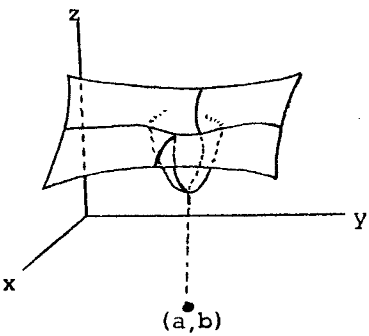
32. $h(r,s,t,u) = \dfrac{7r + 3s^2u^2}{s}$; $h_t(4,3,2,1)$

9.11 Maxima and Minima of Functions of Several Variables:

If $f(x, y)$ is a function of two variables, then we say that $f(x,y)$ has a maximum when $x = a$, $y = b$ if $f(x,y)$ is at most equal to $f(a,b)$ whenever x is near a and y is near b.

Similarly, we say that $f(x,y)$ has a minimum when $x = a$, $y = b$ if $f(x,y)$ is at least equal to $f(a,b)$ whenever x is near a and y is near b.





Suppose that $f(x,y)$ has a maximum at $(x,y) = (a,b)$. Then, keeping y constant at b, $f(x,y)$ becomes a function of the variable x with a maximum at $x = a$. Therefore its derivative with respect to x is zero at $x = a$. That is,

$$\frac{\partial f}{\partial x}(a,b) = 0$$

Similarly, keeping x constant at a, $f(x,y)$ becomes a function of the variable y with a maximum at $y = b$. Therefore its derivative with respect to y is zero at $y = b$. That is,

$$\frac{\partial f}{\partial y}(a,b) = 0$$

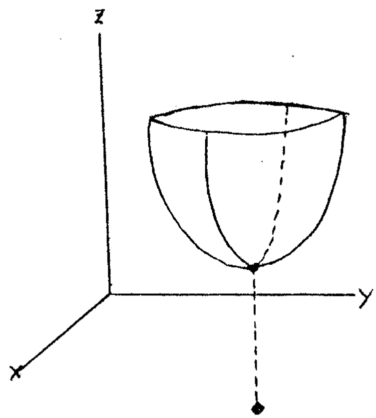
Similarly considerations apply if $f(x,y)$ has a minimum at $(x,y) = (a,b)$. Thus we have the following test for extrema in two variables.

> **First Derivative for Extrema:**
>
> *If $f(x,y)$ has either a maximum or a minimum at $(x,y) = (a,b)$, then*
>
> $$\frac{\partial f}{\partial x}(a,b) = 0$$
>
> *and* $$\frac{\partial f}{\partial y}(a,b) = 0$$

Examples:

1. *The function $f(x,y) = 3x^2 - 4xy + 3y^2 + 8x - 17y + 5$ has the graph as shown in the picture. Find the point (x,y) at which $f(x,y)$ attains its minimum.*

Solution:



$$\frac{\partial f}{\partial x} = 6x - 4y + 8$$

$$\frac{\partial f}{\partial y} = -4x + 6y - 17$$

Setting both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y} = 0$,

we obtain

$$6x - 4y + 8 = 0 \quad \text{or} \quad y = \frac{6x + 8}{4}$$

$$-4x + 6y - 17 = 0 \quad \text{or} \quad y = \frac{4x + 17}{6}$$

By equating these two expressions for y, we have

$$\frac{6x + 8}{4} = \frac{4x + 17}{6}$$

$$x = 1$$

$$y = \frac{7}{2}$$

Therfore $f(x,y)$ has a minimum at $(x,y) = \left(1, \frac{7}{2}\right)$ <u>Ans</u>

2. *A monopolist markets his product in two countries and can charge different amounts in each country. Due to the laws of demand, the monopolist must set the price at $97 - \frac{x}{10}$ dollars in the first country and $83 - \frac{y}{20}$ dollars in the second country in order to sell all the units. (where x = number of units to be sold in the first country and y = number of units to be sold in the second country). The cost of producing these units is $20{,}000 + 3(x+y)$. Find the values of x and y that maximize the profit.*

Solution:

Let f(x,y) be the profit derived from selling x units in the first country and y in the second.

Then $f(x,y)$ = revenue from both countries - cost.

$$= \left(97 - \frac{x}{10}\right)x + \left(83 - \frac{y}{20}\right)y - [20{,}000 + 3(x + y)]$$

$$= 97x - \frac{x^2}{10} + 83y - \frac{y^2}{20} - 20{,}000 - 3x - 3y$$

$$= 94x - \frac{x^2}{10} + 80y - \frac{y^2}{20} - 20{,}000$$

To find where f(x,y) has its maximum value, we look for those values of x and y at which both partial derivatives are zero.

Then
$$\frac{\partial f}{\partial x} = 94 - \frac{x}{5} = 0$$
$$x = 470$$

$$\frac{\partial f}{\partial y} = 80 - \frac{y}{10} = 0$$
$$y = 800$$

Therefore the firm should adjust its prices to levels where it will sell 470 units in the first country and 800 units in the second country Ans

3. *Suppose that we want to design a rectangular building having volume 147,840 cubic feet. Assuming that the daily loss of heat is given by*

$$11xy + 14yz + 15xz,$$

where x, y, and z are, respectively, the length, width, and height of the building, find the dimensions of the building for which the daily heat loss is minimal.

Solution:

We must minimize the function $11xy + 14yz + 15xz$
where x,y, z satisfy $xyz = 147,840$
Let A denote the volume of the building (147,840)

$$\therefore \quad xyz = A$$

$$z = \frac{A}{xy}$$

Substitute this expression for z into the daily heat loss function to obtain a heat loss function g(x,y) of two variables - namely

$$\therefore \quad 11xy + 14yz + 15x\,z = 11xy + 14y\left(\frac{A}{xy}\right) + 15x\left(\frac{A}{xy}\right)$$

$$\text{or } g(x,y) = 11xy + \frac{14A}{x} + \frac{15A}{y}$$

To minimize the function compute $\frac{\partial g}{\partial x}$ and $\frac{dg}{\partial y}$ and equate them to Zero.

$$\frac{\partial g}{\partial x} = 11y - \frac{14A}{x^2} = 0$$

$$y = \frac{14A}{11x^2}$$

$$\frac{\partial g}{\partial x} = 11x - \frac{15A}{y^2} = 0$$

$$11xy^2 = 15A$$

By algebraical compu tation, we can find

$$x = 56 \text{ and } y = 60$$

Finally $\quad z = \frac{A}{xy} = \frac{147,840}{(56)(60)} = 44$

Thus the building should be 56 feet long, 60 feet wide, and 44 feet high in order to minimize the heat loss. Ans

For functions of two variables, there is also test for maxima and minima.

Second Derivative Test for Maxima and Minima *Suppose that $f(x,y)$ is a function and (a,b) is a point at which*

$$\frac{\partial f}{\partial x}(a,b) = 0 \text{ and } \frac{\partial f}{\partial y}(a,b) = 0,$$

and let

$$D(x,y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

1. *If*

$$D(a,b) > 0 \text{ and } \frac{\partial^2 f}{\partial x^2}(a,b) > 0,$$

then $f(x,y)$ has a minimum at (a,b)

2. *If*

$$D(a,b) > 0 \text{ and } \frac{\partial^2 f}{\partial x^2}(a,b) < 0,$$

then $f(x,y)$ has a maximum at (a,b).

3. *If*

$$D(a,b) < 0$$

then $f(x,y)$ has neither a maximum nor a minimum at (a,b)

4. *If*

$$D(a,b) = 0, \text{ then no conclusion is possible.}$$

Examples:

1. *Let $f(x,y) = y^3 - x^2 + 6x - 12y + 5$. Find all maximum and minimum points of $f(x,y)$. Use the second derivative test to determine the nature of each such point.*

Solution:

$$f(x,y) = y^3 - x^2 + 6x - 12y + 5$$

$$\frac{\partial f}{\partial x} = -2x + 6 \,, \quad \frac{\partial f}{\partial y} = 3y^2 - 12,$$

We find that $f(x,y)$ has a potential maximum when

$$-2x + 6 = 0$$
$$3y^2 - 12 = 0$$

From these two equations, we derive $x = 3$ and $y = \pm 2$.

Thus $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both zero when $(x,y) = (3,2)$ and when $(x,y) = (3, -2)$

To apply the second derivative test, we compute

$$\frac{\partial^2 f}{\partial x^2} = -2, \quad \frac{\partial^2 f}{\partial y^2} = 6y, \quad \frac{\partial^2 f}{\partial x \partial y} = 0$$

and

$$D(x,y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = (-2)(6y) - 0 = -12y$$

Since $D(3,2) = -24$ is negative, case 3 of the second derivative test says that $f(x,y)$ has no maximum or minimum at $(3,2)$. But $D(3,-2) = 24 > 0$, $\frac{\partial^2 f}{\partial x^2}(3,-2) = -2 < 0$

Thus by case 2 of the second derivative test, the function $f(x,y)$ has a maximum at $(3,-2)$ Ans

In the case of three or more variables, the test can be handled in a similar fashion.

For instance

If $f(x,y,z)$ has a maximum or a minimum at $(x,y,z) = (a,b,c,)$ $\frac{\partial f}{\partial x}(a,b,c,) = 0$ $\frac{\partial f}{\partial y}(a,b,c) = 0$ $\frac{\partial f}{\partial z}(a,b,c) = 0$

EXERCISE: 9 - 7

Find all points (x,y) where $f(x,y)$ has a possible maximum or minimum.

1. $f(x,y) = x^2 - 3y^2 + 4x + 6y + 8$
2. $f(x,y) = \frac{1}{2}x^2 + y^2 - 3x + 2y - 5$

3. $f(x,y) = x^2 - 5xy + 6y^2 + 3x - 2y + 4$

4. $f(x,y) = -3x^2 + 7xy - 4y^2 + x + y$

5. $f(x,y) = x^3 + y^2 - 3x + 6y$

6. $f(x,y) = x^2 - y^3 + 5x + 12y + 1$

7. $f(x,y) = \frac{1}{3}x^3 - 2y^3 - 5x + 6y - 5$

8. $f(x,y) = x^4 - 8xy + 2y^2 - 3$

9. The function $f(x,y) = 2x + 3y + 9 - x^2 - xy - y^2$ has a maximum at some point (x,y). Find the values of x and y where this situation occurs.

10. The function $f(x,y) = \frac{1}{2}x^2 + 2xy + 3y^2 - x + 2y$ has a minimum at some point (x,y). Find the values of x and y where this situation occurs.

Find all points (x,y) where $f(x,y)$ has a possible maximum or minimum. Then use the second derivative test to determine, if possible, the nature of $f(x,y)$ at each of these points. If the second derivative test is inconclusive, so state.

11. $f(x,y) = x^2 - 2xy + 4y^2$

12. $f(x,y) = 2x^2 + 3xy + 5y^2$

13. $f(x,y) = -2x^2 + 2xy - y^2 + 4x - 6y + 5$

14. $f(x,y) = -x^2 - 8xy - y^2$

15. $f(x,y) = x^2 + 2xy + 5y^2 + 2x + 10y - 3$

16. $f(x,y) = x^2 - 2xy + 3y^2 + 4x - 16y + 22$

17. $f(x,y) = x^3 - y^2 - 3x + 4y$

18. $f(x,y) = x^3 - 2xy + 4y$

19. $f(x,y) = 2x^2 + y^3 - x - 12y + 7$

20. $f(x,y) = x^2 + 4xy + 2y^4$

21. Find the values of x,y,z at which

$$f(x,y,z) = 2x^2 + 3y^2 + z^2 - 2x - y - z$$

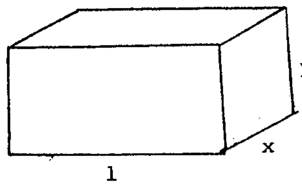
assumes its minimum value.

22. Find the values of x,y,z at which

$$f(x,y,z) = 5 + 8x - 4y + x^2 + y^2 + z^2$$

assumes its maximum value.

23. U.S. postal rules require that the length plus the girth of a package cannot exceed 84 inches in order to be mailed. Find the dimensions of the rectangular package of greatest volume that can be mailed.
[Note: From Fig. we see that 84 = (length) + (girth) = $l + (2x + 2y)$ and volume = xyl].



24. Find the dimensions of the rectangular box of least surface area that has a volume of 100 cubic inches.
[Note: From No.23 we see that the surface area = $2xy+2xl+2yl$ and volume = 100 = xyl.]

25. A company manufactures and sells two products, call them I and II, that sell for \$10 and \$9 per unit, respectively. The cost of producing x units of product I and y units of product II is

$$400 + 2x + 3y + .01(3x^2 + xy + 3y^2).$$

Find the values of x and y that maximize the company's profit.
[Note: Profit = (revenue) - (cost)].

26. A monopolist manufactures and sells two competing products, call them I and II, that cost \$30 and \$20 per unit, respectively, to produce. The revenue from marketing x units of product I and y units of product II is $98x + 112y - .04xy - .1x^2 - .2y^2$. Find the values of x and y that maximize the monopolist's profits.

27. A company manufactures and sells two products, call them I and II, that sell for $\$p_I$ and $\$p_{II}$ per unit, respectively. Let $C(x,y)$ be the cost of producing x units of product I and y units of product II. Show that if the company's profit is maximized when $x = a, y = b$, then

$$\frac{\partial C}{\partial x}(a,b) = p_I \text{ and } \frac{\partial C}{\partial y}(a,b) = p_{II}$$

28. A monopolist manufactures and sells two competing products, call them I and II, that cost $\$p_I$ and $\$p_{II}$ per unit, respectively, to produce. Let $R(x,y)$ be the revenue from marketing x units of product I and y units of product II. Show that if the monopolist's profit is maximized when $x = a, y = b$, then

$$\frac{\partial R}{\partial x}(a,b) = p_I \text{ and } \frac{\partial R}{\partial y}(a,b) = p_{II}.$$

CHAPTER 10

INTEGRATION

Integral calculus is concerned with the opposite process to that of the derivative. That is to say, we are given the derivative of a function and must find the original function. The integral calculus may be subdivided into two parts: *the indefinite integral;* and *the definite integral.*

An important concern of integral calculus is the determination of area which occur between curves and other defined boundaries. Also, if the derivative of an unknown function is known, integral calculus may provide a way of determining the original function.

Antiderivatives:

Given a function f(x), we can find the derivative f'(x). There may be occasions in which we are given the derivative f'(x). Since the process of finding the original function is the reverse of differentiation, f(x) is said to be an *antiderivative of* f'(x).

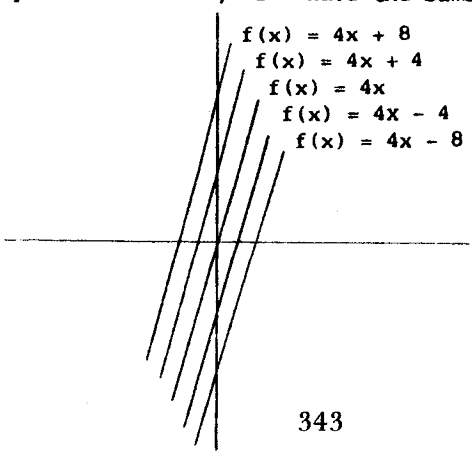
Consider the derivative $f'(x) = 4$

By using a trial-and-error approach, it is not very difficult to conclude that the function $f(x) = 4x$ has a derivative of the form $f'(x) = 4$.

Another function having the same derivative is $f(x) = 4x + 1$

In fact, any function having the form $f(x) = 4x + c$

where C equals a constant, will have the same derivative.



10.1 The Indefinite Integral:

The process of finding antiderivatives is more frequently called *integration*. And the family of function obtained through this process is called the *indefinite integral.*

The notation

$$\int f(x)\ dx$$

is often used to indicate the indefinite integral of the function f(x). The symbol $\int$ is the *integral sign,* f(x) is the *integrand,* or the function for which we want to find the indefinite integral, and dx, indicates the variable with respect to which the integration process is performed. It is read "the indefinite integral of f(x) with respect to x."

The indefinite integral with respect to x of any function f is written $\int f(x)\ dx$ and denotes an arbitrary antiderivative of f. It can be shown that all derivatives of f differ at most by a constant. Thus if F is an antiderivative of f, then

$$\int f(x)\ dx = F(x) + C, \text{ where C is a constant.}$$

10.2 Rules of Integration:

Following are a set of rules for finding the indefinite integral of some functional forms common in business and economics applications.

Rule 1:

The integral of a constant.

$$\boxed{\int k.dx = kx + c}$$ where k is real

Examples:

a) $\int(-2)\ dx \quad = -2x + c$

b) $\int \frac{3}{2} dx \quad = \frac{3}{2} x + c$

c) $\int \sqrt{2}\ dx \quad = \sqrt{2}\ x + c$

d) $\int 0 dx \quad = (0)x + c$

Rule 2:

Power Rule (The integral of a variable to a constant power)

When the integrand is x raised to some real valued exponent increase the exponent of x by 1, divided by the new exponent, and add the constant of integration.

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

Examples:

a) $\int x\, dx = \frac{x^2}{2} + c$

b) $\int x^2 dx = \frac{x^3}{3} + c$

c) $\int \sqrt{x}\, dx = \int x^{\frac{1}{2}}\, dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}\, x^{\frac{3}{2}} + c$

d) $\int \frac{1}{x^3}\, dx = \int x^{-3} dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c$

Rule 3:

The integral of a constant times a function:

The indefinite integral of a constant k times a function f(x) is found by multiplying the constant by the indefinite integral of f(x).

$$\int k.\, f(x)\, dx = k \int f(x)\, dx$$

where k is a real-valued constant.

Examples:

a) $\int 5x \, dx = 5\int x \, dx = 5\left(\frac{x^2}{2} + c_1\right)$

$= \frac{5}{2} x^2 + 5c_1$

$= \frac{5}{2} x^2 + c.$

b) $\int \frac{x^2}{2} \, dx = \int \frac{1}{2} x^2 dx$

$= \frac{1}{2}\int x^2 dx = \frac{1}{2}\left(\frac{x^3}{3} + c_1\right)$

$= \frac{x^3}{6} + \frac{1}{2} c_1$

$= \frac{x^3}{6} + c$

c) $\int \frac{3}{\sqrt{x}} \cdot dx = \int 3x^{-\frac{1}{2}} \, dx$

$= 3\int x^{-\frac{1}{2}} \, dx$

$= \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} + c$

d) $\int 10x^{\frac{3}{2}} \, dx = 10\int x^{\frac{3}{2}} \, dx$

$= 10 \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c$

$= 4x^{\frac{5}{2}} + c$

Rule 4:

Sum or Difference: (The integral of the sum or difference of two or more functions).

The integral of the sum (difference) of two functions is the sum (difference) of their respective integrals.

$$\boxed{\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx}$$

Examples:

a) $$\int (3x - 6)\,dx = \int 3x\,dx - \int 6dx$$
$$= \frac{3x^2}{2} + c_1 - (6x + c_2)$$
$$= \frac{3x^2}{2} - 6x + c_1 - c_2$$
$$= \frac{3x^2}{2} - 6x + c$$

b) $$\int (4x^2 - 7x + 6)\,dx = \int 4x^2\,dx - \int 7x\,dx + \int 6dx$$
$$= \frac{4x^3}{3} - \frac{7x^2}{2} + 6x + c$$

c) $$\int (x^{-3} - x^{-2})\,dx = \int x^{-3}\,dx - \int x^{-2}\,dx$$
$$= -\frac{x^{-2}}{2} + x^{-1} + c$$

d) $$\int y^2\left(y + \frac{2}{3}\right)dy = \int \left(y^3 + \frac{2}{3}y^2\right)dy$$
$$= \frac{y^4}{4} + \left(\frac{2}{3}\right)\frac{y^3}{3} + c$$
$$= \frac{y^4}{4} + \frac{2y^3}{9} + c$$

Rule 5:

Power-Rule Exception:

This is the exception associated with Rule 2 (the power rule) where $n = -1$ for x^n. If $f(x) = \ln x$, $f'(x) = \frac{1}{x} = x^{-1}$.

$$\boxed{\int x^{-1}\,dx = \log_e n + c = \ln x + c}$$

The rule applies when the variable x is raised to the constant power $n = -1$. For x raised to any power other than $n = -1$, the power-rule must be applied.

Examples:

a) $\int (3x^{-1} + 2)\, dx = 3\log_e x + 2x + c$
$= 3 \ln x + 2x + c$

b) $\int 2x^{-1}\, dx = 2 \log_e x + c$
$= 2 \ln x + c$

c) $\int \frac{2}{x}\, dx = \int 2x^{-1}\, dx = 2 \log_e x + c$
$= 2 \ln x + c$

Rule 6:

If the integrand consists of the product of a function f(x) raised to a power n and the derivative of f(x), the indefinite integral is found by increasing the exponent of f(x) by 1 and dividing by the new exponent: This is also called "Integration by Substitution."

$$\boxed{\int [f(x)]^n \; f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c}$$

Examples:

a) $\int (5x - 3)^3 (5)\, dx = \frac{(5x-3)^4}{4} + c$

$$\boxed{\begin{array}{l} f(x) = 5x - 3 \\ f'(x) = 5 \end{array}}$$

b) Evaluate $\int \sqrt{2x^2 - 6}\, .\, (4)\, dx$

$f(x) = 2x^2 - 6$

$f'(x) = 4x$

Since the other factor in the integrand is 4 and not 4x, then the integral cannot be evaluated.

c) Evaluate $\int (x^2 - 2x)^5 (x - 1)\ dx$

$$f(x) = x^2 - 2x$$

$$f'(x) = 2x - 2$$

From the original problem, we notice that the second factor in the integrand is x - 1 and not 2x - 2.

To apply Rule 7, we have to manipulate the integrand into the proper form.

$$\begin{aligned}\int (x^2 - 2x)^5 (x - 1)\ dx &= \frac{2}{2} \int (x^2 - 2x)^5 (x - 1)\ dx \\ &= \frac{1}{2} \int (x^2 - 2x)^5\ 2(x - 1)\ dx \\ &= \frac{1}{2} \int (x^2 - 2x)^5 (2x - 2)\ dx \\ &= \frac{1}{2} \left[\frac{(x^2 - 2x)^6}{6} \right] + c \\ &= \frac{(x^2 - 2x)^6}{12} + c\end{aligned}$$

d) Evaluate $\int (x^4 - 2x^2)^4 (4x^2 - 4)\ dx$

$$f(x) = x^4 - 2x^2$$

$$f'(x) = 4x^3 - 4x$$

The integrand is not quite in the form of Rule 7.

The integral cannot be evaluate..

Do not multiply variable through an integral sign.

Rule 7:

Integration by Substitution.

$$\boxed{\int f'(x)\ e^{f(x)}.\ dx = e^{f(x)} + c}$$

Examples:

a) Evaluate $\int 2x\ e^{x^2}\ dx$

First identify an integrand which contains e raised to a power that is in a function of x.

$$f(x) = x^2 \text{ and } f'(x) = 2x$$

According to Rule 8: the integrand has the appropriate form, and

$$\int 2x\ e^{x^2}\ dx \ = \ e^{x^2} + c \qquad \underline{\text{Ans}}$$

b) Evaluate $\int x^2\ e^{3x^3}\ dx$

$$f(x) = 3x^3 \quad \text{and}$$
$$f'(x) = 9x^2$$

The integrand is currently not in a form which is suitable for using Rule 8. However some manipulation can be done.

Thus $\int x^2 e^{3x}\ dx$

$$= \frac{9}{9} \int x^2 e^{3x^3}\ dx$$
$$= \frac{1}{9} \int 9x^2 e^{3x^3}\ dx$$
$$= \frac{1}{9}\ e^{3x^3} + c \qquad \underline{\text{Ans}}$$

c) Evaluate $\int (x^2 + 1)\ e^{x^3+3x}\ .\ dx$

$$f(x) = x^3 + 3x$$
$$f'(x) = 3x^2 + 3 = 3(x^2 + 1)$$

By some manipulation:

$$\int (x^2 + 1)\ e^{x^{3+3x}}.\ dx = \frac{3}{3} \int (x^2 + 1)\ x^{x^3+3x}.\ dx$$
$$= \frac{1}{3} \int 3(x^2 + 1)\ e^{x^{3+3x}}\ .\ dx$$
$$= \frac{1}{3}\ e^{x^3+3x} + c \qquad \underline{\text{Ans}}$$

d) $a \int x\ e^{-x^2}\ dx$ when a is a constant.

$$f(x) = -x^2$$
$$f'(x) = -2x$$

Therefore, $a \int x\ d^{-x^2}\ dx \ = \ a\left(\frac{-2}{-2}\right) \int x.e^{-x^2}\ dx$

$$= -\frac{a}{2} \int -2x.e^{-x^2}\ dx$$
$$= -\frac{a}{2}\ .\ e^{-x^2} + c \qquad \underline{\text{Ans}}$$

Rule 8:

Integration by substitution:

$$\boxed{\int \frac{f'(x)}{f(x)}\, dx = \ln f(x) + c}$$

Examples:

a) Evaluate $\displaystyle\int \frac{6x}{3x^2 - 10}\, dx$

$f(x) = 3x^2 - 10$

$f'(x) = 6x$

Since the integrand has the form required by Rule 8

$$\therefore \int \frac{6x}{3x^2 - 10}\, dx = \ln (3x^2 - 10) + c \qquad \underline{\text{Ans}}$$

b) Evaluate $\displaystyle\int \frac{x - 1}{4x^2 - 8x + 10}\, dx$

$f(x) = 4x^2 - 8x + 10$

$f'(x) = 8x - 8$

The integrand can be rewritten in the required form.

Thus $\displaystyle\int \frac{x - 1}{4x^2 - 8x + 10}\, dx = \frac{8}{8} \int \frac{x - 1}{4x^2 - 8x + 10}\, dx$

$$= \frac{1}{8} \int \frac{8(x - 1)}{4x^2 - 8x + 10}\, dx$$

$$= \frac{1}{8} \ln (4x^2 - 8x + 10) + c \qquad \underline{\text{Ans}}$$

c) Evaluate $\displaystyle\int \frac{2x}{x^2 + 5}\, dx$

$f(x) = x^2 + 5$

$f'(x) = 2x$

$$\therefore \int \frac{2x}{x^2 + 5}\, dx = \ln (x^2 + 5) + c \qquad \underline{\text{Ans}}$$

d) Evaluate $\int \frac{(2x^3+3x)}{x^4+3x^2+7}\,dx$

$$f(x) = x^4+3x^2+7$$
$$f'(x) = 4x^3+6x = 2(2x^3+3x)$$
$$\therefore \int \frac{(2x^3+3x)\,dx}{x^4+3x^2+7} = \frac{2}{2}\int \frac{(2x^3+3x)\,dx}{x^4+3x^2+7}$$
$$= \frac{1}{2}\int \frac{2(2x^3+3x)\,dx}{x^4+3x^2+7}$$
$$= \frac{1}{2}\ln(x^4+3x^2+7)+c$$
$$\text{or} = \ln(x^4+3x^2+7)^{\frac{1}{2}}+c \qquad \underline{\text{Ans}}$$

Rule 9:

The integral of the exponential function:

$$\boxed{\int a^{kx}\,dx = \frac{a^{kx}}{k\ \ln(a)} + c}$$

Examples:

a) $\int 3^{2x}\,dx = \frac{3^{2x}}{2\ln 3} + c$ <u>Ans</u>

b) $\int 10^{0.5x}.dx = \frac{10^{0.5x}}{0.5\ln 10} + c$ <u>Ans</u>

c) $\int e^x\,dx = \frac{e^x}{\ln e} + c$

$= e^x + c$, since $\ln e = 1$ <u>Ans</u>

d) $\int 4e^{2x}\,dx = \frac{4e^{2x}}{2\ln e} + c$

$= 2e^{2x} + c$ <u>Ans</u>

Rule 10:

The integral of logarithmic function:

$$\boxed{\int \log_a(kx)\,dx = x\log_a(kx) - x\log_a e + c}$$

If this function is ln (kx), the integral becomes:-

$$\boxed{\int \ln(kx)\,dx = x\ln(kx) - x + c}$$

Examples:

a) $\int \ln x\,dx \quad = x\ln x - x + c$

b) $\int \ln(3x)\,dx \quad = x\ln(3x) - x + c$

c) $\int 2(5000)\log_e t\,dt \quad = \int 10{,}000\log_e t\,dt$

$= 10{,}000\,t\,(\log_e t - \log_e e)$

$= 10{,}000t(\log_e t - 1)$ <u>Ans</u>

d) $\int \log_3 x\,dx \quad = x\log_3 x - x\log_3 e + c$

$= x(\log_3 x - \log_3 e) + c$ <u>Ans</u>

Rule 11:

Integration by Parts:

$$\boxed{\int f(x)\,.\,g'(x)\,.\,dx = f(x).\,g(x) - \int g(x)\,f'(x)\,dx}$$

Examples:

a) Determine $\int x.e^{-x}\,dx$

$f(x) = x$

$g'(x) = e^{-x} \quad \therefore g(x) = \int e^{-x}\,dx = -e^{-x}$

By using the formula for integration by parts:

$$\int x.e^{-x}\,dx = x(-e^{-x}) - \int (-e^{-x}).\,1.\,dx$$
$$= -x.e^{-x} - e^{-x} + c$$
$$= -e^{-x}(x+1) + c \qquad \underline{\text{Ans}}$$

b) Determine $\int x^2 e^{-x}\,dx$

$$f(x) = x^2 \qquad f'(x) = 2x$$
$$g'(x) = e^{-x} \quad \therefore\ g(x) = \int e^{-x}\,dx = -e^{-x}$$

Substituting these terms in Rule 11:

$$\int x^2 e^{-x}\,dx = x^2(-e^{-x}) - \int (-e^{-x})(2x)\,dx$$
$$= -x^2e^{-x} + 2\int e^{-x}.x\,dx$$

And from example (a)

$$\int x.e^{-x}\,dx = -e^{-x}(x+1) + c$$

Therefore, $\int x^2 e^{-x}\,dx = -x^2e^{-x} + 2\,[-e^{-x}(x+1) + c_1]$

$$= -x^2e^{-x} - 2e^{-x}(x+1) + c \qquad \underline{\text{Ans}}$$

c) Determine $\int x \ln x\,dx$

$$f(x) = \ln x \qquad \text{and } g'(x) = x$$
$$f'(x) = \frac{1}{x} \qquad \text{and } g(x) = \int x\,dx = \frac{x^2}{2}$$

Using the formula:

$$\int x \ln x\,dx = (\ln x).\,\frac{x^2}{2} - \int \frac{x^2}{2}.\,\frac{1}{x}.\,dx$$
$$= \frac{x^2 \ln x}{2} - \int \frac{x}{2}.\,dx$$
$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c \qquad \underline{\text{Ans}}$$

d) Determine $\int x^2 \ln x.\,dx$

$$f(x) = \ln x \qquad \text{and } g'(x) = x^2$$
$$f'(x) = \frac{1}{x} \qquad \text{and } g(x) = \int x^2\,dx = \frac{x^3}{3}$$

Using the formula:

$$\int x^2 \ln x.\,dx = (\ln x).\left(\frac{x^3}{3}\right) - \int \frac{x^3}{3}.\,\frac{1}{x}.\,dx$$
$$= \frac{x^3 \ln x}{3} - \int \frac{x^2}{3}.\,dx$$
$$= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + c \qquad \underline{\text{Ans}}$$

Rules for integration:

1. $\int k.\ dx \quad = kx + c$

2. $\int x^n\ dx \quad = \dfrac{x^{n+1}}{n+1} + c,\ n \neq 1$

3. $\int k.f(x)\ dx \quad = k\int f(x)\ dx$

4. $\int [f(x) \pm g(x)]\ dx \quad = \int f(x)\ dx \pm \int g(x)\ dx$

5. $\int x^{-1}\ dx \quad = \ln x + c$

6. $\int [f(x)]^n.\ f'(x)\ dx \quad = \dfrac{[f(x)]^{n+1}}{n+1} + c$

7. $\int f'(x).\ e^{f(x)}.\ dx \quad = e^{f(x)} + c$

8. $\int \dfrac{f'(x)}{f(x)}\ .\ dx \quad = \ln f(x) + c$

9. $\int a^{kx}.\ dx \quad = \dfrac{a^{kx}}{k \ln(a)} + c$

10. $\int \log_a(kx)\ dx \quad = x \log_a(kx) - x \log_a e + c$

11. $\int f(x).\ g'(x).dx \quad = f(x).g(x) - \int g(x).f'(x)\ dx$

(Integration by parts)

Note: Determining the integrals of certain functions requires techniques of integration that are beyond the scope of business-oriented texts. Such integrals can most readily be determined by the use of table of integrals. Table contains those integrals most commonly used in business and economic models. For a more extensive reference, the reader is referred *to Standard Mathematical Tables*.

EXERCISE: 10-1

For Exercises 1-30, find the indefinite integral.

1. $\int 100\ dx$
2. $\int -50\ dx$
3. $\int 3x\ dx$
4. $\int \frac{x}{4}\ dx$
5. $\int (2x - 5)\ dx$
6. $\int (10 - 6x)\ dx$
7. $\int (mx + b)\ dx$, m and b constants.

8. $\int x^4 dx$

9. $\int \sqrt{x^3}\, dx$

10. $\int 5/\sqrt[3]{x^2}\, dx$

11. $\int \frac{dx}{x^5}$

12. $\int dx$

13. $\int (x^4 - 3x^3 - 3x^2)\, dx$

14. $\int (ax^2 + bx + c)\, dx$, a, b, and c constants.

15. $\int (x - 10)^5\, dx$

16. $\int (x^2 + 5)^3 (2x)\, dx$

17. $\int \sqrt{x + 25}\, dx$

18. $\int 4x/\sqrt{2x^2 - 5}\, dx$

19. $\int (x^3 + 16)^4 (6x^2)\, dx$

20. $\int (4x^4 - 16x)^{3/2} (x^3 - 1)\, dx$

21. $\int (x^2 - 2x)^7 (x - 1)\, dx$

22. $\int (x^2/6 - 20)^5 (x)\, dx$

23. $\int 3x^2 e^{x^3}\, dx$

24. $\int e^{5x}\, dx$

25. $\int e^{mx}\, dx$, m constant

26. $\int (x - 1)\, e^{2x^2 - 4x}\, dx$

27. $\int \frac{2x}{x^2 + 5}\, dx$

28. $\int \frac{x^2 - 1}{x^3 - 3x}\, dx$

29. $\int \frac{10}{5x - 6}\, dx$

30. $\int \frac{dx}{mx + b}$, m and b constants

Evaluate the following indefinite integrals.

31. $\int 6\, dx$

32. $\int 3x\, dx$

33. $\int x^{-4}\, dx$

34. $\int (x^2 + 2)\, dx$

35. $\int (4x^3 + 3x^2 - 4)\, dx$

36. $\int (x^7 + 1)\, dx$

37. $\int (3\sqrt{x} + x)\, dx.$

38. $\int \left(x - \frac{1}{x^2}\right) dx.$

39. $\int (4x^{3/2} + x^{1/2})\, dx.$

40. $\int x(x - 1)\, dx.$

41. $\int \frac{3x^5 + 1}{x^2}\, dx.$

42. $\int \frac{x^2 + 2x + 1}{x^4}\, dx.$

43. $\int \frac{x^3 - 8}{x - 2}\, dx.$

44. $\int \frac{x^2 - 4}{x + 2}\, dx.$

45. Verify that

a) $\int (x \cdot \sqrt{x})\, dx \neq \int x\, dx \cdot \int \sqrt{x}\, dx$

b) $\int x(x^2 + 1)\, dx \neq x \int (x^2 + 1)\, dx$

c) $\int \frac{x^2 - 1}{x - 1}\, dx \neq \frac{\int (x^2 - 1)\, dx}{\int (x - 1)\, dx}$

46. Given the marginal revenue function $MR = 50x - x^2$, where x is the number of unit sales (in thousands), determine the revenue function and draw its graph. Find the maximum revenue.

47. Suppose that the marginal cost of a product is given by $x^3 - 4x^2 + 8x$ and the fixed cost is known to be 4. Find the cost function.

48. Find the cost function $C(x)$ and draw its graph if the marginal cost function is $MC = 14x - 7$, where x is the number of units produced. Assume the cost is 4300 when the number of units produced is zero. Find the minimum cost.

49. Given $MR = 2 + 7x$, where MR denotes marginal revenue, find the revenue function.

50. Evaluate the following integrals.

a) $\int (3x - 1)^3\, dx.$

b) $\int (3x - 1)^{-3}\, dx.$

c) $\int (x^2 + 2)^6\, 4x\, dx.$

d) $\int (3x + 1)^{70}\, dx.$

e) $\int \frac{dx}{(3 + 5x)^2}$

f) $\int \frac{(\sqrt{x} + 1)^5}{\sqrt{x}}\, dx.$

g) $\int \frac{(x^{1/3} - 1)^6}{x^{2/3}}\, dx$

h) $\int (x^3 - x)(3x^2 - 1)\, dx$

i) $\int \frac{(x + 1)\, dx}{(x^2 + 2x + 3)^{1/4}}$

51. Find a function $f(x)$ satisfying the conditions $f'(x) = 1/\sqrt{x + 1}$ with $f(0) = 1$.

52. Find a function $f(x)$ satisfying the conditions $f'(x) = x/\sqrt{3x^2 - 1}$, with $f(1) = 1 + \sqrt{2}/3$.

53. If the marginal cost is known to be $x\sqrt{3x^2 + 1}$, find the cost function $C(x)$, if it is known that $C(0) = 20$.

54. The rate of increase of pollution in a certain lake is found to obey

$$A'(t) = \frac{3}{4}\,\frac{(t^{\frac{1}{4}} + 3)^2}{t^{\frac{3}{4}}}$$

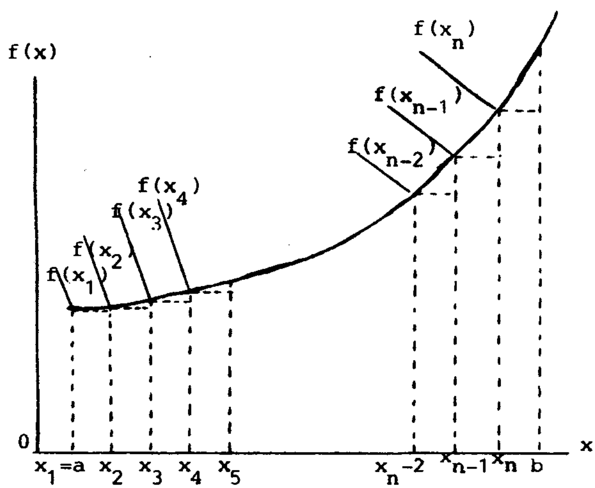
where t is measured in years and the amount A(t) is measured in appropriate units of pollutant. If at time t = 0, the amount of pollutant is 27 units, find the amount of pollutant after 16 years.

55. Use the Power Rule to verify the formula

$$\int (ax + b)^n\,dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + K, \quad a \neq 0,\ n \neq -1$$

10.3 Definite Integrals:

The important thing to remember about the definite integral is that it is the *limit of a sum.*



$$A = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \dots\dots + f(x_n)\Delta x_n$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x_i$$

The actual area under the curve A is the limiting value of the sum of the areas of the n rectangles as the number of rectangles approaches infinity

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x_i$$

Integration, therefore, gives the area under a curve. The area is called the *integral* when the interval is left arbitrary. When the interval is specified, the area is called the *definite integral*.

The definite integral of the function f(x) between the limits x = a and x = b is a numerical value that can be determined from the indefinite integral. If g(x) represents the indefinite integral of the function f(x), then the *fundamental theorem of calculus* states that

$$\boxed{\int_a^b f(x)\,dx = g(x)\Big|_a^b = g(b) - g(a)}$$

This theorem states that the value of the definite integral of the function f(x) between the limits a and b is given by the indefinite integral g(x) evaluated at the upper limit of integration b minus the indefinite integral evaluated at the lower limit of integration a.

The limits a and b are called the *limits of integration*

a = the lower limit and

b = the upper limit

$\int_a^b f(x)\,dx =$ can be verbalized as "the definite integral of f(x) between a lower limit x = a and an upper limit x = b," or more simply "the integral of f(x) between a and b.

When evaluating definite integrals, we always subtract the value of indefinite integral at the lower limit of integration from the value at the upper limit of integration. The constant of integration will always drop out in this computation.

Examples:

a) *Evaluate* $\int_0^3 x^2\,dx$

Solution:

$$\int_0^3 x^2\,dx = \left(\frac{x^3}{3} + c\right)\Bigg|_0^3$$

$$= \left(\frac{3^3}{3} + c\right) - \left(\frac{0^3}{3} + c\right)$$

$$= 9 + c - c$$

$$= 9 \qquad \underline{\text{Ans}}$$

b) *Evaluate* $\int_1^4 (2x^2 - 4x + 5)\,dx$

Solution:

$$\int_1^4 (2x^2 - 4x + 5)\,dx = \frac{2x^3}{3} - \frac{4x^2}{2} + 5x\,\Bigg|_1^4$$

$$= \frac{2x^3}{3} - 2x^2 + 5x\,\Bigg|_1^4$$

$$= \left[\frac{2(4)^3}{3} - 2(4)^2 + 5(4)\right] - \left[\frac{2(1)^3}{3} - 2(1)^2 + 5(1)\right]$$

$$= \left(\frac{128}{3} - 32 + 20\right) - \left(\frac{2}{3} - 2 + 5\right)$$

$$= 30\,\frac{2}{3} - 3\,\frac{2}{3}$$

$$= 27 \qquad \underline{\text{Ans}}$$

c) *Evaluate* $\int_{-2}^{1} e^x\,dx$

Solution:

$$\int_{-2}^{1} e^x dx = e^x\,\Bigg|_{-2}^{1}$$

$$= e^1 - e^{-2}$$

From the table $e = 2.7183$

$e^{-2} = 0.1353$

$$\therefore \int_{-2}^{1} e^x dx = 2.7183 - 0.1353$$

$$= 2.5830 \qquad \underline{\text{Ans}}$$

d) *Evaluate* $\int_{2}^{6} (x + 2)\, dx$

Solution:

$$\int_{2}^{6} (x + 2)\, dx = \frac{x^2}{2} + 2x \Big|_{2}^{6}$$

$$= \left[\frac{6^2}{2} + 2(6)\right] - \left[\frac{2^2}{2} + 2(2)\right]$$

$$= (18 + 12) - (2 + 4)$$

$$= 30 - 6 = 24 \qquad \underline{\text{Ans}}$$

e) *Evaluate* $\int_{1}^{4} 3x^2(4x^3 + 5)\, dx$

Solution:

$$f(x) = 4x^3 + 5$$

$$f'(x) = 12x^2$$

$$\int_{1}^{4} 3x^2(4x^3 + 5)\, dx = \frac{4}{4}\int_{1}^{4} (4x^3 + 5).\ 3x^2 dx$$

$$= \frac{1}{4}\int_{1}^{4} (4x^3 + 5).\ 12x^2 dx$$

$$= \frac{1}{4}\frac{(4x^3 + 5)^2}{2} \Big|_{1}^{4}$$

$$= \frac{[4(4)^3 + 5]^2}{8} - \frac{[4(1)^3 + 5]^2}{8}$$

$$= \frac{68{,}121 - 81}{8} = 8505 \qquad \underline{\text{Ans}}$$

EXERCISE: 10-2

In Problems 1 - 42, evaluate the definite integral.

1. $\int_{0}^{3} 4dx.$

2. $\int_{1}^{3} (2 + e)\, dx.$

3. $\int_{1}^{2} 3x\,dx.$

4. $\int_{0}^{2} -5x\,dx.$

5. $\int_{-1}^{3} \frac{5}{3} x^3 dx.$

6. $\int_{0}^{10} .04x^3\,dx.$

7. $\int_{-2}^{1} (4x - 6)\,dx.$

8. $\int_{-1}^{1} (5y + 2)\,dy.$

9. $\int_{0}^{2} (t^2 + t)\,dt.$

10. $\int_{1}^{3} (2w^2 + 1)\,dw.$

11. $\int_{2}^{3} (y^2 - 2y + 1)\,dy.$

12. $\int_{3}^{2} (2t - t^2)\,dt.$

13. $\int_{-2}^{-1} (3w^2 - w - 1)\,dw.$

14. $\int_{8}^{9} dt.$

15. $\int_{1}^{2} -4t^{-4} dt.$

16. $\int_{1}^{2} \frac{x^{-2}}{2}\,dx.$

17. $\int_{-1}^{1} \sqrt[3]{x^5}\,dx.$

18. $\int_{1/2}^{3/2} (x^2 + x + 1)\,dx.$

19. $\int_{1/2}^{3} \frac{1}{x^2}\,dx.$

20. $\int_{4}^{9} \left(\frac{1}{\sqrt{x}} - 2\right) dx.$

21. $\int_{-1}^{1} (z + 1)^5 dx.$

22. $\int_{1}^{8} (x^{1/3} - x^{-1/3})\,dx.$

23. $\int_{0}^{1} 2x^2 (x^3 - 1)^3\,dx.$

24. $\int_{1}^{3} (x + 3)^3\,dx.$

25. $\int_{1}^{8} \frac{4}{y}\,dx.$

26. $\int_{0}^{e-1} \frac{1}{x + 1}\,dx.$

27. $\int_{0}^{2} x^2 e^{x^3} dx.$

28. $\int_{0}^{1} (3x^2 + 4x)(x^3 + 2x^2)^4\,dx.$

29. $\int_{4}^{5} \frac{2}{(x - 3)^3}\,dx.$

30. $\int_{0}^{6} \sqrt{2x + 4}\,dx.$

31. $\int_{1/3}^{2} \sqrt{10 - 3p}\,dp.$

32. $\int_{-1}^{1} q\sqrt{q^2 + 3}\,dq.$

33. $\int_0^1 x^2 \sqrt[3]{7x^3 + 1}\, dx.$

34. $\int_0^{\sqrt{7}} \left[2x - \frac{x}{(x^2+1)^{5/3}}\right] dx.$

35. $\int_0^1 \frac{2x^3 + x}{x^2 + x^4 + 1}\, dx.$

36. $\int_a^b (m + ny)\, dy$

37. $\int_0^1 (e^x - e^{-2x})\, dx.$

38. $\int_{-2}^1 |x|\, dx.$

39. $\int_1^e (x^{-1} + x^{-2} - x^{-3})\, dx.$

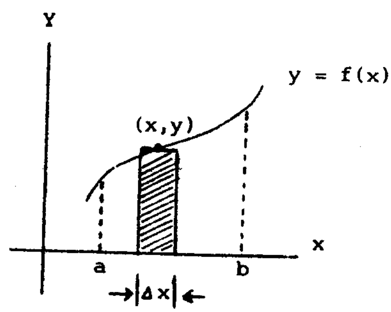
40. $\int_1^2 \left(6\sqrt{x} - \frac{1}{\sqrt{2x}}\right) dx.$

41. $\int_1^3 (x+1)e^{x^2+2x}\, dx.$

42. $\int_3^4 \frac{e^{\ln x}}{x}\, dx.$

10.4 Area:

We can use the definite integral to determine area. Let us consider the area of the region bounded by $y = f(x)$ and the x - axis from $x = a$ to $x = b$ as shown in the figure below.



Since we are summing areas of rectangles, a sample rectangle should be included in the sketch. Such a rectangle is called a *vertical element of area* (or a vertical strip). In the diagram, the width of the vertical element is Δx. The length is the y - value of the curve. Hence the rectangle has area $y.\Delta x$ or $f(x)\Delta x$. We want to add the areas of all such element between x = a and x = b and find the limit of this sum by means of definite integration:

$$\boxed{\sum f(x)\,\Delta x \longrightarrow \int_a^b f(x)\,dx}$$

Examples:

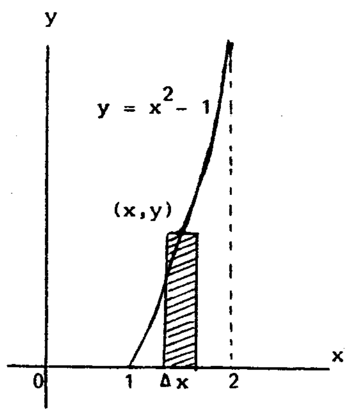
1. *Find the area of the region in the first quadrant that is bounded by the curve $y = x^2 - 1$, the x - axis, and the line $x = 2$*

SOLUTION:

The width of the sample element is Δx and its length is y.

Thus the area of the element is $y.\Delta x$. The limit of the sum of all such areas of elements between x = 1 and x = 2 is found by the definite integral.

The limits of integration are x = 1 and x = 2



$$\sum y.\,\Delta x \longrightarrow \int_1^2 y\,dx$$

To evaluate this integral we must express the integrand in terms of the variable of integration x.

Since $y = x^2 - 1$

$$\text{Therefore, area} = \int_1^2 (x^2 - 1)\,dx$$

$$= \left(\frac{x^3}{3} - x\right)\Bigg|_1^2$$

$$= \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)$$

$$= \frac{4}{3} \text{ sq. units}$$ <u>Ans</u>

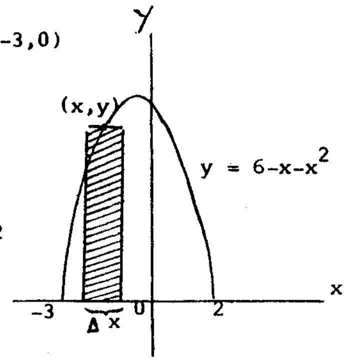
2. Find the area of the region bounded by the curve $y = 6 - x - x^2$ and the x - axis

Solution:

$$y = 6 - x - x^2$$
$$= -(x - 2)(x + 3)$$

The x - intercepts are (2,0) and (-3,0)

The area of the element is



$$= y.\Delta x$$
$$= (6 - x - x^2)\Delta x$$

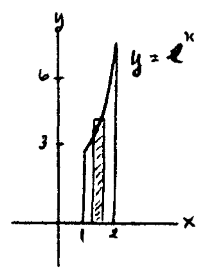
Summing these from x = -3 to x = 2 and taking the limit gives

$$\text{area} = \int_{-3}^{2} (6 - x - x^2)\, dx$$
$$= \left(6x - \frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_{-3}^{2}$$
$$= \left(12 - \frac{4}{2} - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} - \frac{-27}{3}\right)$$
$$= \frac{125}{6} \text{ sq. units}$$

Ans

3. *Find the area between $y = e^x$ and the x - axis from $x = 1$ to $x = 2$*

Solution:

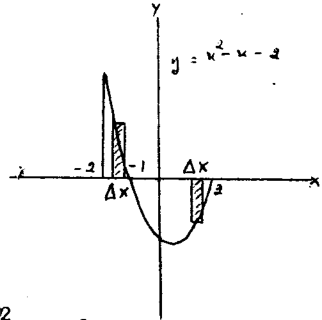


$$\text{area} = \int_{1}^{2} y\, dx$$
$$= \int_{1}^{2} e^x.dx$$
$$= e^x \Big|_{1}^{2} = e^2 - e = e(e - 1) \text{ sq. units}$$

Ans

4. *Find the area of the region bounded by the curves* $y = x^2 - x - 2$ *and* $y = 0$ *(the x - axis) from* $x = -2$ *to* $x = 2$

Solution:



The x - intercepts are (-1,0) and (2,0)
On the interval [-2, -1], the area of the element is

$$y\,\Delta x = (x^2 - x - 2)\,\Delta x$$

On [-1, 2] it is

$$-y\,\Delta x = -(x^2 - x - 2)\,\Delta x$$

Thus Total area $= \int_{-2}^{-1} (x^2 - x - 2)\,dx + \int_{-1}^{2} -(x^2 - x - 2)\,dx$

$$= \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x\right)\Bigg|_{-2}^{-1} - \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x\right)\Bigg|_{-1}^{2}$$

$$= \left[\left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(-\frac{8}{3} - \frac{4}{2} + 4\right)\right] - \left[\left(\frac{8}{3} - \frac{4}{2} - 4\right) - \left(-\frac{1}{3} - \frac{1}{2} + 2\right)\right]$$

$$= \frac{19}{3} \text{ sq. units}$$

<u>Ans</u>

EXERCISE: 10-3

In Problems 1 - 34, find the area of the region bounded by the given curves, the x - axis, and the given lines. In each case first sketch the region.

1. $y = 4x$, $x = 2$.
2. $y = 3x + 1$, $x = 0$, $x = 4$.
3. $y = 3x + 2$, $x = 2$, $x = 3$.
4. $y = x + 5$, $x = 2$, $x = 4$.
5. $y = x - 1$, $x = 5$.
6. $y = 2x^2$, $x = 1$, $x = 2$.
7. $y = x^2$, $x = 2$, $x = 3$.
8. $y = 2x^2 - x$, $x = -2$, $x = -1$.
9. $y = x^2 + 2$, $x = -1$, $x = 2$.
10. $y = 2x + x^3$, $x = 1$.
11. $y = x^2 - 2x$, $x = -3$, $x = -1$.
12. $y = 3x^2 - 4x$, $x = -2$, $x = -1$.
13. $y = 9 - x^2$.
14. $y = \frac{4}{x}$, $x = 1$, $x = 2$.
15. $y = 1 - x - x^3$, $x = -2$, $x = 0$.

16. $y = e^x, x = 1, x = 3.$ 17. $y = 3 + 2x - x^2.$

18. $y = \frac{1}{x^2}, x = 2, x = 3.$ 19. $y = \frac{1}{x}, x = 1, x = e.$

20. $y = \frac{1}{x}, x = 1, x = e^2.$ 21. $y = \sqrt{x + 9}, x = -9, x = 0.$

22. $y = x^2 - 2x, x = 1, x = 3.$ 23. $y = \sqrt{2x - 1}, x = 1, x = 5.$

24. $y = x^3 + 3x^2, x = -2, x = 2.$ 25. $y = \sqrt[3]{x}, x = 2.$

26. $y = x^2 - 4, x = -2, x = 2.$ 27. $y = e^x, x = 0, x = 2.$

28. $y = |x|, x = -2, x = 2.$ 29. $y = x + \frac{2}{x}, x = 1, x = 2.$

30. $y = 6 - x - x^2.$ 31. $y = x^3, x = -2, x = 4.$

32. $y = \sqrt{x - 2}, x = 2, x = 6.$ 33. $y = 2x - x^2, x = 1, x = 3.$

34. $y = x^2 - x + 1, x = 0, x = 1.$

10.5 Area Between Ourves:

Now consider finding the area of a region enclosed by several curves. As before, our procedure will be to draw a sample element of area and use the definite integral to *add together* the areas of all such elements.

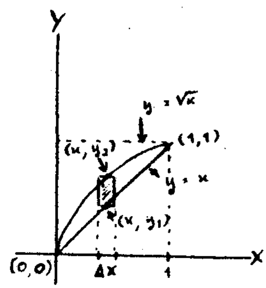
Example 1:

Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = x$

Solution:

First determine where the curves intersect by solving the system formed by the equations $y = \sqrt{x}$ and $y = x$

Therefore,
$$\begin{aligned} \sqrt{x} &= x \\ x &= x^2 \\ x^2 - x &= 0 \\ x(x - 1) &= 0 \\ x &= 0 \text{ or } x = 1 \end{aligned}$$



Thus the curves intersect at (0,0) and (1,1)
The width of the indicated element of area is Δx. The length is the y - value on the upper curve minus the y - value on the lower curve. To distinguish the two curves, write $y_1 = x$ and $y_2 = \sqrt{x}$. Then, the length of the element is

$$y_{upper} - y_{lower} = y_2 - y_1$$
$$= \sqrt{x} - x$$

Thus the area of the element is $(\sqrt{x} - x)\,\Delta x$

Summing all such areas from x = 0 to x = 1 by the definite integral, the area of the entire region is

$$\sum(\sqrt{x} - x)\Delta x \longrightarrow \int_0^1 (\sqrt{x} - x)\,dx$$

$$\text{area} = \int_0^1 (x^{\frac{1}{2}} - x)\,dx$$
$$= \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{2}\right)\Bigg|_0^1$$
$$= \left(\frac{2}{3} - \frac{1}{2}\right) - 0 = \frac{1}{6} \text{ sq.unit} \quad \underline{\text{Ans}}$$

Example 2:

Find the area of the region bounded by the curve $y^2 = 4x$ and the lines $y = 3$ and $x = 0$ (the y - axis)

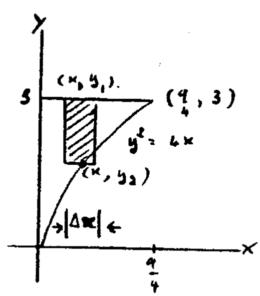
Solution:

a) Method 1: Vertical Strip:

Determine first the point of intersection formed by

$$y = 3 \text{ and } y^2 = 4x$$
$$4x = 9$$
$$x = \frac{9}{4}$$

Thus the point of intersection is $(\frac{9}{4}, 3)$. Since the width of the vertical strip is Δx we integrate with respect to the variable x.



Thus y_{upper} and y_{lower} must be expressed as functions of x.

For the curve $y^2 = 4x$

$$y = \pm 2\sqrt{x}$$

But for the portion of this curve that bounds the region we must have $y \geq 0$, so we use $y = 2\sqrt{x}$.

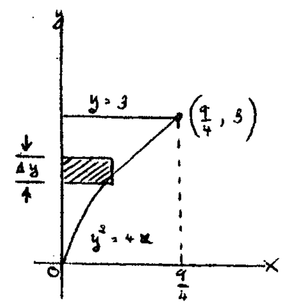
Thus the length of the strip is $y_{upper} - y_{lower} = 3 - 2\sqrt{x}$

Hence the strip has an area of $= (3 - 2\sqrt{x})\Delta x$

And to sum up such areas from $x = 0$ to $x = \frac{9}{4}$, we have

$$\begin{aligned}
\text{area} &= \int_0^{\frac{9}{4}} (3 - 2\sqrt{x})\, dx \\
&= \left(3x - \frac{4x^{\frac{3}{2}}}{3}\right)\Bigg|_0^{\frac{9}{4}} \\
&= \left[3\left(\frac{9}{4}\right) - \frac{4}{3}\left(\frac{9}{4}\right)^{\frac{3}{2}}\right] - (0 - 0) \\
&= \frac{27}{4} - \frac{4}{3}\left[\left(\frac{9}{4}\right)^{\frac{1}{2}}\right]^3 \\
&= \frac{27}{4} - \frac{4}{3}\left(\frac{3}{2}\right)^3 = \frac{9}{4} \text{ sq.units} \qquad \underline{\text{Ans}}
\end{aligned}$$

b) Method 2: Horizontal element of area.



The width of element is Δy. The length of the element is the x - value on the right curve minus the x - value on the left curve.
Thus the area of the element is

$$= \left(x_{right} - x_{left}\right)\Delta y$$

To sum all such areas from

$y = 0$ to $y = 3$, we have

$$\sum\left(x_{right} - x_{left}\right)\Delta y \longrightarrow \int_0^3 \left(x_{right} - x_{left}\right) dy$$

Since the variable of integration is y, we must express $\left(x_{right} - x_{left}\right)$ as functions of y.

The right curve $\quad y^2 = 4x$

Then, $\quad x = \frac{y^2}{4}$

The left curve is $x = 0$

$$\text{Area} = \int_0^3 \left(x_{right} - x_{left}\right) dy$$

$$= \int_0^3 \left(\frac{y^2}{4} - 0\right) dy$$

$$= \int_0^3 \frac{y^2}{4}\, dy = \frac{y^3}{12}\Big|_0^3$$

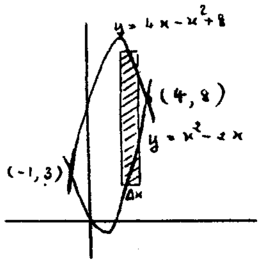
$$= \frac{9}{4} \text{ sq.units} \qquad \underline{\text{Ans}}$$

Example 3:

Find the area of the region bounded by the curves $y = 4x - x^2 + 8$ *and* $y = x^2 - 2x$.

Solution:

Find points of intersection between the two curves.



$$4x - x^2 + 8 = x^2 - 2x$$

$$-2x^2 + 6x + 8 = 0$$

$$x^2 - 3x - 4 = 0$$

$$(x+1)(x-4) = 0$$

$$x = -1 \quad \text{or} \quad x = 4$$

Thus the curves intersect at (-1, 3) and (4,8).

Use the vertical strip. The area of the element is

$$\left(y_{upper} - y_{lower}\right)\Delta x = [(4x - x^2 + 8) - (x^2 - 2x)]\,\Delta x$$

$$= (-2x^2 + 6x + 8)\,\Delta x$$

Summing all such areas from $x = -1$ to $x = 4$, we have

$$\text{Area} = \int_{-1}^{4} (-2x^2 + 6x + 8)\, dx$$

$$= \left(\frac{-2x^3}{3} + \frac{6x^2}{2} + 8x\right)\Big|_{-1}^{4}$$

$$= \left[-\frac{2}{3}(4)^3 + 3(4)^2 + 8(4)\right] - \left[-\frac{2}{3}(-1)^3 + 3(-1)^2 + 8(-1\right.$$

$$= -\frac{128}{3} + 48 + 32 - \frac{2}{3} - 3 + 8$$

$$= 88 - \frac{139}{3} = \frac{264 - 139}{3} = \frac{125}{3}$$

$$= 41\frac{2}{3} \text{ sq.units} \qquad \underline{\text{Ans}}$$

Example 4:

Find the area of the region bounded by $y^2 = x$ and $x - y = 2$.

Solution:

a) Vertical Strip:

The curves interesect when $y^2 = y + 2$

or $y^2 - y - 2 = 0$

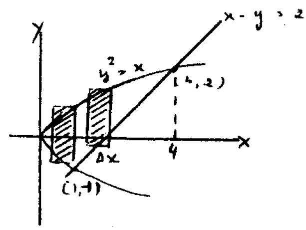
$(y+1)(y-2) = 0$

$y = -1$ or $y = 2$

The points of intersection are $(1, -1)$ and $(4, 2)$

Now $y^2 = x$

$y = \pm\sqrt{x}$



To the left of $x = 1$, the upper end of the element lies on $y = +\sqrt{x}$ and the lower end lies on $y = -\sqrt{x}$

To the right of $x = 1$, the upper curve is $y = \sqrt{x}$ and the lower curve is $x - y = 2$ (or $y = x - 2$)

With this method of vertical strip two integrals are needed to evaluate the area.

$$\text{Area} = \int_0^1 [\sqrt{x} - (-\sqrt{x})]\, dx + \int_1^4 [\sqrt{x} - (x - 2)]\, dx$$

$$= \int_0^1 2\sqrt{x}\, dx + \int_1^4 (x^{\frac{1}{2}} - x + 2)\, dx$$

$$= \left.\frac{2x^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^1 + \left.\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{2} + 2x\right|_1^4$$

$$= \frac{4}{3}(1)^{\frac{3}{2}} - 0 + \left[\frac{2}{3}(4)^{\frac{3}{2}} - \frac{1}{2}(4)^2 + 2(4)\right] - \left[\frac{2}{3}(1)^{\frac{3}{2}} - \frac{1}{2}(1)^2 + 2(1)\right]$$

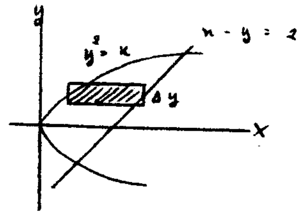
$$= \frac{4}{3} + \frac{16}{3} - 8 + 8 - \frac{2}{3} + \frac{1}{2} - 2$$

$$= \frac{9}{2} \text{ sq. units.} \qquad \underline{\text{Ans}}$$

b) Horizontal Strip:

The width of the strip is Δy
The rightmost curve is always $x - y = 2$ or $x = y + 2$ and the leftmost curve is always $y^2 = x$ or $x = y^2$.

Thus the area of the horizontal strip is $[(y + 2) - y^2]\Delta y$



$$\text{The total area is} = \int_{-1}^{2} (y + 2 - y^2)\, dy$$

$$= \left(\frac{y^2}{2} + 2y - \frac{y^3}{3}\right)\Big|_{-1}^{2}$$

$$= \left[\frac{(2)^2}{2} + 2(2) - \frac{(2)^3}{3}\right] - \left[\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3}\right]$$

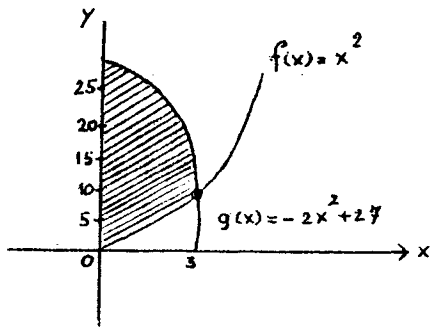
$$= 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$= \frac{9}{2} \text{ sq.units} \qquad \underline{\text{Ans}}$$

Example 5:

a) Determine the shaded area between $f(x) = x^2$ and $g(x) = -2x^2 + 27$

Solution:



Integrate $g(x)$ between $x = 0$ and $x = 3$, and subtract the surplus. The surplus is the area under $f(x)$ between $x = 0$ and $x = 3$ Thus, A, can be determined as

$$A = \int_0^3 g(x)\, dx - \int_0^3 f(x)\, dx$$

$$= \int_0^3 (-2x^2 + 27)\, dx - \int_0^3 x^2\, dx$$

$$= \left(-\frac{2x^3}{3} + 27x\right)\Big|_0^3 - \frac{x^3}{3}\Big|_0^3$$

$$= \left[-\frac{2(3)^3}{3} + 27(3)\right] - [0] - \left[\left(\frac{3^3}{3}\right) - (0)\right]$$

$= (-18 + 81) - 9$

$= 54$ square units Ans

b) Determine the area between the two functions $g(x) = x - 3$ and $h(x) = 3x - x^2$.

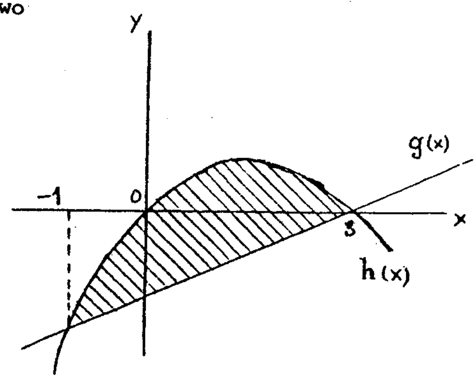
Solution:

The limits of integration are those values of x for which the two functions are equal.

Thus, $3x - x^2 = x - 3$

$x^2 - 2x - 3 = 0$

$x = -1,\ x = 3$



The area between the two functions for x = −1 to x = 3 is

$$A = \int_{-1}^{3} h(x)\,dx - \int_{-1}^{3} g(x)\,dx$$

$$= \int_{-1}^{3} [h(x) - g(x)]\,dx = \int_{-1}^{3} [(3x - x^2) - (x - 3)]\,dx$$

$$= \int_{-1}^{3} (-x^2 + 2x + 3)\,dx = -\frac{x^3}{3} + \frac{2x^2}{2} + 3x \Big|_{-1}^{3}$$

$$= \left[\frac{-(3)^3}{3} + (3)^2 + 3(3)\right] - \left[-\frac{(-1)^3}{3} + (-1)^2 + 3(-1)\right]$$

$$= (-9 + 9 + 9) - \left(+\frac{1}{3} + 1 - 3\right)$$

$$= 9 + 1\frac{2}{3} = 10\frac{2}{3} \text{ square units.}$$ Ans

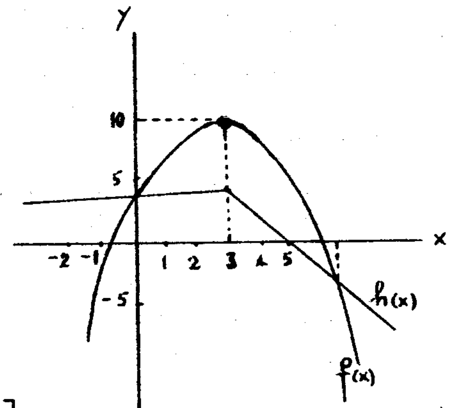
c) Determine the area bounded by $f(x)$ and $h(x)$.

$$f(x) = 4 + 5x - x^2 \quad \text{for } 0 \le x \le 6$$

$$h(x) = \begin{cases} 4 + \frac{2}{3}x & \text{for } 0 \le x \le 3 \\ 12 - 2x & \text{for } 3 \le x \le 6 \end{cases}$$

Solution:

The limits of integration were determined by equating the functions and solving for x = 0, x = 3, and x = 6. The area is found by evaluating the definite integrals.



$$A = \int_0^3 [(4 + 5x - x^2)-(4 + \frac{2}{3} x)]\, dx$$

$$+ \int_3^6 [(4 + 5x - x^2)-(12 - 2x)]\, dx$$

$$= 4x + \frac{5x^2}{2} - \frac{x^3}{3} - 4x - \frac{x^2}{3} \Bigg|_0^3$$

$$+ 4x + \frac{5x^2}{2} - \frac{x^3}{3} - 12x + x^2 \Bigg|_3^6$$

$$= \left[4(3) + \frac{5(3)^2}{2} - \frac{(3)^3}{3} - 4(3) - \frac{(3)^2}{3}\right]$$

$$+ \left\{\left[4(6) + \frac{5(6)^2}{2} - \frac{(6)^3}{3} - 12(6) + (6)^2\right] - \left[4(3) + \frac{5(3)^2}{2} - \frac{(3)^3}{3} - 12(3) + (3)^2\right]\right\}$$

$$= 22.5 + 6.5 = 29 \text{ square units}$$ <u>Ans</u>

EXERCISE: 10-4a

In Problems 1-21, find the area of the region bounded by the graphs of the given equations.

1. $y = x^2$, $y = 2x$.
2. $y = x$, $y = -x + 3$, $y = 0$.
3. $y = x^2$, $x = 0$, $y = 4 (x \geqslant 0)$.
4. $y = x^2$, $y = x$.
5. $y = x^2 + 3$, $y = 9$.
6. $y^2 = x$, $x = 2$.
7. $x = 8 + 2y$, $x = 0$, $y = -1$, $y = 3$.
8. $y = x - 4$, $y^2 = 2x$.
9. $y = 4 - x^2$, $y = -3x$.
10. $x = y^2 + 2$, $x = 6$.
11. $y^2 = x$, $3x - 2y = 1$.
12. $y = x^2$, $y = x + 2$.
13. $2y = 4x - x^2$, $2y = x - 4$.
14. $y = \sqrt{x}$, $y = x^2$.
15. $y^2 = x$, $y = x - 2$.

16. $y = 2 - x^2$, $y = x$.

17. $y = 8 - x^2$, $y = x^2$, $x = -1$, $x = 1$.

18. $y^2 = 4 - x$, $y = x + 2$.

19. $y = x^2$, $y = 2$, $y = 5$.

20. $y = x^3 - x$, x-axis.

21. $y = x^3$, $y = x$.

EXERCISE: 10-4b

In Exercises 1 -16, evaluate the definite integral.

1. $\int_0^2 (2x + 5)\,dx$

2. $\int_1^2 3x^2\,dx$

3. $\int_2^4 (mx + b)\,dx$, m and b constants

4. $\int_5^{10} 10\,dx$

5. $\int_0^2 4x^3\,dx$

6. $\int_0^1 (x - 5)^2\,dx$

7. $\int_4^9 \sqrt{x}\,dx$

8. $\int_{-1}^1 (x^2 - 2x + 5)\,dx$

9. $\int_2^5 3x^5\,dx$

10. $\int_2^3 -6x^2\,dx$

11. $\int_2^4 2e^x\,dx$

12. $\int_0^3 e^x\,dx$

13. $\int_0^1 2xe^{x^2}\,dx$

14. $\int_1^2 dx/x$

15. $\int_3^4 \frac{2x}{x^2 - 4}\,dx$

16. $\int_0^2 (ax^2 + bx + c)\,dx$, a, b, and c constants

In Exercises 17 - 24 , a) sketch f(x) and b) determine the size of the area between f(x) and the x-axis over the indicated interval.

17. $f(x) = -2x + 10$, between $x = 1$ and $x = 4$.

18. $f(x) = x^2$, between $x = 5$ and $x = 10$

19. $f(x) = 4x^3$, between $x = 1$ and $x = 2$

20. $f(x) = 25 - 3x^2$, between $x = 0$ and $x = 2$

21. $f(x) = e^x$, between $x = 2$ and $x = 4$

22. $f(x) = -\frac{1}{2}x^3$, between $x = 2$ and $x = 10$

23. $f(x) = 20x - x^2$, between $x = 0$ and $x = 10$

24. $f(x) = x^2 - 10x$, between $x = 1$ and $x = 4$

Perform the follwing integrations.

25. $\int_0^2 (x^{3/2} + x)\,dx$

26. $\int_0^1 \left(x^4 + \frac{3}{2}x^3\right) dx$

27. $\int (2x^{-3} - 4x^3)\,dx$

28. $\int_2^3 (\frac{1}{2}x^3 + x)\,dx$

29. $\int_{-1}^1 (x^2 - 4)\,dx$

30. $\int x^{19}\,dx$

31. $\int_1^e (x^{-2} + x^{-1} + x^0)\,dx$

32. $\int \left(\frac{4}{x} + \frac{6}{x} + x^2\right) dx$

33. $\int_4^{16} (\log_4 x + x^{-3/2})\,dx$

34. $\int (e^{2x} + 1 - x^{-2} - x^{-1})\,dx$

35. $\int \frac{1}{2} e^{4x}\,dx$

36. $\int_a^b 2^x\,dx$

37. $\int (e^{(x+2)} + x^2)\,dx$

38. $\int 3xe^{(x^2+4)}dx$

39. $\int_0^1 (1 + x)^{7/2}\,dx$

40. $\int (a + bx)^d\,dx$, where a, b, and d are constants, and $d \neq -1$.

41. $\int x(x + 2)^2\,dx$

42. $\int x^2 e^x\,dx$

43. $\int [(3 + \pi)\,x^4 - 17x^{-2}]\,dx$

44. $\int ex^{2/3}\,dx$

45. $\int 0.241x^{0.352}\,dx$

46. Find the values of the constants that are unspecified.

a) $\int_0^a (x + 1)\,dx = 4$; find a.

b) $\int_1^2 b(x + 5)\,dx = 10$; find b.

c) $\int_a^b x\,dx = 4$; find a and b.

47. Find the values for the unspecified constants in the integrals.

a) $\int_1^2 e^{(x+a)}\,dx = e^3(e-1)$; find a.

b) $\int_0^1 e^{bx}\,dx = e - 1$; find b.

c) $\int_1^e \log_a x\,dx = 1/2$; find a.

48. Perform the following integrations. (Part a requires integration by substitution, and part b requires integration by parts.)

a) $\int x^2 \log_e x^3\,dx$

b) $\int x^2 \log_e x^2\,dx$

49. Find f(x), as best you can, in each of the following cases.

a) $f'(x) = 3x + 12x^2$

b) $f''(x) = 7 + 4x^3$

c) $f''(x) = 1 + x + x^4$

Determine the integrals.

50. $\int 6\,dx$

51. $\int k\,dx$

52. $\int \frac{1}{3}\,dx$

53. $\int -2\,dx$

54. $\int (a+b)\,dx$

55. $\int k^2 dx$

56. $\int dx$

57. $\int \frac{k^2}{k+1}\,dx$

58. $\int x\,dx$

59. $\int x^2\,dx$

60. $\int x^{-2}\,dx$

61. $\int x^{1/2}\,dx$

62. $\int \frac{dx}{x^3}$

63. $\int x^{2k}\,dx$

64. $\int \sqrt{x}\,dx$

65. $\int \frac{k^2}{x^2}\,dx$

66. $\int_0^2 x^2\,dx$

67. $\int_1^3 \frac{1}{\sqrt{x}}\,dx$

68. $\int_0^5 dx$

69. $\int_2^6 x^{-1/2}\,dx$

70. $\displaystyle\int_{1}^{10} 3x^3\,dx$

71. $\displaystyle\int_{0}^{5} x^{2.2}\,dx$

Determine the integrals.

72. $\displaystyle\int (3x + 4)\,dx$

73. $\displaystyle\int (x^2 + 2x + 6)\,dx$

74. $\displaystyle\int (2x^3 + x^2)\,dx$

75. $\displaystyle\int (x^{1/2} + x^{1/3})\,dx$

76. $\displaystyle\int (x + 1)(x + 1)\,dx$

77. $\displaystyle\int (x - 2)(x + 3)\,dx$

78. $\displaystyle\int_{1}^{4} (x^2 - 2x + 3)\,dx$

79. $\displaystyle\int_{2}^{5} (x - 3)(x + 1)\,dx$

80. $\displaystyle\int 3e^{3x}\,dx$

81. $\displaystyle\int 5^x\,dx$

82. $\displaystyle\int \frac{1}{2} e^{2x}\,dx$

83. $\displaystyle\int 2^{2x}\,dx$

84. $\displaystyle\int_{0}^{10} 3e^{-3x}\,dx$

85. $\displaystyle\int_{0}^{\infty} e^{-x}\,dx$

86. $\displaystyle\int \ln(x)\,dx$

87. $\displaystyle\int \ln(2x)\,dx$

88. $\displaystyle\int \log(3x)\,dx$

89. $\displaystyle\int \log_5(x)\,dx$

90. $\displaystyle\int_{1}^{10} \ln(x)\,dx$

91. $\displaystyle\int \log_{10}(2x)\,dx$

Determine the integrals. Use the method of substitution.

92. $\displaystyle\int (x - 1)^{5/2}\,dx$

93. $\displaystyle\int x(x^2 - 2)^2\,dx$

94. $\displaystyle\int x^2(4x^3 - 2)^9\,dx$

95. $\displaystyle\int xe^{x^2}\,dx$

96. $\displaystyle\int x^2e^{x^3}\,dx$

97. $\displaystyle\int 4xe^{-x^2-4}\,dx$

98. $\displaystyle\int \frac{1}{5x - 6}\,dx$

99. $\displaystyle\int \frac{dx}{4 - 3x}$

100. $\displaystyle\int_{2}^{5} \frac{2x}{x^2 - 2}\,dx$

101. $\displaystyle\int_{-2}^{2} xe^{x^2}\,dx$

102. $\displaystyle\int x \ln(x^2 - 6)\,dx$

103. $\displaystyle\int x \ln x^2\,dx$

104. $\displaystyle\int_{0}^{1} \frac{2x}{x^2 + 1}\,dx$

105. $\displaystyle\int_{0}^{2} x(x^2 + 1)^2\,dx$

Determine the integrals. Use the method of integration by parts.

106. $\int x^2 \ln x \, dx$

107. $\int x^2 e^{-x} \, dx$

108. $\int x^3 e^{2x} \, dx$

109. $\int x^{\frac{1}{2}} \ln x \, dx$

110. $\int (\ln x)^2 \, dx$

111. $\int x e^{2x} \, dx$

112. $\int x 3^x \, dx$

CHAPTER 11

APPLICATIONS OF INTEGRATION

Applications of Integration:

Integration is used to perform its general function, sum up continuously, in several different areas of application. This chapter examines a few applications of integration, trying to discuss the more important ones, but in no sense is it an exhaustive discussion.

One general characteristic of integration problems is worth noting. If integration is to be useful, we must be given a rate, and need to know the total (sum). For example, if we are given a rate of speed of 40 miles per hour, we can find the total mileage covered in (say) 5 hours, and we can find it using integration. If we know the rate at which someone adds to their bank account, with interest compounded continuously, integration give the total dollars in the bank account after a certain time. If we know the rate of change, the original function (total) is found by integration (a summing-up process).

11.1 Continuous Compounding

In the previous chapter on compounding interest, the compound formula for the continuous compounding is

$$\boxed{F = Pe^{rt}}$$

This formula is used to determine the amount F to which an investment of P will accumulate when invested at interest rate r and compounded continuously for t years.

As an extension of this formula, the intergral of this function can be found.

$$F_n = \int_0^t Pe^{rt}\, dt$$

This integral gives the amount to which a series of n equal payments of P invested at interest rate r and compounded continuously will accumulate. It can be taken as the amount of an annuity assuming continous compounding.

Example 1:

The president of a certain country has called for sacrifices in the consumption of consumer goods in order to shift production to capital goods. The ten-year plan is to increase the capital equipment of the country at a rate of 10 percent per year. The present stock of capital equipment is valued at $10 billion. Assuming continuous compounding

(i) determine the amount that must be invested during the ten-year period.

(ii) determine the investment required during the ninth and tenth years.

Solution:

(i)

$$F_n = \int_0^t Pe^{rt}\, dt$$

$$= \int_0^{10} 10e^{0.10t}\, dt$$

$$= 10 \int_0^{10} \frac{0.10}{0.10} e^{0.10t}\, dt$$

$$= \frac{10}{0.10} e^{0.10t} \Big|_0^{10}$$

$$= \frac{10}{0.10} (e - 1)$$

$$= 100\ (2.718 - 1)$$

$$= \$171.8 \text{ billion}$$

Ans

(ii) The total investment during the final two years can be determined from

$$F = \int_{8}^{10} 10.e^{0.10t}\,dt$$

$$= \frac{10}{0.10}\left(e^{0.10t}\right)\Big|_{8}^{10}$$

$$= 100(e^{1.0} - e^{0.8})$$

$$= 100(2.718 - 2.226)$$

$$= \$49.2 \text{ billion} \qquad \underline{\text{Ans}}$$

Example 2:

An individual deposits \$1000 at the end of each year in a saving account. The account pays 6 percent interest compounded daily.

(i) Determine the amount on deposit at the end of 10 years.

(ii) Assuming that the \$1000 was deposited at the first of the year rather than the last of the year, determine the amount on deposit at the end of 10 years.

Solution:

(i)

$$F = \int_{0}^{t} Pe^{rt}\,dt$$

$$= \int_{0}^{10} 1000\,e^{0.06t}\,dt$$

$$= 1000\int_{0}^{10} \frac{0.06}{0.06}\,e^{0.06t}\,dt$$

$$= \frac{1000}{0.06}\left(e^{0.06t}\right)\Big|_{0}^{10}$$

$$= \frac{1000}{0.06}\left(e^{0.6} - 1\right)$$

$$= \frac{1000}{0.06}(1.822 - 1)$$

$$= \$13{,}700 \qquad \underline{\text{Ans}}$$

(ii) The formula for the amount of an annuity with the payment at the first of the period if continuous compounding is assumed as

$$\boxed{F_n = e^r \int_0^t Pe^{rt}\, dt}$$

$$\therefore\ F_n = 13{,}700(e^{0.06})$$

$$= 13{,}700(1.0618)$$

$$= \$14{,}640 \qquad \underline{\text{Ans}}$$

Example 3:

A hotel makes $1,000 per month above costs (living expenses for the owners are included in the costs). The receipts are placed in a continuously compounded bank account at ½ percent per month. A prospective owner will buy the hotel if he can build up $25,000 in the bank account in 2 years. Will he be able to buy the hotel?

Solution:

Each dollar received earns interest from the time the money is deposited until $t = 2$ years $= 24$ months.
That is, each dollar is worth $e^{r(24-t)} = e^{0.005(24-t)}$ at the end of 24 months, if it is earned at time t.
After 24 months, the bank account will contain

$$= \int_0^{24} 1000\ e^{0.005(24-t)}\ dt$$

$$= \int_0^{24} 1{,}000\ e^{0.12}\ e^{-0.005t}\ dt$$

$$= \frac{1000\ e^{0.12}}{-0.005} \cdot e^{-0.005t} \Big|_0^{24}$$

$$= (200{,}000\ e^{0.12})(-e^{-0.12} + 1)$$

$$= 200{,}000(-1 + e^{0.12}) \approx \$25{,}500$$

He should purchase the hotel. He will reach his goal. <u>Ans</u>

11.2 Present Value of An Annuity If Continuous Compounding

Integral calculus can be used to derive the formula for the present value of an annuity. Based upon continuous compounding, the present value of an amount F received t years hence is

$$\boxed{P = F\, e^{-rt}}$$

Instead of receiving a single payment of F at the end of t years, consider the case in which we receive a payment of F at the end of each year. The present value of this series of payments is found by integrating the above formula ($P = F\, e^{-rt}$) between the limits of integration of t = 0 and t = t. Evaluation of this definite integral gives the present value of an annuity if contnuous compounding is assumed.

The formula is $$\boxed{P = \int_0^t F\, e^{-rt}\, dt}$$

Example 1:

A research and development firm has patented a new photocopy process, which they have licensed to a major manufacturer. The agreement calls for the annual payment of $50,000 for 10 years. If the discount rate is 8 percent and continuous compounding is assumed, determine the present value of this agreement.

Solution:

$$P = \int_0^t F\, e^{-rt}\, dt$$

$$= \int_0^{10} 50{,}000(e^{-0.08t})\, dt$$

$$= 50{,}000 \int_0^{10} \frac{(-0.08)}{-0.08} e^{-0.08t}\, dt$$

$$= \frac{50{,}000}{-0.08} (e^{-0.08t}) \Big|_0^{10}$$

$$= -\frac{50,000}{0.08}(e^{-0.8} - 1)$$

$$= -\frac{50,000}{0.08}(0.449 - 1)$$

$$= -\frac{50,000}{0.08}(-0.551)$$

$$= \$344375 \qquad \underline{\text{Ans}}$$

Example 2:

A company estimates that the rate of revenue produced by a machine at time t will be (5000 - 100t) per year. Find the present value of this continuous stream of income over the next four years at a 6% interest rate.

Solution:

The present value of this income stream is

$$= \int_0^t f(t)\, e^{-rt}\, dt$$

$$= \int_0^4 (5000 - 100t)\, e^{-0.06t}\, dt$$

Using integration by parts we get

$$= (5000 - 100t)\frac{1}{-0.06}e^{-0.06t}\Big|_0^4 - \int_0^4 (-100)\frac{1}{-0.06}e^{-0.06t}\, dt$$

$$\approx 23,025 - \frac{100}{0.06} \cdot \frac{1}{-0.06}e^{-0.06t}\Big|_0^4$$

$$\approx 23,025 - 5927 = \$17,098 \qquad \underline{\text{Ans}}$$

EXERCISE: 11-1

1. Suppose that P dollars is deposited each day in a savings account paying 6% annual interest, compounded continuously. How much money will be in the account at the end of one year? Ans (376.17)$\cdot$P dollars.

2. Suppose that the population density r miles from the center of a city is $\frac{3000}{r}$ people per square mile. Furthermore, suppose that no people live within 1 mile of the center. How many people live within 10 miles of the center of the city?

3. In 1973, about 20.3 billion barrels (bbl.) of oil was used workdwide. The demand for oil seems to be growing at an exponential rate of about 9% per year. Thus, if A(t) denotes the annual rate of consumption of oil at time t (with t = 0 corresponding to 1973), then an approximate formula for A(t) is

$$A(t) = 20.3\, e^{0.09t}$$

Assuming the demand for oil continues to grow at the rate of 9% per year, how much oil will be consumed between 1973 and 1988?

4. Suppose that after t hours of operation, a factory assembly line is producing electric shavers at the rate of p(t) units per hour. If $p(t) = 25 + 2t - \frac{1}{4}t^2$, how many units will be produced between t = 4 and t = 8 hours?

5. If 1 dollar per day is deposited in an account earning 5% interest, compounded continuously, approximately how much money will be in the account after 3 years?

6. If P dollars is deposited daily into an account earning 5% interest, compounded continuously, approximately how much money will be in the account after 2 year?

7. If 3 dollars is deposited daily into an account earning 5% continuous interest, approximately how much money will be in the account after 1 year?

8. Approximately how much money should be deposited daily into an account earning 5% continuous interest so that the account contains $513 at the end of 1 year?

9. Suppose that the population density r miles from the center of a city is $\frac{3000}{\sqrt{r}}$ people per square mile. Also suppose that no

people live within 1 mile of the center. How many people live within 9 miles of the center of the city?

10. A few days after a certain species of fish was released in the center of a large lake, the density of these fish r miles from the center of the lake was expected to be $\frac{200}{e^r}$ per square mile.

    Set up the integral that gives the number of fish within 5 miles of the center of the lake.

11. Suppose that a bacteria culture grows according to the law $N(t) = 50\,e^{.5t}$. What is the average size of the culture over the time interval from t = 0 to t = 10 hours?

12. What is the average amount of money in a savings account earning continuous 5% interest for a 10-year period if the account initially contains $100?

13. Consider a continuous stream of income that is produced by some investment such that, at any given time t, the annual rate of income is K(t) = -500 + 300t. Find the present value of the income over the next ten years, using a 5% interest rate.

11.3 Economic Applications:

The relationship between the integral and the derivative forms the basis for numerous economic models. In this section, basic models in cost theory, capital formulation, and mathematics of finance will be illustrated utilizing this relationship.

11.3.1 Marginal and total cost Functions:

The marginal cost function is given by the derivative of the total cost function. This relationship is reversible; i.e., the integral of the marginal cost function is the total cost function.

If TC = Total cost

MC = Marginal cost

TVE = Total variable cost

FC = Fixed cost

Therefore, $TC = \int MC = TVE + FC$

The integral of marginal cost gives total variable cost plus the constant. The constant represents fixed cost, which does not vary with changes in the quantity of goods produced.

Example 1:

The cost of producing a simple component is composed of a fixed cost and a variable cost. If the fixed cost is k and the variable cost is C per unit, show that the total cost is given by TC = cx + k, where x = total number of units produced.

Solution:

The cost of producing an additional unit is = C

The marginal cost function is

$$MC = c$$

Therefore

$$TC = \int MC = \int c\,dx$$

$$= cx + k \qquad \underline{\text{Ans}}$$

Example 2:

The cost of manufacturing an item consists of \$1000 - fixed cost and \$1.50 per unit variable cost. Determine the total cost function and the cost of manufacturing 2000 units.

Solution:

Let x be the total units manufactured

$$TC = \int MC = TVC + FC$$

$$TC = \int 1.50\,dx = 1.50x + 1000$$

The cost of manufacturing 2000 units is

$$TC = 1.50(2000) + 1000$$

$$= \$4000 \qquad \underline{\text{Ans}}$$

Example 3:

Assume that the marginal cost of manufacturing a product is the following function of output:

$$MC = 3x^2 - 4x + 5$$

Determine the total cost function.

Solution:

$$\begin{aligned} TC &= \int MC \\ &= \int (3x^2 - 4x + 5)\, dx \\ &= \frac{3x^3}{3} - \frac{4x^2}{2} + 5x + k \\ &= x^3 - 2x^2 + 5x + k \end{aligned}$$

where k is the fixed cost of manufacturing the product.

<u>Ans</u>

Example 4:

The marginal cost function for a certain commodity is $MC = 3x^2 - 4x + 5$. Determine the cost of producing the eleventh through the fifteenth units, inclusive.

Solution:

$$\begin{aligned} TC = \int MC &= \int_{10}^{15} (3x^2 - 4x + 5)\, dx \\ &= \frac{3x^3}{3} - \frac{4x^2}{2} + 5x \Big|_{10}^{15} \\ &= x^3 - 2x^2 + 5x \Big|_{10}^{15} \\ &= [(15)^3 - 2(15)^2 + 5(15)] - [(10)^3 - 2(10)^2 + 5(10)] \\ &= \$2150 \end{aligned}$$

The cost of producing units 11 through 15 is \$2150 <u>Ans</u>

Example 5:

A manufacturer's marginal cost function is

$$\frac{dc}{dq} = 0.6q + 2$$

If production is presently set at $q = 80$ units per week, how much more would it cost to increase production to 100 units per week?

Solution:

Since $c = c(q)$ is the total cost function, we want to find the difference $c(100) - c(80)$
However, the rate of change of c is $\frac{dc}{dq}$, and thus

$$c(100) - c(80) = \int_{80}^{100} \frac{dc}{dq} \cdot dq = \int_{80}^{100} (0.6q + 2)\, dq$$

$$= \left[\frac{0.6q^2}{2} + 2q\right] \Bigg|_{80}^{100}$$

$$= (0.3q^2 + 2q) \Bigg|_{80}^{100}$$

$$= [0.3(100)^2 + 2(100)] - [0.3(80)^2 + 2(80)]$$

$$= 3200 - 2080 = 1120$$

If c is in dollars, then the cost of increasing production from 80 units to 100 units is \$1120 Ans

11.3.2 Revenue:

The total revenue function can sometimes be determined by integrating the marginal revenue function.

Example 1:

Assume that the price of a product is constant at a value of \$10 per unit, or the marginal revenue function is $MR = f(x) = 10$, where x equals the number of units sold. Find the total revenue from selling 1,500 units and the additional revenue associated with increasing sales from 1500 to 1,800 units?

Solution:

$$MR = 10$$

$$R = \int_{0}^{1500} 10\, dx$$

$$= 10x \Bigg|_{0}^{1500}$$

$$= 10(1,500) = \$15,000$$ Ans

The additional revenue associated with increasing sales from 1,500 to 1,800 units can be computed as

$$\int_{1500}^{1800} 10dx = 10x \Big|_{1500}^{1800}$$

$$= 18{,}000 - 15{,}000$$

$$= \$3000 \qquad \underline{\text{Ans}}$$

11.3.3 Maintenance Expenditures:

An automobile manufacturer estimates that the annual rate of expenditure r(t) for maintenance on one of its models is represented by the function $r(t) = 100 + 10t^2$ where t is the age of the automobile stated in years and r(t) is measured in dollars per year. Find the expected maintenance expenditures during the automobile's first 5 years. Find also the expenditures expected to be incurred during the fifth year?

Solution:

$$r(t) = 100 + 10t^2$$

$$\int_0^5 (100 + 10t^2)\,dt = 100t + \frac{10t^3}{3} \Big|_0^5$$

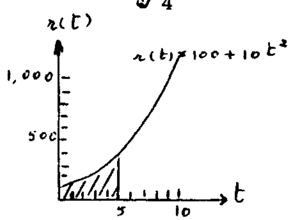
$$= 100(5) + \frac{10(5)^3}{3}$$

$$= 500 + 416.67$$

$$= \$916.67 \qquad \underline{\text{Ans}}$$

Of these expenditures, those expected to be incurred during the fifth year are estimated as

$$\int_4^5 (100 + 10t^2)\,dt = 100t + \frac{10t^3}{3} \Big|_4^5$$



$$= \left[100(5) + \frac{10(5)^3}{3}\right] - \left[100(4) + \frac{10(4)^3}{3}\right]$$

$$= 916.67 - (400 + 213.33)$$

$$= \$303.34 \qquad \underline{\text{Ans}}$$

11.3.4 Fund Raising:

A state civic organization is conducting its annual fund raising campaign for its summer camp program for the disadvantaged. Campaign expenditures will be incurred at a rate of \$10,000 per day. From past experience it is known that contributions will be high during the early stages of the campaign and will tend to fall off as the campaign continues. The function describing the rate at which contributions are received is

$$c(t) = -100t^2 + 20{,}000$$

where t represents the day of the campaign, and c(t) is measured in dollars per day.
The organization wishes to maximize the net proceeds from the campaign.

a) Determine how long the campaign should be conducted in order to maximize net proceeds.
b) What are total campaign expenditures expected to equal?
c) What are total contributions expected to equal?
d) What are the net proceeds (total contributions less total expenditures) expected to equal?

Solution:

a) The function which describes the rate at which expenditures e(t) are incurred is

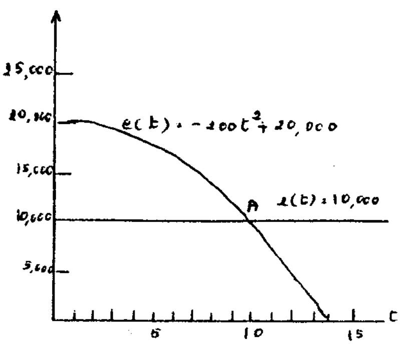
$$e(t) = 10{,}000$$

The function describing the rate at which contributions are received is

$$c(t) = -100t^2 + 20{,}000$$

The two functions intersect when

$$c(t) = e(t)$$



or when

$$\begin{aligned} -100t^2 + 20{,}000 &= 10{,}000 \\ -100t^2 &= -10{,}000 \\ t^2 &= 100 \\ t &= 10 \text{ days} \end{aligned}$$

<u>Ans</u>

b) Total campaign expenditures are represented by the area under e(t) between t = 0 and t = 10. This could be found by integrating e(t) between these limits or more simply by multiplying:

$$\begin{aligned} E &= (\$10{,}000 \text{ per/day})(10 \text{ days}) \\ &= \$100{,}000 \end{aligned}$$

<u>Ans</u>

c) Total contributions during the 10 days are represented by the area under c(t) between t = 0 and t = 10

$$C = \int_0^{10} (-100t^2 + 20,000)\,dt$$

$$= -100\,\frac{t^3}{3} + 20,000t\,\Big|_0^{10}$$

$$= \frac{-100(10)^3}{3} + 20,000(10)$$

$$= -33,333.33 + 200,000$$

$$= \$166,666.67 \qquad \underline{\text{Ans}}$$

d) Net proceeds are expected to equal

$$C - E = \$166,666.67 - \$100,000$$

$$= \$66,666.67 \qquad \underline{\text{Ans}}$$

11.3.5 Nuclear Power:

An electric company has proposed building a nuclear power plant on the outskirts of a major metropolitan area. As might be expected, public opinion is divided and discussions have been heated. One lobbyist group opposing the construction of the plant has presented some disputed data regarding the consequences of a catastrophic accident at the proposed plant. The lobbyist group estimates that the rate at which deaths would occur within the metropolitan area because of radioactive fallout is described by the function

$$r(t) = 200,000e^{-0.1t}$$

where r(t) represents the rate of deaths in persons per hour and t represents time elapsed since the accident, measured in hours. Note: Although the dispute in this example is quite real, the data are all contrived!

The population of the metropolitan area is 1.5 million persons.

a) Determine the expected number of deaths 1 hour after a major accident.

b) How long would it take for all people in the metropolitan area to succumb to the effects of the radioactivity?

Solution:

a) The number of deaths expected during the first hour would be computed as

$$\int_0^1 200{,}000e^{-0.1t}\,dt = \int_0^1 -2{,}000{,}000(-0.1)e^{-0.1t}\,dt$$

$$= -2{,}000{,}000\int_0^1 (-0.1)e^{-0.1t}\,dt$$

$$= -2{,}000{,}000e^{-0.1t}\Big|_0^1$$

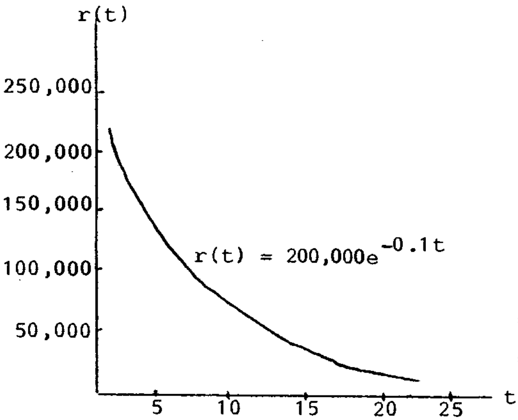
$$= -2{,}000{,}000e^{-0.1} + 2{,}000{,}000e^{0}$$

$$= -2{,}000{,}000(e^{-0.1} - e^{0})$$

$$= -2{,}000{,}000(0.9048 - 1)$$

$$= -2{,}000{,}000(-0.0952)$$

$$= 190{,}400 \text{ people} \qquad \underline{\text{Ans}}$$



b) The entire population would succumb when

$$\int_0^{t_1} 200{,}000e^{-0.1t}\,dt = 1{,}500{,}000$$

or when $-2{,}000{,}000e^{-0.1t}\Big|_0^{t_1} = 1{,}500{,}000$

Solving for t_1, we get

$$-2{,}000{,}000e^{-0.1t_1} = -500{,}000$$

$$e^{-0.1t_1} = 0.25$$

By using mathematical table.

$$-0.1t_1 = -1.39$$

$$t_1 = 13.9 \text{ hours} \qquad \underline{\text{Ans}}$$

11.4 Consumer's and Producer's Surplus:

Suppose the price p, a consumer is willing to pay for a quantity x, of a particular commodity is governed by the demand curve

$$\boxed{p = D(x)}$$

In general the function D(x) is a decreasing function, indicating that, as the price of the commodity increases, the quantity the consumer is willing to buy declines.

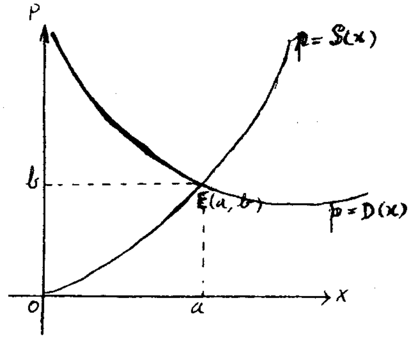
Suppose the price p that a producer is willing to charge for a quantity x of a particular commodity is governed by the supply curve

$$\boxed{p = S(x)}$$

In general, the function S(x) is an increasing function since, as the price p of a commodity increases, the more the producer is willing to supply the commodity

The point of intersection of the demand curve and the supply curve is called the *equilibrium point* E. If the coordinates of E are (a,b) then b, the market price, is the price a consumer is willing to pay for, and a producer is willing to sell for, a quantity "a", the demand level, of the commodity.

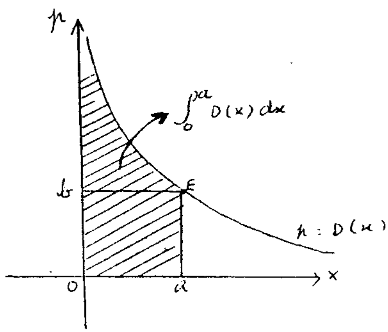
The total revenue of producer at a market price "b", and a demand level "a" is "ab" (the price per unit times the number of units). This revenue can be interpreted geometrically as the area of the rectangle o b E a.

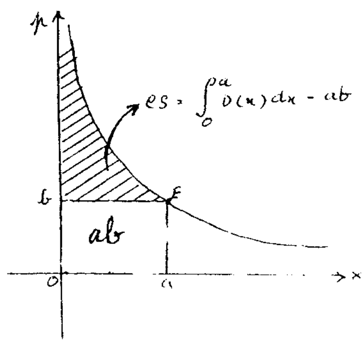


a) Consumer's surplus:

In a free market economy, there are times when some customers would be willing to pay more for a commodity than the market price "b" that they actually do pay. The benefit of this to consumers is called *consumer's surplus (CS)* and is determined by the formula

$$\boxed{CS = \int_0^a D(x)\, dx - ab}$$



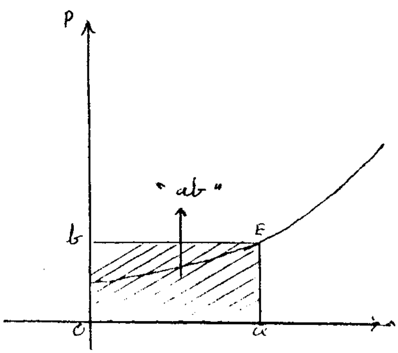


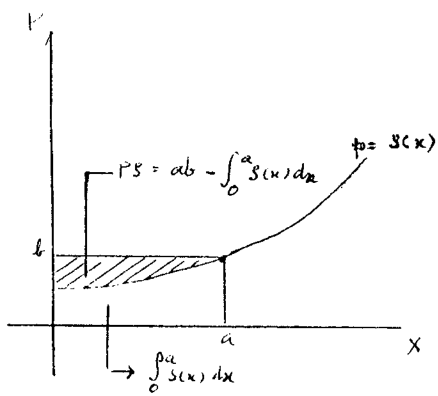
$\int_0^a D(x)\,dx$ is merely the area under the demand curve D(x) from x = 0 to x = a, and represents the total revenue that would have been generated by the willingness of some consumers to pay more. And by subtracting "ab" (the revenue actually achieved), the result is a surplus CS to the customer.

b. Producer's Surplus:

In a free market economy, there are also times when some producers would be willing to sell at a price below the market price "b" that the consumer actually pays. The benefit of this to the producer is called *producer's surplus (PS)* and is calculated by the formula

$$PS = ab - \int_0^a S(x)\,dx$$





$\int_0^a S(x)\,dx$ is merely the area under the suply curve S(x) from x = 0 to x = a, and represents the total revenue that would have been generated by some producer's willingness to sell at a lower price. By subtracting this amount from "ab" (the revenue actually achieved), the result is a surplus to the producer, PS.

Example 1:

Find the CS and PS defined by the demand curve $D(x) = 18 - 3x$ and the supply curve $S(x) = 3x + 6$, where equilibrium price $(b) = D(a) = S(a)$.

Solution:

First determine the equilibrium point E.

$$
\begin{aligned}
D(x) &= S(x) \\
18 - 3x &= 3x + 6 \\
6x &= 12 \\
x &= 2 = a
\end{aligned}
$$

To find b, we compute D(a). Then

$$b = D(a) = D(2) = 18 - 6 = 12$$

To find CS:

$$
\begin{aligned}
CS &= \int_0^a D(x)\,dx - ab \\
&= \int_0^2 (18 - 3x)dx - 2(12) \\
&= \left(18x - \frac{3x^2}{2}\right)\Bigg|_0^2 - 24 \\
&= 36 - 6 - 24 = 6
\end{aligned}
$$

and

$$
\begin{aligned}
PS &= ab - \int_0^a S(x)dx \\
&= 2(12) - \int_0^2 (3x + 6)dx \\
&= 24 - \left(\frac{3x^2}{2} + 6x\right)\Bigg|_0^2 \\
&= 24 - (6 + 12) = 6
\end{aligned}
$$

Thus the consumer's surplus and producers surplus are each equal to \$6 Ans

EXERCISE: 11-2

1. Find CS and PS defined by the demand curve

$$D(x) = 20 - 5x$$

and supply curve

$$S(x) = 4x + 8$$

Sketch the appropriate graphs.

2. Follow the same directions as in problem 1 if

$$D(x) = -0.4 + 15, \quad S(x) = 0.8x + 0.5$$

3. Find the consumer's surplus if the demand curve is

$$D(x) = 50 - 0.025x^2$$

and it is known that the market quantity is 20 units.

Example 2:

The demand function for a product is $p = f(q) = 100 - .05q$, where p is the price per unit (in dollars) for q units. The supply function is $p = g(q) = 10 + .1q$. Determine consumers' surplus and producers' suplus when market equilibrium has been established.

Solution:

Solve for the equilibrium point

$$\begin{aligned} 10 + 0.1q &= 100 - 0.05q \\ 0.15q &= 90 \\ q_o &= 600 \end{aligned}$$

Substituting q_o in the equation, we get $p_o = 70$

Consumers' surplus is $CS = \int_0^{q} D(q)\,dq - p_o q_o$

$$= \int_0^{600} (100 - 0.05q)\,dq - 600(70)$$

$$= \left(100q - \frac{0.05q^2}{2}\right)\Bigg|_0^{600} - 42000$$

$$= \left[100(600) - \frac{0.05(600)^2}{2}\right] - 42000$$

$$= 60000 - 9000 - 42000$$

$$= \$9000 \qquad \underline{\text{Ans}}$$

Producers' surplus is

$$PS = p_o q_o - \int_0^{q_o} S(q)\, dq$$

$$= 42,000 - \int_0^{600} (10 + 0.1q)\, dq$$

$$= 42000 - \left(10q + \frac{0.1q^2}{2}\right)\Bigg|_0^{600}$$

$$= 42000 - 10(600) - \frac{0.1(600)^2}{2}$$

$$= 42000 - 6000 - 18000$$

$$= \$18,000 \qquad \underline{\text{Ans}}$$

Example 3:

The demand equation for a product is $q = f(p) = (90/p) - 2$ and the supply equation is $q = g(p) = p - 1$. Determine the consumers' surplus and producers' surplus when market equilibrium has been established.

Solution:

Determining the equilibrium point, we have

$$p - 1 = \frac{90}{p} - 2$$

$$p^2 + p - 90 = 0$$

$$(p + 10)(p - 9) = 0$$

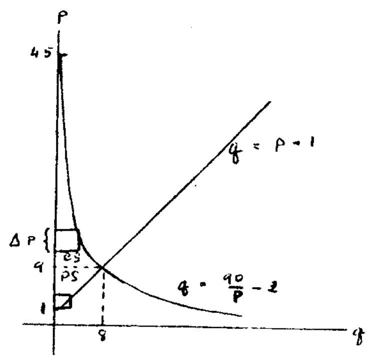
Thus $p_o = 9$ and $q_o = 8$

Consumers' surplus is

$$CS = \int_9^{45} \left(\frac{90}{p} - 2\right) dp$$

$$= (90 \ln |p| - 2p)\Bigg|_9^{45}$$

$$= 90 \ln 5 - 72 \approx 72.85$$



Using horizontal strips for producers' surplus, we have

$$PS = \int_1^9 (p - 1)\, dp$$

$$= \left. \frac{(p - 1)^2}{2} \right|_1^9 = 32 \qquad \underline{\text{Ans}}$$

Example 4:

The quantity demand of a product $q_d = p^2 - 40p + 400$ and the quantity supply of a product $q_s = 10p$. Find the consumer's surplus.

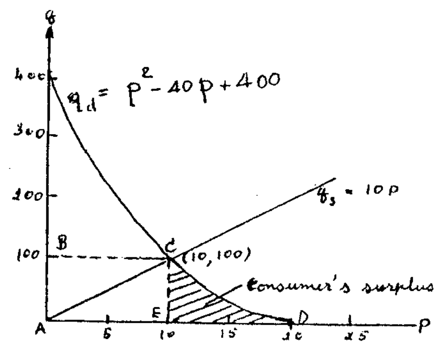
Solution:

Equilibrium occurs at the point of intersection between the two equation.

Therefore,

$$\begin{aligned} p^2 - 40p + 400 &= 10p \\ p^2 - 50p + 400 &= 0 \\ (p - 40)(p - 10) &= 0 \\ p &= 40 \text{ or } 10 \end{aligned}$$

Equilibrium occurs when a price of \$10 is charged and demand equals 100 units.



Consider the nature of the demand function. There would have been a demand for the product at prices higher than \$10. That is, there would have been consumers willing to pay almost \$20 for the product. And, additional consumers would have been drawn into the market at prices between \$10 and \$20.

If we assume that the price these people would be willing to pay is a measure of the utility the products holds for them, they actually receive a bonus when the market price is \$10. (Economists would claim that a measure of the actual utility of the product is the area ABCDE. And when the market is in equilibrium, the extra utility received by consumers, referred to as the consumer's surplus, is represented by the shaded area CDE.

This area can be found as.

$$\int_{10}^{20} (p^2 - 40p + 400)\, dp = \frac{p^3}{3} - 20p^2 + 400p \Big|_{10}^{20}$$

$$= \left[\frac{(20)^3}{3} - 20(20)^2 + 400(20)\right] - \left[\frac{(10)^3}{3} - 20(10)^2 + 400(10)\right]$$

$$= 2{,}666.67 - 2{,}333.33$$

$$= \$333.34 \qquad \underline{\text{Ans}}$$

EXERCISE: 11-3

1. The marginal revenue function for a firm's product is

$$MR = -0.02x + 20$$

where x equals the number of units sold.

(a) Determine the total revenue from selling 100 units of the product.
(b) What is the added revenue associated with an increase in sales from 100 to 150 units?

2. A manufacturer of jet engines estimates that the rate at which maintenance costs are incurred on its engines is a function of the number of hours of operation of the engine. For one engine used on commercial aircraft, the function is

$$r(x) = 50 + .025x^2$$

where x equals the number of hours of operation and r(x) equals the rate at which repair costs are incurred in dollars per hour of operation.

(a) Determine the rate at which costs are being incurred after 100 hours of operation.
(b) What are total maintenance costs expected to equal during the first 100 hours of operation?

. A company specializing in a mail-order sales approach is beginning a promotional campaign. Advertising expenditures will cost the firm $5,950 per day. Marketing specialists estimate that the rate at which profit (exclusive of advertising costs) will be generated from the promotion campaign decreases over the length of the campaign. Specifically, the rate r(t) for this campaign is estimated by the function

$$r(t) = -50t^2 + 10{,}000$$

where t represents the day of the campaign and $r(t)$ is measured in dollars per day. In order to maximize *net* profit, the firm should conduct the campaign as long as $r(t)$ exceeds the daily advertising cost.

(a) Graph the function $r(t)$ and the function $c(t) = 5,950$ which describes the rate at which advertising expenses are incurred.

(b) How long should the campaign be conducted?
(c) What are total advertising expenditures expected to equal during the campaign?
(d) What *net* profit will be expected?

4. You are given the demand function

$$q_d = p^2 - 30p + 200$$

and the supply function

$$q_s = 15p$$

where p is stated in dollars, q_d and q_s are stated in units, and $0 \leqslant p \leqslant 9$.

(a) Sketch the two functions.
(b) Determine the equilibrium price and quantity.
(c) Determine the value of the consumer's surplus if the market is in equilibrium.

5. *Energy Conservation.* A small business is considering buying an energy-saving device which will reduce its consumption of fuel. The device will cost $49,250. Engineering estimates suggest that savings from using the device will occur at a rate of $s(t)$ dollars per year where

$$s(t) = 50,000e^{-t}$$

and t equals time measured in years. Determine how long it will take for the firm to recover the cost of the device (that is, when the accumulated fuel savings equal the purchase cost).

6. *Blood Bank Management.* A hospital blood bank conducts an annual blood drive to replenish its inventory of blood. The hospital estimates that blood will be donated at a rate of $d(t)$ pints per day where

$$d(t) = 500e^{-0.4t}$$

and t equals the length of the blood drive in days. If the goal for the blood drive is 1,000 pints, when will the hospital reach its goal?

7. *Forest Management.* The demand for commercial forestland timber has been increasing rapidly over the past three to four decades. The function describing the rate of demand for timber is

$$d(t) = 12 + 0.005t^2$$

where d(t) is stated in billions of cubic feet per year and t equals time in years (t = 0 corresponds to January 1, 1965).

(a) Determine the rate of demand at the beginning of 1965.
(b) Determine the rate of demand at the beginning of 1980.
(c) Determine the *total* demand for timber during the period 1965 through 1980. (Hint: Integrate d(t) between t = 0 and t = 16).

8. *Solid Waste Management.* The rate w(t) at which solid waste is being generated in a major United States city is described by the function

$$w(t) = 2e^{0.075t}$$

where w(t) is stated in billions of tons per year and t equals time measured in years (t = 0 corresponds to Januray 1, 1976).

(a) Determine the rate at which solid waste is expected to be generated at the beginning of 1986.
(b) What total tonnage is expected to be generated during the 10-year period from 1976 through 1985?

9. Maintenance expenses for a piece of industrial equipment are estimated to be incurred at a rate described by the function

$$r(t) = 100 + 50t^2$$

where r(t) is measured in dollars per year and t equals the age of the machine in years.

(a) Determine the rate at which costs are being incurred when the machine is 3 years old.
(b) What are total maintenance expenses expected to equal during the first 5 years of operation?

10. *Oil Consumption.* In 1976 the amount of oil used in a particular region of the United States was 5 billion barrels. The demand for oil was growing at an exponential rate of 10 percent per year. The function describing annual rate of consumption c(t) at time t is

$$c(t) = 5e^{0.1t}$$

where t is measured in years, t = 0 corresponds to January 1, 1976, and c(t) is measured in billions of barrels per year. If the demand for oil continues to grow at this rate, how much oil is expected to be consumed in the 20-year period January 1, 1976, to January 1, 1996?

11. The demand for a product has been decreasing exponentially. The annual rate of demand d(t) is

$$d(t) = 250{,}000e^{-0.15t}$$

where t = 0 corresponds to January 1, 1977. The demand continues to decrease at the same rate.

(a) Determine the annual rate of demand at t = 4.
(b) How many total units are expected to be demanded over the time interval 1977 through 1986 (t = 0 to t = 10)?

12. An automobile manufacturer estimates that the annual rate of expenditure r(t) for maintenance on one of its models is represented by the function

$$r(t) = 120 + 8t^2$$

where t is the age of the automobile stated in years and r(t) is measured in dollars per year.

(a) At what annual rate are maintenance costs being incurred when the car is 4 years old?
(b) What are total maintenance costs expected to equal during the first 3 years?

13. If marginal revenue is given by $dr/dq = 100 - (3/2)\sqrt{2q}$, determine the corresponding demand equation.

14. If marginal cost is given by $dc/dq = q^2 + 7q + 6$, and fixed costs are 2500, determine the total cost for producing 6 units.

15. A manufacturer's marginal revenue function is $dr/dq = 275 - q - .3q^2$. If r is in dollars, find the increase in the manufacturer's total revenue if production is increased from 10 to 20 units.

16. A manufacturer's marginal cost function is $dc/dq = 500/\sqrt{2q + 25}$. If c is in dollars, determine the cost involved to increase production from 100 to 300 units.

17. For a product the demand equation is $p = .01q^2 - 1.1q + 30$ and its supply equation is $p = .01q^2 + 8$. Determine consumers' surplus and producers' surplus when market equilibrium has been established.

11.5 Maximizing Profit over Time:

The model introduced here is concerned with business operations of a special character. In oil drilling, mining, and other depletion operations, the initial revenue rate is generally higher than the revenue rate after a period of time has passed. That is, revenue rate, as a function of time, is a decreasing function (this is because depletion is occurring).

The cost rate of such operations generally increases with time because of inflation and other reasons. That is, cost rate, as a function of time, is an increasing function. The problem that management faces is to determine the time t^* that maximizes the profit function $P(t)$

Let $C(t)$ = denote the cost function.

$R(t)$ = denote the revenue function.

and t = denote time.

We make the natural assumption that the revenue rate is greater than cost rate at the beginning of the business operation under consideration. Also, as time goes on, we assume the cost rate increases to revenue rate, and thereafter exceeds it. The optimum time at which the business operation should terminate is that point in time where the rates are equal. That is, the optimum time t^* obeys

$$C'(t^*) = R'(t^*)$$

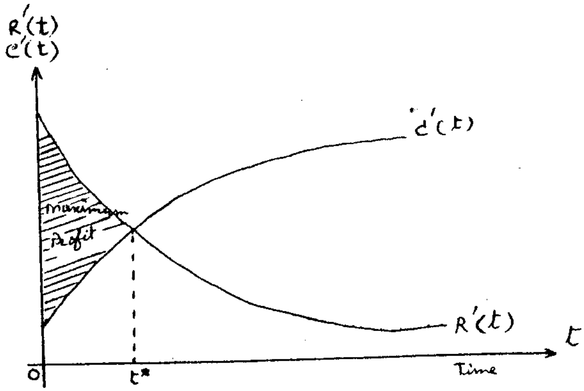
The profit rate $P'(t)$ is the difference between the revenue rate and the cost rate. That is,

$$P'(t) = R'(t) - C'(t)$$

Hence,

$$P(t) = \int_0^t [R'(t) - C'(t)]\, dt$$

The maximum profit is obtained when $t = t^*$, since $P'(t^*) = R'(t^*) - C'(t^*) = 0$. Thus, the maximum profit is $P(t^*)$. Genometrically, the maximum profit $P(t^*)$ is the area bounded by the curves $C'(t)$ and $R'(t)$ from $t = 0$ to $t = t^*$



Example:

The G - B Oil Company's revenue rate, in millions of dollars per year, at time t years is

$$R'(t) = 9 - t^{1/3}$$

and the corresponding cost rate function, also in millions of dollars, is

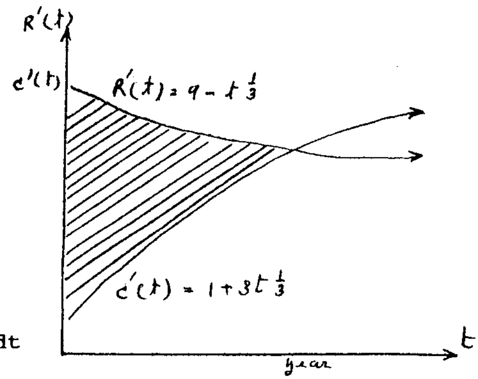
$$C'(t) = 1 + 3t^{1/3}$$

Determine how long the oil company should continue to operate and what total profit will be at the end of the operation.

Solution:

The time t* of optimal termination is found when

$$C'(t) = R'(t)$$
$$9 - t^{1/3} = 1 + 3t^{1/3}$$
$$8 = 4t^{1/3}$$
$$t^{1/3} = 2$$
$$t^* = 8 \text{ years} \quad \underline{\text{Ans}}$$



At $t^* = 8$ both revenue and cost rates are 7 million dollars per year.
The profit $P(t^*)$ is

$$P(t^*) = \int_0^8 [R'(t) - C'(t)]\, dt$$

$$= \int_0^8 \left[\left(9 - t^{1/3}\right) - \left(1 + 3t^{1/3}\right)\right] dt$$

$$= (8t - 3t^{4/3}) \Big|_0^8 = 16 \text{ millions of dollars} \qquad \underline{\text{Ans}}$$

EXERCISE: 11-4

1. Find the area bounded by $f(x) = 2x - 5$ and the x-axis
   (a) from $x = 3$ to $x = 6$
   (b) from $x = 0$ to $x = 1$
   (c) from $x = 2$ to $x = 4$

2. Find the area between the curves $f(x) = x^2$ and $g(x) = x^3$.

3. Find the area bounded by $f(x) = x^3 - x$ and the x-axis from $x = -1$ to $x = 2$.

4. Find the area between $f(x) = 2x/(x^2 + 1)$ and the x-axis from $x = 0$ to $x = 2$.

5. Find the area between $f(x) = xe^{3x^2}$ and the x-axis from $x = 0$ to $x = 1$.

6. Find the area between the curves $2x + 3y = 6$ and $y = 1/x$.

7. Find the area between $f(x) = x^2$ and $g(x) = \sqrt{x}$.

8. Find the area between the curves $f(x) = -x^2 + 2x + 2$ and $g(x) = x^2 - 4x + 2$.

11.6 The Learning Curve:

Quite often the managerial planning and control component of a production industry is faced with the problem of predicting labor time requirements and cost per unit of product. The tool used to achieve such prediction is the so-called *learning curve*.

The basic assumption made here is that, in certain production industries such as assembling of televisions and cars, the worker learns from experience. As a result, the more often he repeats an operation, the more efficient he becomes and hence, his direct labor input per unit of product declines. If the rate of improvement is regular enough, the learning curve can be used to predict future reductions in labor requirements.

The general form of the function describing such a situation is

$$\boxed{f(x) = Cx^k}$$

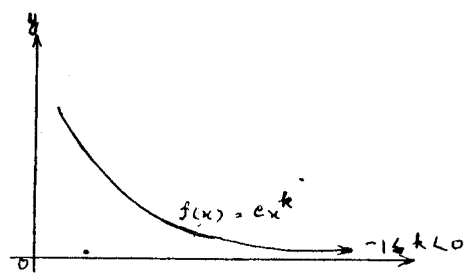
where f(x) is the number of hours of direct labor required to produce the x^{th} unit.

$$c > 0$$

$$\text{and} \quad -1 \leqslant k < 0$$

The choice of x^k, with $-1 \leqslant k < 0$, guarantees that, as the number x of units produced increases, the direct labor input decreases.

The function $f(x) = Cx^k$, describes a rate of learning per unit produced. This rate is measured in terms of labor hours per unit; thus the function f(x) shows that the number of direct labor hours declines as more items are produced.



Once a learning curve has been determined for a gross production process, it can be used as a predictor to determine the number of production hours for future work.

If a learning curve is known, the total number of labor hours required to produce units numbered "a" through "b" is

$$\boxed{N = \int_a^b f(x)\,dx = \int_a^b Cx^k\,dx}$$

Example:

The Air Conditioning Company manufactures air conditioners on an assembly line. From experience it was determined the first 100 air conditioners required 1272 labor hours. For each subsequent 100 air conditioners (1 unit), less labor hours were required according to the learning curve

$$f(x) = 1272x^{-0.25}$$

Where $f(x)$ is the rate of labor hours required to assemble the x^{th} unit (each unit being 100 air conditioners). This curve was determined after 30 units had been manufactured. The company is in the process of bidding for a large contract involving 5000 additional air conditioners or 50 additional units. What would be the labor hours required to assemble these units.

Solution:

$$N = \int_a^b Cx^k\,dx$$

$$= \int_{30}^{80} 1272x^{-0.25}dx$$

$$= \left.\frac{1272x^{0.75}}{0.75}\right|_{30}^{80}$$

$$= 1696\ (80^{0.75} - 30^{0.75})$$

$$= 1696(26.75 - 12.82)$$

$$= 1696(13.93) = 23{,}625.28$$

Thus the company can bid estimating the total labor hours needed as 23,625.28 Ans

EXERCISE: 11-5

1. After producing 35 units, a company determines that its production facility is following a learning curve of the form

$$f(x) = 1000x^{-0.5}$$

where f(x) is the rate of labor hours required to assemble the x^{th} unit. How many total labor hours should they estimate are required to produce an additional 25 units.

2. Danny's Auto Shop has found that, after tuning up 50 cars, a learning curve of the form

$$f(x) = 1000x^{-1}$$

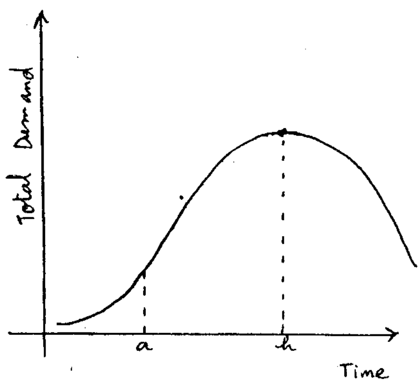
is being followed. How many total labor hours should they estimate are required to tune up an additional 50 cars?

3. In the construction of a learning curve, how would you interpret the situation in which $k \longrightarrow 0^{-}$.

11.7 A New Product Application:

In some situation it is easier to estimate the rate of change of demand for a new product than to estimate the demand pattern itself. This situation can occur when a product is very new (early in its "life cycle"), since we can observe how fast the demand is increasing.

Suppose we believe that for all new products of a certain type, sales follow an S-shaped curve with time, as shown in the figure. The firm may observe data up to point "a", then try to guess what the demand up to point "h" will be. For example, suppose that a firm, after market testing and observing the early returns, believes that they are marketing a product that has an S-shaped curve. The curve, S(t), is S-shaped since $\frac{ds}{dt}$ is known to them to be



$$\frac{ds}{dt} = b + ct - dt^2$$

where t is time and sales are in thousands. Their research group has estimated that

$$b = 0, \ c = 10, \ d = 1$$

Thus $\dfrac{ds}{dt} = 10t - t^2$

How many units do they expect to sell during the first 10 years if they have sold $4\frac{2}{3}$ (thousand) to date and t = 1 currently?

Solution:

$$\frac{ds}{dt} = 10t - t^2$$

$$\therefore S(t) = \int_1^t (10t - t^2)\,dt + 4\,\frac{2}{3}$$

$$S(10) = \int_1^{10} (10t - t^2)\,dt + 4\,\frac{2}{3}$$

$$= \frac{10t^2}{2} - \frac{t^3}{3}\,\Bigg|_1^{10} + 4\,\frac{2}{3}$$

$$= \left(\frac{10(10^2)}{2} - \frac{10^3}{3}\right) - \left(\frac{10}{2} - \frac{1}{3} + 4\,\frac{2}{3}\right)$$

$$= 166\,\frac{2}{3} \text{ (thousand)}$$

Therefore, they expect to sell $166\,\frac{2}{3}$ (thousand) units over the first 10 years of the product. Ans

Note: They can use $S(t) = 5t^2 - \frac{t^3}{3} + 4\,\frac{2}{3}$ as a guideline to check their progress. Every so often they can examine total demand to date and revise the estimate of the 10-year total if they are running high or low.

11.8 Advertising Effectiveness:

Another application of integration to what can be called "marketing" problems is in determining *the effect of advertising on sales*. In many such situations the rate of change of sales with respect to a change in advertising is known. From that we can obtain sales as a function of advertising expenditure, and this function can be used to make advertising (as well as other) decisions.

The dollar effect of advertising on sales usually decreases as more and more is spent. That is, the first dollar of advertising adds more to demand than the one-thousandth dollar. In fact, the effect of additional advertising may even become negative at some point; that is, the dollar expenditure may even cause sales revenues to decline.

Several functions can be used to describe the suggested relationship between advertising and sales. Some examples are:

i) $\frac{dS}{dA} = a - bA$ iii) $\frac{dS}{dA} = bA\left(\frac{a - S}{a}\right)$

ii) $\frac{dS}{dA} = \frac{a}{bA}$ iv) be^{-aA}.

where a and b are constants

Example:

Suppose that a small manufacturing firm wants to expand its market and is considering an increase in advertising as one possible means.

Let $S \equiv$ *total sales in a given month (in thousands of units)*
and $A \equiv$ *advertising expenditure per month (in thousands of dollars).*

The current level of advertising is \$1,000 per month and current sales are 10,000 units per month. During the last 2 years, the firm has tried several levels of advertising. The firm's market research group has examined the data (consisting of advertising expenditures, sales, and the change in sales from the previous month) for the different periods and believes that

$$\frac{dS}{dA} = \frac{a}{A^b}$$

best fits their situation. They ran some statistical routines (called regressions) to estimate a and b using several years of data for ΔS, ΔA, *and* A, *and they have estimated* $a = 2$ *and* $b = 1$.

The firm wants to consider the following:-

i) They are considering raising their advertising expenditure to \$2,700. What will their sales be then?

ii) The president wants to have total sales of 14,000 per month. How much advertising is needed to reach that goal? The president believes that goal is necessary to the long-run strength of the firm.

iii) If they sell the item for \$3, and it costs \$1 to produce, what is the optimal (short-run) advertising expenditure?

Solution:

First find sales as a function of advertising

$$\text{Since} \quad \frac{dS}{dA} = \frac{2}{A}$$

$$\text{or} \quad dS = 2\left(\frac{dA}{A}\right)$$

$$\int dS = \int \frac{2}{A}\, dA$$

$$S(A) = 2\log_e A + C$$

Evaluate C. $S(A) = 10$ when $A = 1$. Thus

$$10 = 2\log_e 1 + C$$

$$= 2(0) + C$$

$$C = 10$$

$$\therefore \quad S(A) = 2\log_e A + 10$$

(i) If the firm increases its advertising expenditure to 2.7 thousand, then sales will increase to 12,000 units. That is,

$$S(2.7) = 2\log_e(2.7) + 10$$

$$= 12 \qquad \underline{\text{Ans}}$$

(ii) The total sales should be 14,000 per month.

$$14 = 2\log_e A + 10$$

$$\text{or} \quad \log_e A = 2$$

$$A = e^2 = 7.3$$

The firm must expend 7.3 thousand dollars to reach the president's goal. <u>Ans</u>

iii) They sell the item for \$3, and it costs \$1 to produce. The total profit function $f(A) = (3 - 1)\, S(A) - A$

$$= (2)\, 2\log_e A - A$$

$$= 4 \log_e A - A$$

To maximize f(A), take the first derivative, set it equal to zero, and solve.

$$f'(A) = \frac{4}{A} - 1 = 0$$
$$A = 4$$

The optimal level of adverising, to maximize the short-run total profit, is 4 thousand dollars. Ans

11.9 A Production Problem:

Suppose that a manufacturer is faced with the problem of choosing between two methods of producing a given product. When x units of the product are being produced, methods 1 and 2 have respective marginal costs

$$C_1(x) = 5 - \left(\frac{x}{1000}\right)^2 e^{-x/1000}$$

and $$C_2(x) = 3 + \frac{400}{x+2}$$

Which production method is cheaper if 4000 units are to be produced?

Solution:

The graphs of $C_1(x)$ and $C_2(x)$ are sketched in Fig 1.

Note: that in using production method 1, the initial cost per unit is 5. The cost per additional unit falls until 2000 units are produced, after which the per unit cost of additional units increases, eventually approaching 5. On the other hand, with method 2, the first few units are prohibitively expensive. But the cost of additional units continues to fall, eventually approaching 3.

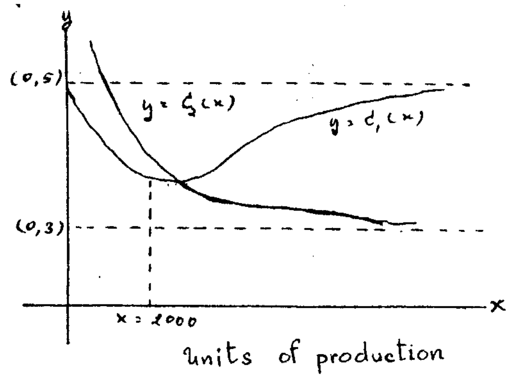


Fig 1.

As we have seen before , the total cost of producing 4000 units by methods 1 and 2, respectively, is (ignoring fixed costs)

$$T_1 = \int_0^{4000} C_1(x)\, dx$$

$$T_2 = \int_0^{4000} C_2(x)\, dx$$

However,

$$\begin{aligned} T_1 &= \int_0^{4000} \left[5 - \left(\frac{x}{1000}\right)^2 e^{-x/1000}\right] dx \\ &= 20{,}000 - \int_0^{4000} \left(\frac{x}{1000}\right)^2 e^{-x/1000}\, dx \\ &= 20{,}000 - 1000 \int_0^{4} u^2 e^{-u}\, du, \end{aligned}$$

where, in the last integral, we have made the substitution $u = \frac{x}{1000}$.

$$\begin{aligned} T_1 &= 20{,}000 - 1000[-u^2e^{-u} - 2ue^{-u} - 2e^{-u}]\Big|_0^4 \\ &= 20{,}000 + 1000[26e^{-4} - 2] \\ &\approx 18{,}476. \end{aligned}$$

Now

$$\begin{aligned} T_2 &= \int_0^{4000} \left[3 + \frac{400}{x + 2}\right] dx \\ &= 3(4000) + 400 \int_0^{4000} \frac{dx}{x + 2} \\ &= 12{,}000 + 400 \ln(x + 2)\Big|_0^{4000} \\ &= 12{,}000 + 400 \ln(4002) - 400 \ln 2 \\ &\approx 15{,}041. \end{aligned}$$

Hence we see that the second production method leads to a lower overall cost.

EXERCISE: 11-6

1. A firm has two potential investments, each of which costs \$1,000,000 initially, and nothing thereafter. They only have funds to invest in one of them. The data on the two investments are shown below:

Investment A	Investment B
Earns money at a rate of \$150,000 per year for 10 years, and nothing thereafter	Earns money at a rate of \$270,000 per year for 5 years, and nothing thereafter

   a) If the firm has a discount rate of 0.10, which, if either, investment should they choose?
   b) If the firm has a discount rate of zero, which should they choose?

2. A firm has a new product for which it is certain the new product will follow a known pattern of sales growth. (They have observed many products of this type before.) The pattern, for the first 2 years, is: annual rate of sales at a time t, where sales are in thousands and t is measured in years, is equal to

$$\text{ARS} = a\left(2t - \frac{t^2}{2} + \frac{t^3}{6}\right)$$

   where a is a constant that depends on how well the product "catches on." You also know:

   1) The constant a can be estimated very early in the 2-year period.
   2) A product is considered a success only if the total sales volume for 2 years is at least 10,000. If they expect less than that, they will abort the product.
   3) There have not been any sales of the prduct before time zero.

      a. What value of a will make them abort the product?
      b. If a equals 6, what is their expected first year's sales?

3. The Ace Hot Dog Company believes that their sales are growing exponentially, while their costs are growing linearly. In particular:

$$\text{sales(in tons)} = 1{,}000e^{0.05t}dt, \quad 0 < t < 10 \text{ (t is in years)}$$
$$\text{cost (in dollars)} = 500{,}000 + 50{,}000t, \quad 0 < t < 10.$$

   They sell hot dogs for \$1000 per ton, so profit is equal to sales times \$1000 minus cost.

a) How much profit will they make in the next 10 years?
b) What is the present value of the revenue stream if the firm's discount rate is 10 percent? That is, what is the present value of the \$1000 times the sales stream of revenue?
c) In part b, what is the answer if the discount rate is 5 percent?

4. A firm is considering two investments, which generate revenues at the following rates:

Investment A	Investement B
$100{,}000 + 10{,}000t,\ 0 < t < 5$	$120{,}000,\ 0 < t < 5$

Time, t, is measured in years.

a) Which investment generates the most total revenue over the first 3 years?
b) Which investment generates the most total revenue over the first 5 years?
c) Which investment is the preferable investment?

5. Suppose that the rate of change of sales with respect to advertising is given by

$$\frac{dS}{dA} = a - bA$$

where S is measured in units, $b > 0$, and A is measured in dollars.

a) At what point are sales maximized? How do you know?
b) At what point does an increase in advertising lead to a decrease in sales?
c) If the profit margin on each sale is \$c, at what advertising level is profit maximized?

6. A marketing manager knows that his rate of change of sales with respect to advertising is a decreasing function, as follows:

$$\frac{dS}{dA} = \frac{4}{A}, \quad A > 0,$$

where $A \equiv$ advertising in thousands of dollars and $S \equiv$ sales in hundreds.

a) Find S(A), sales as a function of advertising, if the manager knows that he can sell 5(hundred) when $A = 1$.
b) If each unit sells for p dollars above variable cost, find the function for profit (sales revenues minus advertising cost).
c) Find the optimal level of advertising for a price of p dollars.

7. Suppose that a person deposits money into a savings account at the rate of \$2000 per year for 5 years and \$3000 per year for the following 5 years. The bank pays interest at the rate of 5 percent per year. What is the value of the account after 10 years) (Leave the answer in terms of e to a power.)

8. What is the average amount of money in a savings account earning continuous 5% interest for a 10-year period if the account initially contains \$100?

9. Find CS and PS defined by the demand curve

$$D(x) = 20 - 5x$$

and supply curve

$$S(x) = 4x + 8$$

Sketch an appropriate graph.

10. Find the consumer's surplus if the demand curve is

$$D(x) = 50 - 0.025x^2$$

and it is known that the market quantity is 20 units.

11. A company whose annual sales are currently \$10,000,000 has reason to believe that sales will experience an increase of 8% per year for the next 5 years. What will the total sales be in 5 years?

12. Find the present value, to the nearest dollar, of a continuous annity at an annual rate of r for T years if the payment at time t is at the annual rate of f(t) dollars given that

a) $r = .06$, $T = 10$, $f(t) = 5000$.
b) $r = .05$, $T = 8$, $f(t) = 200t$.

13. Find the accumulated amount, to the nearest dollar, of a continuous annuity at an annual rate of r for T years if the payment at time t is at an annual rate of f(t) dollars given that

a) $r = .06$, $T = 10$, $f(t) = 400$.
b) $r = .04$, $T = 5$, $f(t) = 40t$.

14. Over the next five years the profits of a business at time t are estimated to be 20,000t dollars per year. The business is to be sold at a price equal to the present value of these future profits. If interest is compounded continously at the annual rate of 6 percent, to the nearest ten dollars at what price should the business be sold?

15. For a business the present value of all future profits at an annual interest rate r compounded continuously is given by

$$\int_0^{\infty} p(t)e^{-rt}\,dt,$$

where p(t) is the profit per year in dollars at time t. If $p(t) = 240{,}000$ and $r = .06$, evaluate the above integral.

16. After producing 35 units, a company determines that its production facility is following a learning curve of the form

$$f(x) = 1000x^{-0.5}$$

where f(x) is the rate of labor hours required to assemble the x^{th} unit. How many total labor hours should they estimate are required to produce an additional 25 units.

17. The revenue and the cost rate of Gold Star mining operation are, respectively,

$$R'(t) = 19 - t^{\frac{1}{2}}$$

and

$$C'(t) = 3 + 3t^{\frac{1}{2}}$$

where t is measured in years and R, C are measured in millions of dollars. Determine how long the operation should continue and the profit that will be generated during this period.

APPENDIXES

Appendix A

Table of the number of each day of the year*

Day of month	Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Day of month
1	1	32	60	91	121	152	182	213	244	274	305	335	1
2	2	33	61	92	122	153	183	214	245	275	306	336	2
3	3	34	62	93	123	154	184	215	246	276	307	337	3
4	4	35	63	94	124	155	185	216	247	277	308	338	4
5	5	36	64	95	125	156	186	217	248	278	309	339	5
6	6	37	65	96	126	157	187	218	249	279	310	340	6
7	7	38	66	97	127	158	188	219	250	280	311	341	7
8	8	39	67	98	128	159	189	220	251	281	312	342	8
9	9	40	68	99	129	160	190	221	252	282	313	343	9
10	10	41	69	100	130	161	191	222	253	283	314	344	10
11	11	42	70	101	131	162	192	223	254	284	315	345	11
12	12	43	71	102	132	163	193	224	255	285	316	346	12
13	13	44	72	103	133	164	194	225	256	286	317	347	13
14	14	45	73	104	134	165	195	226	257	287	318	348	14
15	15	46	74	105	135	166	196	227	258	288	319	349	15
16	16	47	75	106	136	167	197	228	259	289	320	350	16
17	17	48	76	107	137	168	198	229	260	290	321	351	17
18	18	49	77	108	138	169	199	230	261	291	322	352	18
19	19	50	78	109	139	170	200	231	262	292	323	353	19
20	20	51	79	110	140	171	201	232	263	293	324	354	20
21	21	52	80	111	141	172	202	233	264	294	325	355	21
22	22	53	81	112	142	173	203	234	265	295	326	356	22
23	23	54	82	113	143	174	204	235	266	296	327	357	23
24	24	55	83	114	144	175	205	236	267	297	328	358	24
25	25	56	84	115	145	176	206	237	268	298	329	359	25
26	26	57	85	116	146	177	207	238	269	299	330	360	26
27	27	58	86	117	147	178	208	239	270	300	331	361	27
28	28	59	87	118	148	179	209	240	271	301	332	362	28
29	29		88	119	149	180	210	241	272	302	333	363	29
30	30		89	120	150	181	211	242	273	303	334	364	30
31	31		90		151		212	243		304		365	31

* For leap year add 1 to the tabulated number after Feb.28. The number designating a leap year is divisible by 4.

* Source : Taken from, MATHEMATICS OF FINANCE, Third Ed. : Hummel and Seebeck, 1971, p. 252.

Appendix B

Compound Interest Table

Values of $(1 + i)^n$

n	$\frac{1}{4}\%$	$\frac{7}{24}\%$	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$
1	1.0025 0000	1.0029 1667	1.0033 3333	1.0041 6667	1.0050 0000
2	1.0050 0625	1.0058 4184	1.0066 7778	1.0083 5069	1.0100 2500
3	1.0075 1877	1.0087 7555	1.0100 3337	1.0125 5216	1.0150 7513
4	1.0100 3756	1.0117 1781	1.0134 0015	1.0167 7112	1.0201 5050
5	1.0125 6266	1.0146 6865	1.0167 7815	1.0210 0767	1.0252 5125
6	1.0150 9406	1.0176 2810	1.0201 6741	1.0252 6187	1.0303 7751
7	1.0176 3180	1.0205 9618	1.0235 6797	1.0295 3379	1.0355 2940
8	1.0201 7588	1.0235 7292	1.0269 7986	1.0338 2352	1.0407 0704
9	1.0227 2632	1.0265 5834	1.0304 0313	1.0381 3111	1.0459 1058
10	1.0252 8313	1.0295 5247	1.0338 3780	1.0424 5666	1.0511 4013
11	1.0278 4634	1.0325 5533	1.0372 8393	1.0468 0023	1.0563 9583
12	1.0304 1596	1.0355 6695	1.0407 4154	1.0511 6190	1.0616 7781
13	1.0329 9200	1.0385 8736	1.0442 1068	1.0555 4174	1.0669 8620
14	1.0355 7448	1.0416 1657	1.0476 9138	1.0599 3983	1.0723 2113
15	1.0381 6341	1.0446 5462	1.0511 8369	1.0643 5625	1.0776 8274
16	1.0407 5882	1.0477 0153	1.0546 8763	1.0687 9106	1.0830 7115
17	1.0433 6072	1.0507 5732	1.0582 0326	1.0732 4436	1.0884 8651
18	1.0459 6912	1.0538 2203	1.0617 3060	1.0777 1621	1.0939 2894
19	1.0485 8404	1.0568 9568	1.0652 6971	1.0822 0670	1.0993 9858
20	1.0512 0550	1.0599 7829	1.0688 2060	1.0867 1589	1.1048 9558
21	1.0538 3352	1.0630 6990	1.0723 8334	1.0912 4387	1.1104 2006
22	1.0564 6810	1.0661 7052	1.0759 5795	1.0957 9072	1.1159 7216
23	1.0591 0927	1.0692 8018	1.0795 4448	1.1003 5652	1.1215 5202
24	1.0617 5704	1.0723 9891	1.0831 4296	1.1049 4134	1.1271 5978
25	1.0644 1144	1.0755 2674	1.0867 5344	1.1095 4526	1.1327 9558
26	1.0670 7247	1.0786 6370	1.0903 7595	1.1141 6836	1.1384 5955
27	1.0697 4015	1.0818 0980	1.0940 1053	1.1188 1073	1.1441 5185
28	1.0724 1450	1.0849 6508	1.0976 5724	1.1234 7244	1.1498 7261
29	1.0750 9553	1.0881 2956	1.1013 1609	1.1281 5358	1.1556 2197
30	1.0777 8327	1.0913 0327	1.1049 8715	1.1328 5422	1.1614 0008
31	1.0804 7773	1.0944 8624	1.1086 7044	1.1375 7444	1.1672 0708
32	1.0831 7892	1.0976 7849	1.1123 6601	1.1423 1434	1.1730 4312
33	1.0858 8687	1.1008 8005	1.1160 7389	1.1470 7398	1.1789 0833
34	1.0886 0159	1.1040 9095	1.1197 9414	1.1518 5346	1.1848 0288
35	1.0913 2309	1.1073 1122	1.1235 2679	1.1566 5284	1.1907 2689
36	1.0940 5140	1.1105 4088	1.1272 7187	1.1614 7223	1.1966 8052
37	1.0967 8653	1.1137 7995	1.1310 2945	1.1663 1170	1.2026 6393
38	1.0995 2850	1.1170 2848	1.1347 9955	1.1711 7133	1.2086 7725
39	1.1022 7732	1.1202 8648	1.1385 8221	1.1760 5121	1.2147 2063
40	1.1050 3301	1.1235 5398	1.1423 7748	1.1809 5142	1.2207 9424
41	1.1077 9559	1.1268 3101	1.1461 8541	1.1858 7206	1.2268 9821
42	1.1105 6508	1.1301 1760	1.1500 0603	1.1908 1319	1.2330 3270
43	1.1133 4149	1.1334 1378	1.1538 3938	1.1957 7491	1.2391 9786
44	1.1161 2485	1.1367 1957	1.1576 8551	1.2007 5731	1.2453 9385
45	1.1189 1516	1.1400 3500	1.1615 4446	1.2057 6046	1.2516 2082
46	1.1217 1245	1.1433 6010	1.1654 1628	1.2107 8446	1.2578 7892
47	1.1245 1673	1.1466 9490	1.1693 0100	1.2158 2940	1.2641 6832
48	1.1273 2802	1.1500 3943	1.1731 9867	1.2208 9536	1.2704 8916
49	1.1301 4634	1.1533 9371	1.1771 0933	1.2259 8242	1.2768 4161
50	1.1329 7171	1.1567 5778	1.1810 3303	1.2310 9068	1.2832 2581

$(1 + i)^n$ (continued)

n	¼%	7⁄24%	⅓%	5⁄12%	½%
51	1.1358 0414	1.1601 3165	1.1849 6981	1.2362 2022	1.2896 4194
52	1.1386 4365	1.1635 1537	1.1889 1971	1.2413 7114	1.2960 9015
53	1.1414 9026	1.1669 0896	1.1928 8277	1.2465 4352	1.3025 7060
54	1.1443 4398	1.1703 1244	1.1968 5905	1.2517 3745	1.3090 8346
55	1.1472 0484	1.1737 2585	1.2008 4858	1.2569 5302	1.3156 2887
56	1.1500 7285	1.1771 4922	1.2048 5141	1.2621 9033	1.3222 0702
57	1.1529 4804	1.1805 8257	1.2088 6758	1.2674 4946	1.3288 1805
58	1.1558 3041	1.1840 2594	1.2128 9714	1.2727 3050	1.3354 6214
59	1.1587 1998	1.1874 7935	1.2169 4013	1.2780 3354	1.3421 3946
60	1.1616 1678	1.1909 4283	1.2209 9659	1.2833 5868	1.3488 5015
61	1.1645 2082	1.1944 1641	1.2250 6658	1.2887 0601	1.3555 9440
62	1.1674 3213	1.1979 0013	1.2291 5014	1.2940 7561	1.3623 7238
63	1.1703 5071	1.2013 9400	1.2332 4730	1.2994 6760	1.3691 8424
64	1.1732 7658	1.2048 9807	1.2373 5813	1.3048 8204	1.3760 3016
65	1.1762 0977	1.2084 1235	1.2414 8266	1.3103 1905	1.3829 1031
66	1.1791 5030	1.2119 3689	1.2456 2093	1.3157 7872	1.3898 2486
67	1.1820 9817	1.2154 7171	1.2497 7300	1.3212 6113	1.3967 7399
68	1.1850 5342	1.2190 1683	1.2539 3891	1.3267 6638	1.4037 5785
69	1.1880 1605	1.2225 7230	1.2581 1871	1.3322 9458	1.4107 7664
70	1.1909 8609	1.2261 3813	1.2623 1244	1.3378 4580	1.4178 3053
71	1.1939 6356	1.2297 1437	1.2665 2015	1.3434 2016	1.4249 1968
72	1.1969 4847	1.2333 0104	1.2707 4188	1.3490 1774	1.4320 4428
73	1.1999 4084	1.2368 9816	1.2749 7769	1.3546 3865	1.4392 0450
74	1.2029 4069	1.2405 0578	1.2792 2761	1.3602 8298	1.4464 0052
75	1.2059 4804	1.2441 2393	1.2834 9170	1.3659 5082	1.4536 3252
76	1.2089 6291	1.2477 5262	1.2877 7001	1.3716 4229	1.4609 0069
77	1.2119 8532	1.2513 9190	1.2920 6258	1.3773 5746	1.4682 0519
78	1.2150 1528	1.2550 4179	1.2963 6945	1.3830 9645	1.4755 4622
79	1.2180 5282	1.2587 0233	1.3006 9068	1.3888 5935	1.4829 2395
80	1.2210 9795	1.2623 7355	1.3050 2632	1.3946 4627	1.4903 3857
81	1.2241 5070	1.2660 5547	1.3093 7641	1.4004 5729	1.4977 9026
82	1.2272 1108	1.2697 4813	1.3137 4099	1.4062 9253	1.5052 7921
83	1.2302 7910	1.2734 5156	1.3181 2013	1.4121 5209	1.5128 0561
84	1.2333 5480	1.2771 6580	1.3225 1386	1.4180 3605	1.5203 6964
85	1.2364 3819	1.2808 9086	1.3269 2224	1.4239 4454	1.5279 7148
86	1.2395 2928	1.2846 2680	1.3313 4532	1.4298 7764	1.5356 1134
87	1.2426 2811	1.2883 7362	1.3357 8314	1.4358 3546	1.5432 8940
88	1.2457 3468	1.2921 3138	1.3402 3575	1.4418 1811	1.5510 0585
89	1.2488 4901	1.2959 0010	1.3447 0320	1.4478 2568	1.5587 6087
90	1.2519 7114	1.2996 7980	1.3491 8554	1.4538 5829	1.5665 5468
91	1.2551 0106	1.3034 7054	1.3536 8283	1.4599 1603	1.5743 8745
92	1.2582 3882	1.3072 7233	1.3581 9510	1.4659 9902	1.5822 5939
93	1.2613 8441	1.3110 8520	1.3627 2242	1.4721 0735	1.5901 7069
94	1.2645 3787	1.3149 0920	1.3672 6483	1.4782 4113	1.5981 2154
95	1.2676 9922	1.3187 4435	1.3718 2238	1.4844 0047	1.6061 1215
96	1.2708 6847	1.3225 9069	1.3763 9512	1.4905 8547	1.6141 4271
97	1.2740 4564	1.3264 4825	1.3809 8310	1.4967 9624	1.6222 1342
98	1.2772 3075	1.3303 1706	1.3855 8638	1.5030 3289	1.6303 2449
99	1.2804 2383	1.3341 9715	1.3902 0500	1.5092 9553	1.6387 7611
100	1.2836 2489	1.3380 8856	1.3948 3902	1.5155 8426	1.6466 6849

$(1 + i)^n$ *(continued)*

n	¼%	7/24%	⅓%	5/12%	½%
101	1.2868 3395	1.3419 9131	1.3994 8848	1.5218 9919	1.6549 0183
102	1.2900 5104	1.3459 0546	1.4041 5344	1.5282 4044	1.6631 7634
103	1.2932 7616	1.3498 3101	1.4088 3395	1.5346 0811	1.6714 9223
104	1.2965 0935	1.3537 6802	1.4135 3007	1.5410 0231	1.6798 4969
105	1.2997 5063	1.3577 1651	1.4182 4183	1.5474 2315	1.6882 4894
106	1.3030 0000	1.3616 7652	1.4229 6931	1.5538 7075	1.6966 9018
107	1.3062 5750	1.3656 4807	1.4277 1254	1.5603 4521	1.7051 7363
108	1.3095 2315	1.3696 3121	1.4324 7158	1.5668 4665	1.7136 9950
109	1.3127 9696	1.3736 2597	1.4372 4649	1.5733 7518	1.7222 6800
110	1.3160 7895	1.3776 3238	1.4420 3731	1.5799 3091	1.7308 7934
111	1.3193 6915	1.3816 5047	1.4468 4410	1.5865 1395	1.7395 3373
112	1.3226 6757	1.3856 8029	1.4516 6691	1.5931 2443	1.7482 3140
113	1.3259 7424	1.3897 2186	1.4565 0580	1.5997 6245	1.7569 7256
114	1.3292 8917	1.3937 7521	1.4613 6082	1.6064 2812	1.7657 5742
115	1.3326 1240	1.3978 4039	1.4662 3202	1.6131 2157	1.7745 8621
116	1.3359 4393	1.4019 1742	1.4711 1946	1.6198 4291	1.7834 5914
117	1.3392 8379	1.4060 0635	1.4760 2320	1.6265 9226	1.7923 7644
118	1.3426 3200	1.4101 0720	1.4809 4327	1.6333 6973	1.8013 3832
119	1.3459 8858	1.4142 2001	1.4858 7975	1.6401 7543	1.8103 4501
120	1.3493 5355	1.4183 4482	1.4908 3268	1.6470 0950	1.8193 9673
121	1.3527 2693	1.4224 8166	1.4958 0212	1.6538 7204	1.8284 9372
122	1.3561 0875	1.4266 3057	1.5007 8813	1.6607 6317	1.8376 3619
123	1.3594 9902	1.4307 9157	1.5057 9076	1.6676 8302	1.8468 2437
124	1.3628 9777	1.4349 6471	1.5108 1006	1.6746 3170	1.8560 5849
125	1.3663 0501	1.4391 5003	1.5158 4609	1.6816 0933	1.8653 3878
126	1.3697 2077	1.4433 4755	1.5208 9892	1.6886 1603	1.8746 6548
127	1.3731 4508	1.4475 5731	1.5259 6858	1.6956 5193	1.8840 3880
128	1.3765 7794	1.4517 7935	1.5310 5514	1.7027 1715	1.8934 5900
129	1.3800 1938	1.4560 1371	1.5361 5866	1.7098 1181	1.9029 2629
130	1.3834 6943	1.4602 6042	1.5412 7919	1.7169 3602	1.9124 4092
131	1.3869 2811	1.4645 1951	1.5464 1678	1.7240 8992	1.9220 0313
132	1.3903 9543	1.4687 9103	1.5515 7151	1.7312 7363	1.9316 1314
133	1.3938 7142	1.4730 7500	1.5567 4341	1.7384 8727	1.9412 7121
134	1.3973 5609	1.4773 7147	1.5619 3256	1.7457 3097	1.9509 7757
135	1.4008 4948	1.4816 8047	1.5671 3900	1.7530 0485	1.9607 3245
136	1.4043 5161	1.4860 0204	1.5723 6279	1.7603 0903	1.9705 3612
137	1.4078 6249	1.4903 3621	1.5776 0400	1.7676 4365	1.9803 8880
138	1.4113 8214	1.4946 8302	1.5828 6268	1.7750 0884	1.9902 9074
139	1.4149 1060	1.4990 4252	1.5881 3889	1.7824 0471	2.0002 4219
140	1.4184 4787	1.5034 1472	1.5934 3269	1.7898 3139	2.0102 4340
141	1.4219 9399	1.5077 9968	1.5987 4413	1.7972 8902	2.0202 9462
142	1.4255 4898	1.5121 9743	1.6040 7328	1.8047 7773	2.0303 9609
143	1.4291 1285	1.5166 0801	1.6094 2019	1.8122 9763	2.0405 4808
144	1.4326 8563	1.5210 3145	1.6147 8492	1.8198 4887	2.0507 5082
145	1.4362 6735	1.5254 6779	1.6201 6754	1.8274 3158	2.0610 0457
146	1.4398 5802	1.5299 1707	1.6255 6810	1.8350 4588	2.0713 0959
147	1.4434 5766	1.5343 7933	1.6309 8666	1.8426 9190	2.0816 6614
148	1.4470 6631	1.5388 5460	1.6364 2328	1.8503 6978	2.0920 7447
149	1.4506 8397	1.5433 4293	1.6418 7802	1.8580 7966	2.1025 3484
150	1.4543 1068	1.5478 4434	1.6473 5095	1.8658 2166	2.1130 4752

$(1 + i)^n$ (continued)

n	¼%	7/24%	⅓%	5/12%	½%
151	1.4579 4646	1.5523 5889	1.6528 4212	1.8735 9591	2.1236 1276
152	1.4615 9132	1.5568 8660	1.6583 5160	1.8814 0256	2.1342 3082
153	1.4652 4530	1.5614 2752	1.6638 7943	1.8892 4174	2.1449 0197
154	1.4689 0842	1.5659 8169	1.6694 2570	1.8971 1358	2.1556 2648
155	1.4725 8069	1.5705 4913	1.6749 9045	1.9050 1822	2.1664 0462
156	1.4762 6214	1.5751 2990	1.6805 7375	1.9129 5580	2.1772 3664
157	1.4799 5279	1.5797 2403	1.6861 7567	1.9209 2645	2.1881 2282
158	1.4836 5268	1.5843 3156	1.6917 9625	1.9289 3031	2.1990 6344
159	1.4873 6181	1.5889 5253	1.6974 3557	1.9369 6752	2.2100 587[illegible]
160	1.4910 8021	1.5935 8697	1.7030 9369	1.9450 3821	2.2211 0905
161	1.4948 0791	1.5982 3493	1.7087 7067	1.9531 4254	2.2322 1459
162	1.4985 4493	1.6028 9645	1.7144 6657	1.9612 8063	2.2433 756[illegible]
163	1.5022 9129	1.6075 7157	1.7201 8146	1.9694 5264	2.2545 9254
164	1.5060 4702	1.6122 6032	1.7259 1540	1.9776 5869	2.2658 6551
165	1.5098 1214	1.616[illegible] 6274	1.7316 6845	1.9858 9893	2.2771 9483
166	1.5135 8667	1.6216 7888	1.7374 4068	1.9941 7351	2.2885 8081
167	1.5173 7064	1.6264 0878	1.7432 3215	2.0024 8257	2.3000 2371
168	1.5211 6406	1.6311 5247	1.7490 4292	2.0108 2625	2.3115 2383
169	1.5249 6697	1.6359 1000	1.7548 7306	2.0192 0469	2.3230 8145
170	1.5287 7939	1.6406 8140	1.7607 2264	2.0276 1804	2.3346 9686
171	1.5326 0134	1.6454 6673	1.7665 9172	2.0360 6645	2.3463 7034
172	1.5364 3284	1.6502 6600	1.7724 8035	2.0445 5006	2.3581 0219
173	1.5402 7393	1.6550 7928	1.7783 8862	2.0530 6902	2.3698 9270
174	1.5441 2461	1.6599 0659	1.7843 1658	2.0616 2347	2.3817 4217
175	1.5479 8492	1.6647 4799	1.7902 6431	2.0702 1357	2.3936 5088
176	1.5518 5488	1.6696 0350	1.7962 3185	2.0788 3946	2.4056 1913
177	1.5557 3452	1.6744 7318	1.8022 1929	2.0875 0129	2.4176 4723
178	1.5596 2386	1.6793 5706	1.8082 2669	2.0961 9921	2.4297 3546
179	1.5635 2292	1.6842 5518	1.8142 5411	2.1049 3338	2.4418 8414
180	1.5674 3172	1.6891 6760	1.8203 0163	2.1137 0393	2.4540 9356
181	1.5713 5030	1.6940 9433	1.8263 6930	2.1225 1103	2.4663 6403
182	1.5752 7868	1.6990 3544	1.8324 5720	2.1313 5483	2.4786 9585
183	1.5792 1688	1.7039 9096	1.8385 6539	2.1402 3547	2.4910 8933
184	1.5831 6492	1.7089 6094	1.8446 9394	2.1491 5312	2.5035 4478
185	1.5871 2283	1.7139 4541	1.8508 4292	2.1581 0793	2.5160 6250
186	1.5910 9064	1.7189 4441	1.8570 1240	2.1671 0004	2.5286 4281
187	1.5950 6836	1.7239 5800	1.8632 0244	2.1761 2963	2.5412 8603
188	1.5990 5604	1.7289 8621	1.8694 1311	2.1851 9683	2.5539 9246
189	1.6030 5368	1.7340 2909	1.8756 4449	2.1943 0182	2.5667 6242
190	1.6070 6131	1.7390 8667	1.8818 9664	2.2034 4474	2.5795 9623
191	1.6110 7896	1.7441 5901	1.8881 6963	2.2126 2576	2.5924 9421
192	1.6151 0666	1.7492 4614	1.8944 6352	2.2218 4504	2.6054 5668
193	1.6191 4443	1.7543 4811	1.9007 7840	2.2311 0272	2.6184 8397
194	1.6231 9229	1.7594 6496	1.9071 1433	2.2403 9899	2.6315 7639
195	1.6272 5027	1.7645 9673	1.9134 7138	2.2497 3398	2.6447 3427
196	1.6313 1839	1.7697 4347	1.9198 4962	2.2591 0787	2.6579 5794
197	1.6353 9669	1.7749 0522	1.9262 4912	2.2685 2082	2.6712 4773
198	1.6394 8518	1.7800 8203	1.9326 6995	2.2779 7299	2.6846 0397
199	1.6435 8390	1.7852 7393	1.9391 1218	2.2874 6455	2.6980 2699
200	1.6476 9285	1.7904 8098	1.9455 7589	2.2969 9565	2.7115 1712

$(1+i)^n$ (continued)

n	$\frac{7}{12}\%$	$\frac{2}{3}\%$	$\frac{3}{4}\%$	$\frac{7}{8}\%$	1%
1	1.0058 3333	1.0066 6667	1.0075 0000	1.0087 5000	1.0100 0000
2	1.0117 0069	1.0133 7778	1.0150 5625	1.0175 7656	1.0201 0000
3	1.0176 0228	1.0201 3363	1.0226 6917	1.0264 8036	1.0303 0100
4	1.0235 3830	1.0269 3452	1.0303 3919	1.0354 6206	1.0406 0401
5	1.0295 0894	1.0337 8075	1.0380 6673	1.0445 2235	1.0510 1005
6	1.0355 1440	1.0406 7262	1.0458 5224	1.0536 6192	1.0615 2015
7	1.0415 5490	1.0476 1044	1.0536 9613	1.0628 8147	1.0721 3535
8	1.0476 3064	1.0545 9451	1.0615 9885	1.0721 8168	1.0828 5671
9	1.0537 4182	1.0616 2514	1.0695 6084	1.0815 6327	1.0936 8527
10	1.0598 8865	1.0687 0264	1.0775 8255	1.0910 2695	1.1046 2213
11	1.0660 7133	1.0758 2732	1.0856 6441	1.1005 7343	1.1156 6835
12	1.0722 9008	1.0829 9951	1.0938 0690	1.1102 0345	1.1268 2503
13	1.0785 4511	1.0902 1950	1.1020 1045	1.1199 1773	1.1380 9328
14	1.0848 3662	1.0974 8763	1.1102 7553	1.1297 1701	1.1494 7421
15	1.0911 6483	1.1048 0422	1.1186 0259	1.1396 0203	1.1609 6896
16	1.0975 2996	1.1121 6958	1.1269 9211	1.1495 7355	1.1725 7864
17	1.1039 3222	1.1195 8404	1.1354 4455	1.1596 3232	1.1843 0443
18	1.1103 7182	1.1270 4794	1.1439 6039	1.1697 7910	1.1961 4748
19	1.1168 4899	1.1345 6159	1.1525 4009	1.1800 1467	1.2081 0895
20	1.1233 6395	1.1421 2533	1.1611 8414	1.1903 3980	1.2201 9004
21	1.1299 1690	1.1497 3950	1.1698 9302	1.2007 5527	1.2323 9194
22	1.1365 0808	1.1574 0443	1.1786 6722	1.2112 6188	1.2447 1586
23	1.1431 3771	1.1651 2046	1.1875 0723	1.2218 6042	1.2571 6302
24	1.1498 0602	1.1728 8793	1.1964 1353	1.2325 5170	1.2697 3465
25	1.1565 1322	1.1807 0718	1.2053 8663	1.2433 3653	1.2824 3200
26	1.1632 5955	1.1885 7857	1.2144 2703	1.2542 1572	1.2952 5631
27	1.1700 4523	1.1965 0242	1.2235 3523	1.2651 9011	1.3082 0888
28	1.1768 7049	1.2044 7911	1.2327 1175	1.2762 6052	1.3212 9097
29	1.1837 3557	1.2125 0897	1.2419 5709	1.2874 2780	1.3345 0388
30	1.1906 4069	1.2205 9236	1.2512 7176	1.2986 9280	1.3478 4892
31	1.1975 8610	1.2287 2964	1.2606 5630	1.3100 5636	1.3613 2740
32	1.2045 7202	1.2369 2117	1.2701 1122	1.3215 1935	1.3749 4068
33	1.2115 9869	1.2451 6731	1.2796 3706	1.3330 8265	1.3886 9009
34	1.2186 6634	1.2534 6843	1.2892 3434	1.3447 4712	1.4025 7699
35	1.2257 7523	1.2618 2489	1.2989 0359	1.3565 1366	1.4166 0276
36	1.2329 2559	1.2702 3705	1.3086 4537	1.3683 8315	1.4307 6878
37	1.2401 1765	1.2787 0530	1.3184 6021	1.3803 5650	1.4450 7647
38	1.2473 5167	1.2872 3000	1.3283 4866	1.3924 3462	1.4595 2724
39	1.2546 2789	1.2958 1153	1.3383 1128	1.4046 1843	1.4741 2251
40	1.2619 4655	1.3044 5028	1.3483 4861	1.4169 0884	1.4888 6373
41	1.2693 0791	1.3131 4661	1.3584 6123	1.4293 0679	1.5037 5237
42	1.2767 1220	1.3219 0092	1.3686 4969	1.4418 1322	1.5187 8989
43	1.2841 5969	1.3307 1360	1.3789 1456	1.4544 2909	1.5339 7779
44	1.2916 5062	1.3395 8502	1.3892 5642	1.4671 5534	1.5493 1757
45	1.2991 8525	1.3485 1559	1.3996 7584	1.4799 9295	1.5648 1075
46	1.3067 6383	1.3575 0569	1.4101 7341	1.4929 4289	1.5804 5885
47	1.3143 8662	1.3665 5573	1.4207 4971	1.5060 0614	1.5962 6344
48	1.3220 5388	1.3756 6610	1.4314 0533	1.5191 8370	1.6122 2608
49	1.3297 6586	1.3848 3721	1.4421 4087	1.5324 7655	1.6283 4834
50	1.3375 2283	1.3940 6946	1.4529 5693	1.5458 8572	1.6446 3182

$(1 + i)^n$ (continued)

n	7/12 %	2/3 %	3/4 %	7/8 %	1 %
51	1.3453 2504	1.4033 6325	1.4638 5411	1.5594 1222	1.6610 7814
52	1.3531 7277	1.4127 1901	1.4748 3301	1.5730 5708	1.6776 8892
53	1.3610 6628	1.4221 3713	1.4858 9426	1.5868 2133	1.6944 6581
54	1.3690 0583	1.4316 1805	1.4970 3847	1.6007 0602	1.7114 1047
55	1.3769 9170	1.4411 6217	1.5082 6626	1.6147 1219	1.7285 2457
56	1.3850 2415	1.4507 6992	1.5195 7825	1.6288 4093	1.7458 0982
57	1.3931 0346	1.4604 4172	1.5309 7509	1.6430 9328	1.7632 6792
58	1.4012 2990	1.4701 7799	1.5424 5740	1.6574 7035	1.7809 0060
59	1.4094 0374	1.4799 7918	1.5540 2583	1.6719 7322	1.7987 0960
60	1.4176 2526	1.4898 4571	1.5656 8103	1.6866 0298	1.8166 9670
61	1.4258 9474	1.4997 7801	1.5774 2363	1.7013 6076	1.8348 6367
62	1.4342 1246	1.5097 7653	1.5892 5431	1.7162 4766	1.8532 1230
63	1.4425 7870	1.5198 4171	1.6011 7372	1.7312 6483	1.8717 4443
64	1.4509 9374	1.5299 7399	1.6131 8252	1.7464 1340	1.8904 6187
65	1.4594 5787	1.5401 7381	1.6252 8139	1.7616 9452	1.9093 6649
66	1.4679 7138	1.5504 4164	1.6374 7100	1.7771 0934	1.9284 6015
67	1.4765 3454	1.5607 7792	1.6497 5203	1.7926 5905	1.9477 4475
68	1.4851 4766	1.5711 8310	1.6621 2517	1.8083 4482	1.9672 2220
69	1.4938 1102	1.5816 5766	1.6745 9111	1.8241 6783	1.9868 9442
70	1.5025 2492	1.5922 0204	1.6871 5055	1.8401 2930	2.0067 6337
71	1.5112 8965	1.6028 1672	1.6998 0418	1.8562 3043	2.0268 3100
72	1.5201 0550	1.6135 0217	1.7125 5271	1.8724 7245	2.0470 9931
73	1.5289 7279	1.6242 5885	1.7253 9685	1.8888 5658	2.0675 7031
74	1.5378 9179	1.6350 8724	1.7383 3733	1.9053 8408	2.0882 4601
75	1.5468 6283	1.6459 8782	1.7513 7486	1.9220 5619	2.1091 2847
76	1.5558 8620	1.6569 6107	1.7645 1017	1.9388 7418	2.1302 1975
77	1.5649 6220	1.6680 0748	1.7777 4400	1.9558 3933	2.1515 2195
78	1.5740 9115	1.6791 2753	1.7910 7708	1.9729 5292	2.1730 3717
79	1.5832 7334	1.6903 2172	1.8045 1015	1.9902 1626	2.1947 6754
80	1.5925 0910	1.7015 9053	1.8180 4398	2.0076 3066	2.2167 1522
81	1.6017 9874	1.7129 3446	1.8316 7931	2.0251 9742	2.2388 8237
82	1.6111 4257	1.7243 5403	1.8454 1691	2.0429 1790	2.2612 7119
83	1.6205 4090	1.7358 4972	1.8592 5753	2.0607 9343	2.2838 8390
84	1.6299 9405	1.7474 2205	1.8732 0196	2.0788 2537	2.3067 2274
85	1.6395 0235	1.7590 7153	1.8872 5098	2.0970 1510	2.3297 8997
86	1.6490 6612	1.7707 9868	1.9014 0536	2.1153 6398	2.3530 8787
87	1.6586 8567	1.7826 0400	1.9156 6590	2.1338 7341	2.3766 1875
88	1.6683 6134	1.7944 8803	1.9300 3339	2.1525 4481	2.4003 8494
89	1.6780 9344	1.8064 5128	1.9445 0865	2.1713 7957	2.4243 8879
90	1.6878 8232	1.8184 9429	1.9590 9246	2.1903 7914	2.4486 3267
91	1.6977 2830	1.8306 1758	1.9737 8565	2.2095 4496	2.4731 1900
92	1.7076 3172	1.8428 2170	1.9885 8905	2.2288 7848	2.4978 5019
93	1.7175 9290	1.8551 0718	2.0035 0346	2.2483 8117	2.5228 2869
94	1.7276 1219	1.8674 7456	2.0185 2974	2.2680 5450	2.5480 5698
95	1.7376 8993	1.8799 2439	2.0336 6871	2.2878 9998	2.5735 3755
96	1.7478 2646	1.8924 5722	2.0489 2123	2.3079 1910	2.5992 7293
97	1.7580 2211	1.9050 7360	2.0642 8814	2.3281 1340	2.6252 6565
98	1.7682 7724	1.9177 7409	2.0797 7030	2.3484 8439	2.6515 1831
99	1.7785 9219	1.9305 5925	2.0953 6858	2.3690 3363	2.6780 3349
100	1.7889 6731	1.9434 2965	2.1110 8384	2.3897 6267	2.7048 1383

$(1 + i)^n$ (continued)

n	$\frac{7}{12}\%$	$\frac{2}{3}\%$	$\frac{3}{4}\%$	$\frac{7}{8}\%$	1%
101	1.7994 0295	1.9563 8585	2.1269 1697	2.4106 7309	2.7318 6197
102	1.8098 9947	1.9694 2842	2.1428 6885	2.4317 6648	2.7591 8059
103	1.8204 5722	1.9825 5794	2.1589 4036	2.4530 4444	2.7867 7239
104	1.8310 7655	1.9957 7499	2.1751 3242	2.4745 0858	2.8146 4012
105	1.8417 5783	2.0090 8016	2.1914 4591	2.4961 6053	2.8427 8652
106	1.8525 0142	2.0224 7403	2.2078 8175	2.5180 0193	2.8712 1438
107	1.8633 0768	2.0359 5719	2.2244 4087	2.5400 3445	2.8999 2653
108	1.8741 7697	2.0495 3024	2.2411 2417	2.5622 5975	2.9289 2579
109	1.8851 0967	2.0631 9377	2.2579 3260	2.5846 7953	2.9582 1505
110	1.8961 0614	2.0769 4840	2.2748 6710	2.6072 9547	2.9877 9720
111	1.9071 6676	2.0907 9472	2.2919 2860	2.6301 0931	3.0176 7517
112	1.9182 9190	2.1047 3335	2.3091 1807	2.6531 2276	3.0478 5192
113	1.9294 8194	2.1187 6491	2.3264 3645	2.6763 3759	3.0783 3044
114	1.9407 3725	2.1328 9000	2.3438 8472	2.6997 5554	3.1091 1375
115	1.9520 5822	2.1471 0927	2.3614 6386	2.7233 7840	3.1402 0489
116	1.9634 4522	2.1614 2333	2.3791 7484	2.7472 0796	3.1716 0693
117	1.9748 9865	2.1758 3282	2.3970 1865	2.7712 4603	3.2033 2300
118	1.9864 1890	2.1903 3837	2.4149 9629	2.7954 9444	3.2353 5623
119	1.9980 0634	2.2049 4063	2.4331 0876	2.8199 5501	3.2677 0980
120	2.0096 6138	2.2196 4023	2.4513 5708	2.8446 2962	3.3003 8689
121	2.0213 8440	2.2344 3784	2.4697 4226	2.8695 2013	3.3333 9076
122	2.0331 7581	2.2493 3409	2.4882 6532	2.8946 2843	3.3667 2467
123	2.0450 3600	2.2643 2965	2.5069 2731	2.9199 5643	3.4003 9192
124	2.0569 6538	2.2794 2518	2.5257 2927	2.9455 0605	3.4343 9584
125	2.0689 6434	2.2946 2135	2.5446 7224	2.9712 7922	3.4687 3980
126	2.0810 3330	2.3099 1882	2.5637 5728	2.9972 7792	3.5034 2719
127	2.0931 7266	2.3253 1828	2.5829 8546	3.0235 0410	3.5384 6147
128	2.1053 8284	2.3408 2040	2.6023 5785	3.0499 5976	3.5738 4608
129	2.1176 6424	2.3564 2587	2.6218 7553	3.0766 4691	3.6095 8454
130	2.1300 1728	2.3721 3538	2.6415 3960	3.1035 6757	3.6456 8039
131	2.1424 4238	2.3879 4962	2.6613 5115	3.1307 2378	3.6821 3719
132	2.1549 3996	2.4038 6928	2.6813 1128	3.1581 1762	3.7189 5856
133	2.1675 1044	2.4198 9507	2.7014 2112	3.1857 5115	3.7561 4815
134	2.1801 5425	2.4360 2771	2.7216 8177	3.2136 2647	3.7937 0963
135	2.1928 7182	2.4522 6789	2.7420 9439	3.2417 4570	3.8316 4673
136	2.2056 6357	2.4686 1635	2.7626 6009	3.2701 1098	3.8699 6319
137	2.2185 2994	2.4850 7379	2.7833 8005	3.2987 2445	3.9086 6282
138	2.2314 7137	2.5016 4095	2.8042 5540	3.3275 8829	3.9477 4945
139	2.2444 8828	2.5183 1855	2.8252 8731	3.3567 0468	3.9872 2695
140	2.2575 8113	2.5351 0734	2.8464 7697	3.3860 7585	4.0270 9922
141	2.2707 5036	2.5520 0806	2.8678 2554	3.4157 0401	4.0673 7021
142	2.2839 9640	2.5690 2145	2.8893 3424	3.4455 9142	4.1080 4391
143	2.2973 1971	2.5861 4826	2.9110 0424	3.4757 4035	4.1491 2435
144	2.3107 2074	2.6033 8924	2.9328 3677	3.5061 5308	4.1906 1559
145	2.3241 9995	2.6207 4517	2.9548 3305	3.5368 3192	4.2325 2175
146	2.3377 5778	2.6382 1681	2.9769 9430	3.5677 7919	4.2748 4697
147	2.3513 9470	2.6558 0492	2.9993 2175	3.5989 9726	4.3175 9544
148	2.3651 1117	2.6735 1028	3.0218 1667	3.6304 8849	4.3607 7139
149	2.3789 0765	2.6913 3369	3.0444 8029	3.6622 5526	4.4043 7910
150	2.3927 8461	2.7092 7591	3.0673 1389	3.6943 0000	4.4484 2290

n	7/12%	2/3%	3/4%	7/8%	1%
151	2.4067 4252	2.7273 3775	3.0903 1875	3.7266 2512	4.4929 0712
152	2.4207 8186	2.7455 2000	3.1134 9614	3.7592 3309	4.5378 3620
153	2.4349 0308	2.7638 2347	3.1368 4736	3.7921 2638	4.5832 1456
154	2.4491 0668	2.7822 4896	3.1603 7372	3.8253 0749	4.6290 4670
155	2.4633 9314	2.8007 9729	3.1840 7652	3.8587 7893	4.6753 3717
156	2.4777 6293	2.8194 6927	3.2079 5709	3.8925 4324	4.7220 9054
157	2.4922 1655	2.8382 6573	3.2320 1677	3.9266 0300	4.7693 1145
158	2.5067 5448	2.8571 8750	3.2562 5690	3.9609 6077	4.8170 0456
159	2.5213 7721	2.8762 3542	3.2806 7882	3.9956 1918	4.8651 7461
160	2.5360 8525	2.8954 1032	3.3052 8391	4.0305 8085	4.9138 2635
161	2.5508 7908	2.9147 1306	3.3300 7354	4.0658 4843	4.9629 6462
162	2.5657 5921	2.9341 4448	3.3550 4910	4.1014 2460	5.0125 9426
163	2.5807 2614	2.9537 0544	3.3802 1196	4.1373 1207	5.0627 2021
164	2.5957 8037	2.9733 9681	3.4055 6355	4.1735 1355	5.1133 4741
165	2.6109 2242	2.9932 1945	3.4311 0528	4.2100 3179	5.1644 8088
166	2.6261 5280	3.0131 7425	3.4568 3857	4.2468 6957	5.2161 2569
167	2.6414 7203	3.0332 6208	3.4827 6486	4.2840 2968	5.2682 8695
168	2.6568 8062	3.0534 8383	3.5088 8560	4.3215 1494	5.3209 6982
169	2.6723 7909	3.0738 4038	3.5352 0224	4.3593 2819	5.3741 7952
170	2.6879 6796	3.0943 3265	3.5617 1625	4.3974 7232	5.4279 2131
171	2.7036 4778	3.1149 6154	3.5884 2913	4.4359 5020	5.4822 0052
172	2.7194 1906	3.1357 2795	3.6153 4234	4.4747 6476	5.5370 2253
173	2.7352 8233	3.1566 3280	3.6424 5741	4.5139 1896	5.5923 9275
174	2.7512 3815	3.1776 7702	3.6697 7584	4.5534 1575	5.6483 1668
175	2.7672 8704	3.1988 6153	3.6972 9916	4.5932 5813	5.7047 9985
176	2.7834 2954	3.2201 8728	3.7250 2891	4.6334 4914	5.7618 4785
177	2.7996 6622	3.2416 5519	3.7529 6662	4.6739 9182	5.8194 6633
178	2.8159 9760	3.2632 6623	3.7811 1387	4.7148 8925	5.8776 6099
179	2.8324 2426	3.2850 2134	3.8094 7223	4.7561 4453	5.9364 3760
180	2.8489 4673	3.3069 2148	3.8380 4327	4.7977 6080	5.9958 0198
181	2.8655 6559	3.3289 6762	3.8668 2859	4.8397 4120	6.0557 6000
182	2.8822 8139	3.3511 6074	3.8958 2981	4.8820 8894	6.1163 1760
183	2.8990 9469	3.3735 0181	3.9250 4853	4.9248 0722	6.1774 8077
184	2.9160 0608	3.3959 9182	3.9544 8639	4.9678 9928	6.2392 5558
185	2.9330 1612	3.4186 3177	3.9841 4504	5.0113 6840	6.3016 4813
186	2.9501 2538	3.4414 2265	4.0140 2613	5.0552 1787	6.3646 6462
187	2.9673 3444	3.4643 6546	4.0441 3133	5.0994 5103	6.4283 1126
188	2.9846 4389	3.4874 6123	4.0744 6231	5.1440 7123	6.4925 9437
189	3.0020 5431	3.5107 1097	4.1050 2078	5.1890 8185	6.5575 2032
190	3.0195 6630	3.5341 1571	4.1358 0843	5.2344 8631	6.6230 9552
191	3.0371 8043	3.5576 7649	4.1668 2700	5.2802 8807	6.6893 2648
192	3.0548 9732	3.5813 9433	4.1980 7820	5.3264 9059	6.7562 1974
193	3.0727 1755	3.6052 7029	4.2295 6379	5.3730 9738	6.8237 8194
194	3.0906 4174	3.6293 0543	4.2612 8551	5.4201 1199	6.8920 1976
195	3.1086 7048	3.6535 0080	4.2932 4516	5.4675 3797	6.9609 3996
196	3.1268 0440	3.6778 5747	4.3254 4449	5.5153 7892	7.0305 4936
197	3.1450 4409	3.7023 7652	4.3578 8533	5.5636 3849	7.1008 5485
198	3.1633 9018	3.7270 5903	4.3905 6947	5.6123 2033	7.1718 6340
199	3.1818 4329	3.7519 0609	4.4234 9874	5.6614 2813	7.2435 8203
200	3.2004 0404	3.7769 1880	4.4566 7498	5.7109 6562	7.3160 1785

$(1 + i)^n$ (continued)

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
1	1.0112 5000	1.0125 0000	1.0137 5000	1.0150 0000	1.0175 0000
2	1.0226 2656	1.0251 5625	1.0276 8906	1.0302 2500	1.0353 0625
3	1.0341 3111	1.0379 7070	1.0418 1979	1.0456 7838	1.0534 2411
4	1.0457 6509	1.0509 4534	1.0561 4481	1.0613 6355	1.0718 5903
5	1.0575 2994	1.0640 8215	1.0706 6680	1.0772 8400	1.0906 1656
6	1.0694 2716	1.0773 8318	1.0853 8847	1.0934 4326	1.1097 0235
7	1.0814 5821	1.0908 5047	1.1003 1256	1.1098 4491	1.1291 2215
8	1.0936 2462	1.1044 8610	1.1154 4186	1.1264 9259	1.1488 8178
9	1.1059 2789	1.1182 9218	1.1307 7918	1.1433 8998	1.1689 8721
10	1.1183 6958	1.1322 7083	1.1463 2740	1.1605 4083	1.1894 4449
11	1.1309 5124	1.1464 2422	1.1620 8940	1.1779 4894	1.2102 5977
12	1.1436 7444	1.1607 5452	1.1780 6813	1.1956 1817	1.2314 3931
13	1.1565 4078	1.1752 6395	1.1942 6656	1.2135 5244	1.2529 8950
14	1.1695 5186	1.1899 5475	1.2106 8773	1.2317 5573	1.2749 1682
15	1.1827 0932	1.2048 2918	1.2273 3469	1.2502 3207	1.2972 2786
16	1.1960 1480	1.2198 8955	1.2442 1054	1.2689 8555	1.3199 2935
17	1.2094 6997	1.2351 3817	1.2613 1843	1.2880 2033	1.3430 2811
18	1.2230 7650	1.2505 7739	1.2786 6156	1.3073 4064	1.3665 3111
19	1.2368 3611	1.2662 0961	1.2962 4316	1.3269 5075	1.3904 4540
20	1.2507 5052	1.2820 3723	1.3140 6650	1.3468 5501	1.4147 7820
21	1.2648 2146	1.2980 6270	1.3321 3492	1.3670 5783	1.4395 3681
22	1.2790 5071	1.3142 8848	1.3504 5177	1.3875 6370	1.4647 2871
23	1.2934 4003	1.3307 1709	1.3690 2048	1.4083 7715	1.4903 6146
24	1.3079 9123	1.3473 5105	1.3878 4451	1.4295 0281	1.5164 4279
25	1.3227 0613	1.3641 9294	1.4069 2738	1.4509 4535	1.5429 8054
26	1.3375 8657	1.3812 4535	1.4262 7263	1.4727 0953	1.5699 8269
27	1.3526 3442	1.3985 1092	1.4458 8388	1.4948 0018	1.5974 5739
28	1.3678 5156	1.4159 9230	1.4657 6478	1.5172 2218	1.6254 1290
29	1.3832 3989	1.4336 9221	1.4859 1905	1.5399 8051	1.6538 5762
30	1.3988 0134	1.4516 1336	1.5063 5043	1.5630 8022	1.6828 0013
31	1.4145 3785	1.4697 5853	1.5270 6275	1.5865 2642	1.7122 4913
32	1.4304 5140	1.4881 3051	1.5480 5986	1.6103 2432	1.7422 1349
33	1.4465 4398	1.5067 3214	1.5693 4569	1.6344 7918	1.7727 0223
34	1.4628 1760	1.5255 6629	1.5909 2419	1.6589 9637	1.8037 2452
35	1.4792 7430	1.5446 3587	1.6127 9940	1.6838 8132	1.8352 8970
36	1.4959 1613	1.5639 4382	1.6349 7539	1.7091 3954	1.8674 0727
37	1.5127 4519	1.5834 9312	1.6574 5630	1.7347 7663	1.9000 8689
38	1.5297 6357	1.6032 8678	1.6802 4633	1.7607 9828	1.9333 3841
39	1.5469 7341	1.6233 2787	1.7033 4971	1.7872 1025	1.9671 7184
40	1.5643 7687	1.6436 1946	1.7267 7077	1.8140 1841	2.0015 9734
41	1.5819 7611	1.6641 6471	1.7505 1387	1.8412 2868	2.0366 2530
42	1.5997 7334	1.6849 6677	1.7745 8343	1.8688 4712	2.0722 6624
43	1.6177 7079	1.7060 2885	1.7989 8396	1.8968 7982	2.1085 3090
44	1.6359 7071	1.7273 5421	1.8237 1999	1.9253 3302	2.1454 3019
45	1.6543 7538	1.7489 4614	1.8487 9614	1.9542 1301	2.1829 7522
46	1.6729 8710	1.7708 0797	1.8742 1708	1.9835 2621	2.2211 7728
47	1.6918 0821	1.7929 4306	1.8999 8757	2.0132 7910	2.2600 4789
48	1.7108 4105	1.8153 5485	1.9261 1240	2.0434 7829	2.2995 9872
49	1.7300 8801	1.8380 4679	1.9525 9644	2.0741 3046	2.3398 4170
50	1.7495 5150	1.8610 2237	1.9794 4464	2.1052 4242	2.3807 8893

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
51	1.7692 3395	1.8842 8515	2.0066 6201	2.1368 2106	2.4224 5274
52	1.7891 3784	1.9078 3872	2.0342 5361	2.1688 7337	2.4648 4566
53	1.8092 6564	1.9316 8670	2.0622 2460	2.2014 0647	2.5079 8046
54	1.8296 1988	1.9558 3279	2.0905 8019	2.2344 2757	2.5518 7012
55	1.8502 0310	1.9802 8070	2.1193 2566	2.2679 4398	2.5965 2785
56	1.8710 1788	2.0050 3420	2.1484 6639	2.3019 6314	2.6419 6708
57	1.8920 6684	2.0300 9713	2.1780 0780	2.3364 9259	2.6882 0151
58	1.9133 5259	2.0554 7335	2.2079 5541	2.3715 3998	2.7352 4503
59	1.9348 7780	2.0811 6676	2.2383 1480	2.4071 1308	2.7831 1182
60	1.9566 4518	2.1071 8135	2.2690 9163	2.4432 1978	2.8318 1628
61	1.9786 5744	2.1335 2111	2.3002 9164	2.4798 6807	2.8813 7306
62	2.0009 1733	2.1601 9013	2.3319 2065	2.5170 6609	2.9317 9709
63	2.0234 2765	2.1871 9250	2.3639 8456	2.5548 2208	2.9831 0354
64	2.0461 9121	2.2145 3241	2.3964 8934	2.5931 4442	3.0353 0785
65	2.0692 1087	2.2422 1407	2.4294 4107	2.6320 4158	3.0884 2574
66	2.0924 8949	2.2702 4174	2.4628 4589	2.6715 2221	3.1424 7319
67	2.1160 2999	2.2986 1976	2.4967 1002	2.7115 9504	3.1974 6647
68	2.1398 3533	2.3273 5251	2.5310 3978	2.7522 6896	3.2534 2213
69	2.1639 0848	2.3564 4442	2.5658 4158	2.7935 5300	3.3103 5702
70	2.1882 5245	2.3858 9997	2.6011 2190	2.8354 5629	3.3682 8827
71	2.2128 7029	2.4157 2372	2.6368 8732	2.8779 8814	3.4272 3331
72	2.2377 6508	2.4459 2027	2.6731 4453	2.9211 5796	3.4872 0990
73	2.2629 3994	2.4764 9427	2.7099 0026	2.9649 7533	3.5482 3607
74	2.2883 9801	2.5074 5045	2.7471 6139	3.0094 4996	3.6103 3020
75	2.3141 4249	2.5387 9358	2.7849 3486	3.0545 9171	3.6735 1098
76	2.3401 7659	2.5705 2850	2.8232 2771	3.1004 1059	3.7377 9742
77	2.3665 0358	2.6026 6011	2.8620 4710	3.1469 1674	3.8032 0888
78	2.3931 2675	2.6351 9336	2.9014 0024	3.1941 2050	3.8697 6503
79	2.4200 4942	2.6681 3327	2.9412 9450	3.2420 3230	3.9374 8592
80	2.4472 7498	2.7014 8494	2.9817 3730	3.2906 6279	4.0063 9192
81	2.4748 0682	2.7352 5350	3.0227 3618	3.3400 2273	4.0765 0378
82	2.5026 4840	2.7694 4417	3.0642 9881	3.3901 2307	4.1478 4260
83	2.5308 0319	2.8040 6222	3.1064 3291	3.4409 7492	4.2204 2984
84	2.5592 7473	2.8391 1300	3.1491 4637	3.4925 8954	4.2942 8737
85	2.5880 6657	2.8746 0191	3.1924 4713	3.5449 7838	4.3694 3740
86	2.6171 8232	2.9105 3444	3.2363 4328	3.5981 5306	4.4459 0255
87	2.6466 2562	2.9469 1612	3.2808 4300	3.6521 2535	4.5237 0584
88	2.6764 0016	2.9837 5257	3.3259 5459	3.7069 0723	4.6028 7070
89	2.7065 0966	3.0210 4948	3.3716 8646	3.7625 1084	4.6834 2093
90	2.7369 5789	3.0588 1260	3.4180 4715	3.8189 4851	4.7653 8080
91	2.7677 4867	3.0970 4775	3.4650 4530	3.8762 3273	4.8487 7496
92	2.7988 8584	3.1357 6085	3.5126 8967	3.9343 7622	4.9336 2853
93	2.8303 7331	3.1749 5786	3.5609 8916	3.9933 9187	5.0199 6703
94	2.8622 1501	3.2146 4483	3.6099 5276	4.0532 9275	5.1078 1645
95	2.8944 1492	3.2548 2789	3.6595 8961	4.1140 9214	5.1972 0324
96	2.9269 7709	3.2955 1324	3.7099 0897	4.1758 0352	5.2881 5429
97	2.9599 0559	3.3367 0716	3.7609 2021	4.2384 4057	5.3806 9699
98	2.9932 0452	3.3784 1600	3.8126 3287	4.3020 1718	5.4748 5919
99	3.0268 7807	3.4206 4620	3.8650 5657	4.3665 4744	5.5706 6923
100	3.0609 3045	3.4634 0427	3.9182 0110	4.4320 4565	5.6681 5594

$(1 + i)^n$ (continued)

n	2%	2¼%	2½%	2¾%	3%
1	1.0200 0000	1.0225 0000	1.0250 0000	1.0275 0000	1.0300 0000
2	1.0404 0000	1.0455 0625	1.0506 2500	1.0557 5625	1.0609 0000
3	1.0612 0800	1.0690 3014	1.0768 9063	1.0847 8955	1.0927 2700
4	1.0824 3216	1.0930 8332	1.1038 1289	1.1146 2126	1.1255 0881
5	1.1040 8080	1.1176 7769	1.1314 0821	1.1452 7334	1.1592 7407
6	1.1261 6242	1.1428 2544	1.1596 9342	1.1767 6836	1.1940 5230
7	1.1486 8567	1.1685 3901	1.1886 8575	1.2091 2949	1.2298 7387
8	1.1716 5938	1.1948 3114	1.2184 0290	1.2423 8055	1.2667 7008
9	1.1950 9257	1.2217 1484	1.2488 6297	1.2765 4602	1.3047 7318
10	1.2189 9442	1.2492 0343	1.2800 8454	1.3116 5103	1.3439 1638
11	1.2433 7431	1.2773 1050	1.3120 8666	1.3477 2144	1.3842 3387
12	1.2682 4179	1.3060 4999	1.3448 8882	1.3847 8378	1.4257 6089
13	1.2936 0663	1.3354 3611	1.3785 1104	1.4228 6533	1.4685 3371
14	1.3194 7876	1.3654 8343	1.4129 7382	1.4619 9413	1.5125 8972
15	1.3458 6834	1.3962 0680	1.4482 9817	1.5021 9896	1.5579 6742
16	1.3727 8571	1.4276 2146	1.4845 0562	1.5435 0944	1.6047 0644
17	1.4002 4142	1.4597 4294	1.5216 1826	1.5859 5595	1.6528 4763
18	1.4282 4625	1.4925 8716	1.5596 5872	1.6295 6973	1.7024 3306
19	1.4568 1117	1.5261 7037	1.5986 5019	1.6743 8290	1.7535 0605
20	1.4859 4740	1.5605 0920	1.6386 1644	1.7204 2843	1.8061 1123
21	1.5156 6634	1.5956 2066	1.6795 8185	1.7677 4021	1.8602 9457
22	1.5459 7967	1.6315 2212	1.7215 7140	1.8163 5307	1.9161 0341
23	1.5768 9926	1.6682 3137	1.7646 1068	1.8663 0278	1.9735 8651
24	1.6084 3725	1.7057 6658	1.8087 2595	1.9176 2610	2.0327 9411
25	1.6406 0599	1.7441 4632	1.8539 4410	1.9703 6082	2.0937 7793
26	1.6734 1811	1.7833 8962	1.9002 9270	2.0245 4575	2.1565 9127
27	1.7068 8648	1.8235 1588	1.9478 0002	2.0802 2075	2.2212 8901
28	1.7410 2421	1.8645 4499	1.9964 9502	2.1374 2682	2.2879 2768
29	1.7758 4469	1.9064 9725	2.0464 0739	2.1962 0606	2.3565 6551
30	1.8113 6158	1.9493 9344	2.0975 6758	2.2566 0173	2.4272 6247
31	1.8475 8882	1.9932 5479	2.1500 0677	2.3186 5828	2.5000 8035
32	1.8845 4059	2.0381 0303	2.2037 5694	2.3824 2138	2.5750 8276
33	1.9222 3140	2.0839 6034	2.2588 5086	2.4479 3797	2.6523 3524
34	1.9606 7603	2.1308 4945	2.3153 2213	2.5152 5626	2.7319 0530
35	1.9998 8955	2.1787 9356	2.3732 0519	2.5844 2581	2.8138 6245
36	2.0398 8734	2.2278 1642	2.4325 3532	2.6554 9752	2.8982 7833
37	2.0806 8509	2.2779 4229	2.4933 4870	2.7285 2370	2.9852 2668
38	2.1222 9879	2.3291 9599	2.5556 8242	2.8035 5810	3.0747 8348
39	2.1647 4477	2.3816 0290	2.6195 7448	2.8806 5595	3.1670 2698
40	2.2080 3966	2.4351 8897	2.6850 6384	2.9598 7399	3.2620 3779
41	2.2522 0046	2.4899 8072	2.7521 9043	3.0412 7052	3.3598 9893
42	2.2972 4447	2.5460 0528	2.8209 9520	3.1249 0546	3.4606 9589
43	2.3431 8936	2.6032 9040	2.8915 2008	3.2108 4036	3.5645 1677
44	2.3900 5314	2.6618 6444	2.9638 0808	3.2991 3847	3.6714 5227
45	2.4378 5421	2.7217 5639	3.0379 0328	3.3898 6478	3.7815 9584
46	2.4866 1129	2.7829 9590	3.1138 5086	3.4830 8606	3.8950 4372
47	2.5363 4352	2.8456 1331	3.1916 9713	3.5788 7093	4.0118 9503
48	2.5870 7039	2.9096 3961	3.2714 8956	3.6772 8988	4.1322 5188
49	2.6388 1179	2.9751 0650	3.3532 7680	3.7784 1535	4.2562 1944
50	2.6915 8803	3.0420 4640	3.4371 0872	3.8823 2177	4.3839 0602

n	2%	2¼%	2½%	2¾%	3%
51	2.7454 1979	3.1104 9244	3.5230 3644	3.9890 8562	4.5154 2320
52	2.8003 2819	3.1804 7852	3.6111 1235	4.0987 8547	4 6508 8590
53	2.8563 3475	3.2520 3929	3.7013 9016	4.2115 0208	4.7904 1247
54	2.9134 6144	3.3252 1017	3.7939 2491	4.3273 1838	4.9341 2485
55	2.9717 3067	3.4000 2740	3.8887 7303	4.4463 1964	5.0821 4859
56	3.0311 6529	3.4765 2802	3.9859 9236	4.5685 9343	5.2346 1305
57	3.0917 8859	3.5547 4990	4.0856 4217	4.6942 2975	5.3916 5144
58	3.1536 2436	3.6347 3177	4.1877 8322	4.8233 2107	5.5534 0098
59	3.2166 9685	3.7165 1324	4.2924 7780	4.9559 6239	5.7200 0301
60	3.2810 3079	3.8001 3479	4.3997 8975	5.0922 5136	5.8916 0310
61	3.3466 5140	3.8856 3782	4.5097 8449	5.2322 8827	6.0683 5120
62	3.4135 8443	3.9730 6467	4.6225 2910	5.3761 7620	6.2504 0173
63	3.4818 5612	4.0624 5862	4.7380 9233	5.5240 2105	6.4379 1379
64	3.5514 9324	4.1538 6394	4.8565 4464	5.6759 3162	6.6310 5120
65	3.6225 2311	4.2473 2588	4.9779 5826	5.8320 1974	6.8299 8273
66	3.6949 7357	4.3428 9071	5.1024 0721	5.9924 0029	7.0348 8222
67	3.7688 7304	4.4406 0576	5.2299 6739	6.1571 9130	7.2459 2868
68	3.8442 5050	4.5405 1939	5.3607 1658	6.3265 1406	7.4633 0654
69	3.9211 3551	4.6426 8107	5.4947 3449	6.5004 9319	7.6872 0574
70	3.9995 5822	4.7471 4140	5.6321 0286	6.6792 5676	7.9178 2191
71	4.0795 4939	4.8539 5208	5.7729 0543	6.8629 3632	8.1553 5657
72	4.1611 4038	4.9631 6600	5.9172 2806	7.0516 6706	8.4000 1727
73	4.2443 6318	5.0748 3723	6.0651 5876	7.2455 8791	8.6520 1778
74	4.3292 5045	5.1890 2107	6.2167 8773	7.4448 4158	8.9115 7832
75	4.4158 3546	5.3057 7405	6.3722 0743	7.6495 7472	9.1789 2567
76	4.5041 5216	5.4251 5396	6.5315 1261	7.8599 3802	9.4542 9344
77	4.5942 3521	5.5472 1993	6.6948 0043	8.0760 8632	9.7379 2224
78	4.6861 1991	5.6720 3237	6.8621 7044	8.2981 7869	10.0300 5991
79	4.7798 4231	5.7996 5310	7.0337 2470	8.5263 7861	10.3309 6171
80	4.8754 3916	5.9301 4530	7.2095 6782	8.7608 5402	10.6408 9056
81	4.9729 4794	6.0635 7357	7.3898 0701	9.0017 7751	10.9601 1727
82	5.0724 0690	6.2000 0397	7.5745 5219	9.2493 2639	11.2889 2079
83	5.1738 5504	6.3395 0406	7.7639 1599	9.5036 8286	11.6275 8842
84	5.2773 3214	6.4821 4290	7.9580 1389	9.7650 3414	11.9764 1607
85	5.3828 7878	6.6279 9112	8.1569 6424	10.0335 7258	12.3357 0855
86	5.4905 3636	6.7771 2092	8.3608 8834	10.3094 9583	12.7057 7981
87	5.6003 4708	6.9296 0614	8.5699 1055	10.5930 0696	13.0869 5320
88	5.7123 5402	7.0855 2228	8.7841 5832	10.8843 1465	13.4795 6180
89	5.8266 0110	7.2449 4653	9.0037 6228	11.1836 3331	13.8839 4865
90	5.9431 3313	7.4079 5782	9.2288 5633	11.4911 8322	14.3004 6711
91	6.0619 9579	7.5746 3688	9.4595 7774	11.8071 9076	14.7294 8112
92	6.1832 3570	7.7450 6621	9.6960 6718	12.1318 8851	15.1713 6556
93	6.3069 0042	7.9193 3020	9.9384 6886	12.4655 1544	15.6265 0652
94	6.4330 3843	8.0975 1512	10.1869 3058	12.8083 1711	16.0953 0172
95	6.5616 9920	8.2797 0921	10.4416 0385	13.1605 4584	16.5781 6077
96	6.6929 3318	8.4660 0267	10.7026 4395	13.5224 6085	17.0755 0559
97	6.8267 9184	8.6564 8773	10.9702 1004	13.8943 2852	17.5877 7076
98	6.9633 2768	8.8512 5871	11.2444 6530	14.2764 2255	18.1154 0388
99	7.1025 9423	9.0504 1203	11.5255 7693	14.6690 2417	18.6588 6600
100	7.2446 4612	9.2540 4630	11.8137 1635	15.0724 2234	19.2186 3198

$(1 + i)^n$ (continued)

n	3½%	4%	4½%	5%	5½%
1	1.0350 0000	1.0400 0000	1.0450 0000	1.0500 0000	1.0550 0000
2	1.0712 2500	1.0816 0000	1.0920 2500	1.1025 0000	1.1130 2500
3	1.1087 1788	1.1248 6400	1.1411 6613	1.1576 2500	1.1742 4138
4	1.1475 2300	1.1698 5856	1.1925 1860	1.2155 0625	1.2388 2465
5	1.1876 8631	1.2166 5290	1.2461 8194	1.2762 8156	1.3069 6001
6	1.2292 5533	1.2653 1902	1.3022 6012	1.3400 9564	1.3788 4281
7	1.2722 7926	1.3159 3178	1.3608 6183	1.4071 0042	1.4546 7916
8	1.3168 0904	1.3685 6905	1.4221 0061	1.4774 5544	1.5346 8651
9	1.3628 9735	1.4233 1181	1.4860 9514	1.5513 2822	1.6190 9427
10	1.4105 9876	1.4802 4428	1.5529 6942	1.6288 9463	1.7081 4446
11	1.4599 6972	1.5394 5406	1.6228 5305	1.7103 3936	1.8020 9240
12	1.5110 6866	1.6010 3222	1.6958 8143	1.7958 5633	1.9012 0749
13	1.5639 5606	1.6650 7351	1.7721 9610	1.8856 4914	2.0057 7390
14	1.6186 9452	1.7316 7645	1.8519 4492	1.9799 3160	2.1160 9146
15	1.6753 4883	1.8009 4351	1.9352 8244	2.0789 2818	2.2324 7649
16	1.7339 8604	1.8729 8125	2.0223 7015	2.1828 7459	2.3552 6270
17	1.7946 7555	1.9479 0050	2.1133 7681	2.2920 1832	2.4848 0215
18	1.8574 8920	2.0258 1652	2.2084 7877	2.4066 1923	2.6214 6627
19	1.9225 0132	2.1068 4918	2.3078 6031	2.5269 5020	2.7656 4691
20	1.9897 8886	2.1911 2314	2.4117 1402	2.6532 9771	2.9177 5749
21	2.0594 3147	2.2787 6807	2.5202 4116	2.7859 6259	3.0782 3415
22	2.1315 1158	2.3699 1879	2.6336 5201	2.9252 6072	3.2475 3703
23	2.2061 1448	2.4647 1554	2.7521 6635	3.0715 2376	3.4261 5157
24	2.2833 2849	2.5633 0416	2 8760 1383	3.2250 9994	3.6145 8990
25	2.3632 4498	2.6658 3633	3.0054 3446	3.3863 5494	3.8133 9235
26	2.4459 5856	2.7724 6978	3.1406 7901	3.5556 7269	4.0231 2893
27	2.5315 6711	2.8833 6858	3.2820 0956	3.7334 5632	4.2444 0102
28	2.6201 7196	2.9987 0332	3.4296 9999	3.9201 2914	4.4778 4307
29	2.7118 7798	3.1186 5145	3.5840 3649	4.1161 3560	4.7241 2444
30	2.8067 9370	3.2433 9751	3.7453 1813	4.3219 4238	4.9839 5129
31	2.9050 3148	3.3731 3341	3.9138 5745	4.5380 3949	5.2580 6861
32	3.0067 0759	3.5080 5875	4.0899 8104	4.7649 4147	5.5472 6238
33	3.1119 4235	3.6483 8110	4.2740 3018	5.0031 8854	5.8523 6181
34	3.2208 6033	3.7943 1634	4.4663 6154	5.2533 4797	6.1742 4171
35	3.3335 9045	3.9460 8899	4.6673 4781	5.5160 1537	6.5138 2501
36	3.4502 6611	4.1039 3255	4.8773 7846	5.7918 1614	6.8720 8538
37	3.5710 2543	4.2680 8986	5.0968 6049	6.0814 0694	7.2500 5008
38	3.6960 1132	4.4388 1345	5.3262 1921	6.3854 7729	7.6488 0283
39	3.8253 7171	4.6163 6599	5.5658 9908	6.7047 5115	8.0694 8699
40	3.9592 5972	4.8010 2063	5.8163 6454	7.0399 8871	8.5133 0877
41	4.0978 3381	4.9930 6145	6.0781 0094	7.3919 8815	8.9815 4076
42	4.2412 5799	5.1927 8391	6.3516 1548	7.7615 8756	9.4755 2550
43	4.3897 0202	5.4004 9527	6.6374 3818	8.1496 6693	9.9966 7940
44	4.5433 4160	5.6165 1508	6.9361 2290	8.5571 5028	10.5464 9677
45	4.7023 5855	5.8411 7568	7.2482 4843	8.9850 0779	11.1265 5409
46	4.8669 4110	6.0748 2271	7.5744 1961	9.4342 5818	11.7385 1456
47	5.0372 8404	6.3178 1562	7.9152 6849	9.9059 7109	12.3841 3287
48	5.2135 8898	6.5705 2824	8.2714 5557	10.4012 6965	13.0652 6017
49	5.3960 6459	6.8333 4937	8.6436 7107	10.9213 3313	13.7838 4948
50	5.5849 2686	7.1066 8335	9.0326 3627	11.4673 9979	14.5419 6120

$(1 + i)^n$ (continued)

n	3½%	4%	4½%	5%	5½%
51	5.7803 9930	7.3909 5068	9.4391 0490	12.0407 6978	15.3417 6907
52	5.9827 1327	7.6865 8871	9.8638 6463	12.6428 0826	16.1855 6637
53	6.1921 0824	7.9940 5226	10.3077 3853	13.2749 4868	17.0757 7252
54	6.4088 3202	8.3138 1435	10.7715 8677	13.9386 9611	18.0149 4001
55	6.6331 4114	8.6463 6692	11.2563 0817	14.6356 3092	19.0057 6171
56	6.8653 0108	8.9922 2160	11.7628 4204	15.3674 1246	20.0510 7860
57	7.1055 8662	9.3519 1046	12.2921 6993	16.1357 8309	21.1538 8793
58	7.3542 8215	9.7259 8688	12.8453 1758	16.9425 7224	22.3173 5176
59	7.6116 8203	10.1150 2635	13.4233 5687	17.7897 0085	23.5448 0611
60	7.8780 9090	10.5196 2741	14.0274 0793	18.6791 8589	24.8397 7045
61	8.1538 2408	10.9404 1250	14.6586 4129	19.6131 4519	26.2059 5782
62	8.4392 0793	11.3780 2900	15.3182 8014	20.5938 0245	27.6472 8550
63	8.7345 8020	11.8331 5016	16.0076 0275	21.6234 9257	29.1678 8620
64	9.0402 9051	12.3064 7617	16.7279 4487	22.7046 6720	30.7721 1994
65	9.3567 0068	12.7987 3522	17.4807 0239	23.8399 0056	32.4645 8654
66	9.6841 8520	13.3106 8463	18.2673 3400	25.0318 9559	34.2501 3880
67	10.0231 3168	13.8431 1201	19.0893 6403	26.2834 9037	36.1338 9643
68	10.3739 4129	14.3968 3649	19.9483 8541	27.5976 6488	38.1212 6074
69	10.7370 2924	14.9727 0995	20.8460 6276	28.9775 4813	40.2179 3008
70	11.1128 2526	15.5716 1835	21.7841 3558	30.4264 2554	42.4299 1623
71	11.5017 7414	16.1944 8308	22.7644 2168	31.9477 4681	44.7635 6163
72	11.9043 3624	16.8422 6241	23.7888 2066	33.5451 3415	47.2255 5751
73	12.3209 8801	17.5159 5290	24.8593 1759	35.2223 9086	49.8229 6318
74	12.7522 2259	18.2165 9102	25.9779 8688	36.9835 1040	52.5632 2615
75	13.1985 5038	18.9452 5466	27.1469 9629	38.8326 8592	55.4542 0359
76	13.6604 9964	19.7030 6485	28.3686 1112	40.7743 2022	58.5041 8479
77	14.1386 1713	20.4911 8744	29.6451 9862	42.8130 3623	61.7219 1495
78	14.6334 6873	21.3108 3494	30.9792 3256	44.9536 8804	65.1166 2027
79	15.1456 4013	22.1632 6834	32.3732 9802	47.2013 7244	68.6980 3439
80	15.6757 3754	23.0497 9907	33.8300 9643	49.5614 4107	72.4764 2628
81	16.2243 8835	23.9717 9103	35.3524 5077	52.0395 3132	76.4626 2973
82	16.7922 4195	24.9306 6267	36.9433 1106	54.6414 8878	80.6680 7436
83	17.3799 7041	25.9278 8918	38.6057 6006	57.3735 6322	85.1048 1845
84	17.9882 6938	26.9650 0475	40.3430 1926	60.2422 4138	89.7855 8347
85	18.6178 5881	28.0436 0494	42.1584 5513	63.2543 5344	94.7237 9056
86	19.2694 8387	29.1653 4914	44.0555 8561	66.4170 7112	99.9335 9904
87	19.9439 1580	30.3319 6310	46.0380 8696	69.7379 2467	105.4299 4698
88	20.6419 5285	31.5452 4163	48.1098 0087	73.2248 2091	111.2285 9407
89	21.3644 2120	32.8070 5129	50.2747 4191	76.8860 6195	117.3461 6674
90	22.1121 7595	34.1193 3334	52.5371 0530	80.7303 6505	123.8002 0591
91	22.8861 0210	35.4841 0668	54.9012 7503	84.7668 8330	130.6092 1724
92	23.6871 1568	36.9034 7094	57.3718 3241	89.0052 2747	137.7927 2419
93	24.5161 6473	38.3796 0978	59.9535 6487	93.4554 8884	145.3713 2402
94	25.3742 3049	39.9147 9417	62.6514 7529	98.1282 6328	153.3667 4684
95	26.2623 2856	41.5113 8594	65.4707 9168	103.0346 7645	161.8019 1791
96	27.1815 1006	43.1718 4138	68.4169 7730	108.1864 1027	170.7010 2340
97	28.1328 6291	44.8987 1503	71.4957 4128	113.5957 3078	180.0895 7969
98	29.1175 1311	46.6946 6363	74.7130 4964	119.2755 1732	189.9945 0657
99	30.1366 2607	48.5624 5018	78.0751 3687	125.2392 9319	200.4442 0443
100	31.1914 0798	50.5049 4818	81.5885 1803	131.5012 5785	211.4686 3567

$(1 + i)^n$ (continued)

n	6%	6½%	7%	7½%	8%
1	1.0600 0000	1.0650 0000	1.0700 0000	1.0750 0000	1.0800 0000
2	1.1236 0000	1.1342 2500	1.1449 0000	1.1556 2500	1.1664 0000
3	1.1910 1600	1.2079 4963	1.2250 4300	1.2422 9688	1.2597 1200
4	1.2624 7696	1.2864 6635	1.3107 9601	1.3354 6914	1.3604 8896
5	1.3382 2558	1.3700 8666	1.4025 5173	1.4356 2933	1.4693 2808
6	1.4185 1911	1.4591 4230	1.5007 3035	1.5433 0153	1.5868 7432
7	1.5036 3026	1.5539 8655	1.6057 8148	1.6590 4914	1.7138 2427
8	1.5938 4807	1.6549 9567	1.7181 8618	1.7834 7783	1.8509 3021
9	1.6894 7896	1.7625 7039	1.8384 5921	1.9172 3866	1.9990 0463
10	1.7908 4770	1.8771 3747	1.9671 5136	2.0610 3156	2.1589 2500
11	1.8982 9856	1.9991 5140	2.1048 5195	2.2156 0893	2.3316 3900
12	2.0121 9647	2.1290 9624	2.2521 9159	2.3817 7960	2.5181 7012
13	2.1329 2826	2.2674 8750	2.4098 4500	2.5604 1307	2.7196 2373
14	2.2609 0396	2.4148 7418	2.5785 3415	2.7524 4405	2.9371 9362
15	2.3965 5819	2.5718 4101	2.7590 3154	2.9588 7735	3.1721 6911
16	2.5403 5168	2.7390 1067	2.9521 6375	3.1807 9315	3.4259 4264
17	2.6927 7279	2.9170 4637	3.1588 1521	3.4193 5264	3.7000 1805
18	2.8543 3915	3.1066 5438	3.3799 3228	3.6758 0409	3.9960 1950
19	3.0255 9950	3.3085 8691	3.6165 2754	3.9514 8940	4.3157 0106
20	3.2071 3547	3.5236 4506	3.8696 8446	4.2478 5110	4.6609 5714
21	3.3995 6360	3.7526 8199	4.1405 6237	4.5664 3993	5.0338 3372
22	3.6035 3742	3.9966 0632	4.4304 0174	4.9089 2293	5.4365 4041
23	3.8197 4966	4.2563 8573	4.7405 2986	5.2770 9215	5.8714 6365
24	4.0489 3464	4.5330 5081	5.0723 6695	5.6728 7406	6.3411 8074
25	4.2918 7072	4.8276 9911	5.4274 3264	6.0983 3961	6.8484 7520
26	4.5493 8296	5.1414 9955	5.8073 5292	6.5557 1508	7.3963 5321
27	4.8223 4594	5.4756 9702	6.2138 6763	7.0473 9371	7.9880 6147
28	5.1116 8670	5.8316 1733	6.6488 3836	7.5759 4824	8.6271 0639
29	5.4183 8790	6.2106 7245	7.1142 5705	8.1441 4436	9.3172 7490
30	5.7434 9117	6.6143 6616	7.6122 5504	8.7549 5519	10.0626 5689
31	6.0881 0064	7.0442 9996	8.1451 1290	9.4115 7683	10.8676 6944
32	6.4533 8668	7.5021 7946	8.7152 7080	10.1174 4509	11.7370 8300
33	6.8405 8988	7.9898 2113	9.3253 3975	10.8762 5347	12.6760 4964
34	7.2510 2528	8.5091 5950	9.9781 1354	11.6919 7248	13.6901 3361
35	7.6860 8679	9.0622 5487	10.6765 8148	12.5688 7042	14.7853 4429
36	8.1472 5200	9.6513 0143	11.4239 4219	13.5115 3570	15.9681 7184
37	8.6360 8712	10.2786 3603	12.2236 1814	14.5249 0088	17.2456 2558
38	9.1542 5235	10.9467 4737	13.0792 7141	15.6142 6844	18.6252 7563
39	9.7035 0749	11.6582 8595	13.9948 2041	16.7853 3858	20.1152 9768
40	10.2857 1794	12.4160 7453	14.9744 5784	18.0442 3897	21.7245 2150
41	10.9028 6101	13.2231 1938	16.0226 6989	19.3975 5689	23.4624 8322
42	11.5570 3267	14.0826 2214	17.1442 5678	20.8523 7366	25.3394 8187
43	12.2504 5463	14.9979 9258	18.3443 5475	22.4163 0168	27.3666 4042
44	12.9854 8191	15.9728 6209	19.6284 5959	24.0975 2431	29.5559 7166
45	13.7646 1083	17.0110 9813	21.0024 5176	25.9048 3863	31.9204 4939
46	14.5904 8748	18.1168 1951	22.4726 2338	27.8477 0153	34.4740 8534
47	15.4659 1673	19.2944 1278	24.0457 0702	29.9362 7915	37.2320 1217
48	16.3938 7173	20.5485 4961	25.7289 0651	32.1815 0008	40.2105 7314
49	17.3775 0403	21.8842 0533	27.5299 2997	34.5951 1259	43.4274 1899
50	18.4201 5427	23.3066 7868	29.4570 2506	37.1897 4603	46.9016 1251

$(1+i)^n$ (continued)

n	8½%	9%	9½%	10%	10½%
1	1.0850 0000	1.0900 0000	1.0950 0000	1.1000 0000	1.1050 0000
2	1.1772 2500	1.1881 0000	1.1990 2500	1.2100 0000	1.2210 2500
3	1.2772 8913	1.2950 2900	1.3129 3238	1.3310 0000	1.3492 3263
4	1.3858 5870	1.4115 8161	1.4376 6095	1.4641 0000	1.4909 0205
5	1.5036 5669	1.5386 2395	1.5742 3874	1.6105 1000	1.6474 4677
6	1.6314 6751	1.6771 0011	1.7237 9142	1.7715 6100	1.8204 2868
7	1.7701 4225	1.8280 3912	1.8875 5161	1.9487 1710	2.0115 7369
8	1.9206 0434	1.9925 6264	2.0668 6901	2.1435 8881	2.2227 8892
9	2.0838 5571	2.1718 9328	2.2632 2156	2.3579 4769	2.4561 8176
10	2.2609 8344	2.3673 6367	2.4782 2761	2.5937 4246	2.7140 8085
11	2.4531 6703	2.5804 2641	2.7136 5924	2.8531 1671	2.9990 5934
12	2.6616 8623	2.8126 6478	2.9714 5686	3.1384 2838	3.3139 6057
13	2.8879 2956	3.0658 0461	3.2537 4527	3.4522 7121	3.6619 2643
14	3.1334 0357	3.3417 2703	3.5628 5107	3.7974 9834	4.0464 2870
15	3.3997 4288	3.6424 8246	3.9013 2192	4.1772 4817	4.4713 0371
16	3.6887 2102	3.9703 0588	4.2719 4750	4.5949 7299	4.9407 9060
17	4.0022 6231	4.3276 3341	4.6777 8251	5.0544 7028	5.4595 7362
18	4.3424 5461	4.7171 2042	5.1221 7185	5.5599 1731	6.0328 2885
19	4.7115 6325	5.1416 6125	5.6087 7818	6.1159 0904	6.6662 7588
20	5.1120 4612	5.6044 1077	6.1416 1210	6.7274 9995	7.3662 3484
21	5.5465 7005	6.1088 0774	6.7250 6525	7.4002 4994	8.1396 8950
22	6.0180 2850	6.6586 0043	7.3639 4645	8.1402 7494	8.9943 5690
23	6.5295 6092	7.2578 7447	8.0635 2137	8.9543 0243	9.9387 6437
24	7.0845 7360	7.9110 8317	8.8295 5590	9.8497 3268	10.9823 3463
25	7.6867 6236	8.6230 8066	9.6683 6371	10.8347 0594	12.1354 7977
26	8.3401 3716	9.3991 5792	10.5868 5826	11.9181 7654	13.4097 0514
27	9.0490 4881	10.2450 8213	11.5926 0979	13.1099 9419	14.8177 2418
28	9.8182 1796	11.1671 3952	12.6939 0772	14.4209 9361	16.3735 8522
29	10.6527 6649	12.1721 8208	13.8998 2896	15.8630 9297	18.0928 1167
30	11.5582 5164	13.2676 7847	15.2203 1271	17.4494 0227	19.9925 5690
31	12.5407 0303	14.4617 6953	16.6662 4241	19.1943 4250	22.0917 7537
32	13.6066 6279	15.7633 2879	18.2495 3544	21.1137 7675	24.4114 1178
33	14.7632 2913	17.1820 2838	19.9832 4131	23.2251 5442	26.9746 1002
34	16.0181 0360	18.7284 1093	21.8816 4924	25.5476 6986	29.8069 4407
35	17.3796 4241	20.4139 6792	23.9604 0591	28.1024 3685	32.9366 7320
36	18.8569 1201	22.2512 2503	26.2366 4448	30.9126 8053	36.3950 2389
37	20.4597 4953	24.2538 3528	28.7291 2570	34.0039 4859	40.2165 0140
38	22.1988 2824	26.4366 8046	31.4583 9264	37.4043 4344	44.4392 3404
39	24.0857 2865	28.8159 8170	34.4469 3994	41.1447 7779	49.1053 5362
40	26.1330 1558	31.4094 2005	37.7193 9924	45.2592 5557	54.2614 1575
41	28.3543 2190	34.2362 6786	41.3027 4216	49.7851 8113	59.9588 6440
42	30.7644 3927	37.3175 3197	45.2265 0267	54.7636 9924	66.2545 4516
43	33.3794 1660	40.6761 0984	49.5230 2042	60.2400 6916	73.2112 7240
44	36.2166 6702	44.3369 5973	54.2277 0736	66.2640 7608	80.8984 5601
45	39.2950 8371	48.3272 8610	59.3793 3956	72.8904 8369	89.3927 9389
46	42.6351 6583	52.6767 4185	65.0203 7682	80.1795 3205	98.7790 3724
47	46.2591 5492	57.4176 4862	71.1973 1262	88.1974 8526	109.1508 3616
48	50.1911 8309	62.5852 3700	77.9610 5732	97.0172 3378	120.6116 7395
49	54.4574 3365	68.2179 0833	85.3673 5777	106.7189 5716	133.2758 9972
50	59.0863 1551	74.3575 2008	93.4772 5675	117.3908 5288	147.2693 6919

* **Source : Taken from, MATHEMATICS OF FINANCE,**

Third Ed. : Hummel and Seeback, 1971, pp. 260 - 275.

Appendix C

Compound Interest Table

Values of $(1 + i)^{-n}$

n	$\frac{1}{4}$%	$\frac{7}{24}$%	$\frac{1}{3}$%	$\frac{5}{12}$%	$\frac{1}{2}$%
1	0.9975 0623	0.9970 9182	0.9966 7774	0.9958 5062	0.9950 2488
2	0.9950 1869	0.9941 9209	0.9933 6652	0.9917 1846	0.9900 7450
3	0.9925 3734	0.9913 0079	0.9900 6630	0.9876 0345	0.9851 4876
4	0.9900 6219	0.9884 1791	0.9867 7704	0.9835 0551	0.9802 4752
5	0.9875 9321	0.9855 4341	0.9834 9871	0.9794 2457	0.9753 7067
6	0.9851 3038	0.9826 7727	0.9802 3127	0.9753 6057	0.9705 1808
7	0.9826 7370	0.9798 1946	0.9769 7469	0.9713 1343	0.9656 8963
8	0.9802 2314	0.9769 6996	0.9737 2893	0.9672 8308	0.9608 8520
9	0.9777 7869	0.9741 2875	0.9704 9395	0.9632 6946	0.9561 0468
10	0.9753 4034	0.9712 9581	0.9672 6972	0.9592 7249	0.9513 4794
11	0.9729 0807	0.9684 7110	0.9640 5620	0.9552 9211	0.9466 1487
12	0.9704 8187	0.9656 5461	0.9608 5335	0.9513 2824	0.9419 0534
13	0.9680 6171	0.9628 4631	0.9576 6115	0.9473 8082	0.9372 1924
14	0.9656 4759	0.9600 4617	0.9544 7955	0.9434 4978	0.9325 5646
15	0.9632 3949	0.9572 5418	0.9513 0852	0.9395 3505	0.9279 1688
16	0.9608 3740	0.9544 7031	0.9481 4803	0.9356 3657	0.9233 0037
17	0.9584 4130	0.9516 9453	0.9449 9803	0.9317 5426	0.9187 0684
18	0.9560 5117	0.9489 2683	0.9418 5851	0.9278 8806	0.9141 3616
19	0.9536 6700	0.9461 6718	0.9387 2941	0.9240 3790	0.9095 8822
20	0.9512 8878	0.9434 1555	0.9356 1071	0.9202 0372	0.9050 6290
21	0.9489 1649	0.9406 7192	0.9325 0236	0.9163 8544	0.9005 6010
22	0.9465 5011	0.9379 3627	0.9294 0435	0.9125 8301	0.8960 7971
23	0.9441 8964	0.9352 0858	0.9263 1663	0.9087 9636	0.8916 2160
24	0.9418 3505	0.9324 8882	0.9232 3916	0.9050 2542	0.8871 8567
25	0.9394 8634	0.9297 7697	0.9201 7192	0.9012 7013	0.8827 7181
26	0.9371 4348	0.9270 7301	0.9171 1487	0.8975 3042	0.8783 7991
27	0.9348 0646	0.9243 7691	0.9140 6798	0.8938 0623	0.8740 0986
28	0.9324 7527	0.9216 8865	0.9110 3121	0.8900 9749	0.8696 6155
29	0.9301 4990	0.9190 0821	0.9080 0453	0.8864 0414	0.8653 3488
30	0.9278 3032	0.9163 3557	0.9049 8790	0.8827 2611	0.8610 2973
31	0.9255 1653	0.9136 7069	0.9019 8130	0.8790 6335	0.8567 4600
32	0.9232 0851	0.9110 1357	0.8989 8468	0.8754 1578	0.8524 8358
33	0.9209 0624	0.9083 6417	0.8959 9802	0.8717 8335	0.8482 4237
34	0.9186 0972	0.9057 2248	0.8930 2128	0.8681 6599	0.8440 2226
35	0.9163 1892	0.9030 8848	0.8900 5444	0.8645 6365	0.8398 2314
36	0.9140 3384	0.9004 6213	0.8870 9745	0.8609 7624	0.8356 4492
37	0.9117 5445	0.8978 4342	0.8841 5028	0.8574 0373	0.8314 8748
38	0.9094 8075	0.8952 3232	0.8812 1290	0.8538 4604	0.8273 5073
39	0.9072 1272	0.8926 2882	0.8782 8528	0.8503 0311	0.8232 3455
40	0.9049 5034	0.8900 3289	0.8753 6739	0.8467 7488	0.8191 3886
41	0.9026 9361	0.8874 4451	0.8724 5920	0.8432 6129	0.8150 6354
42	0.9004 4250	0.8848 6366	0.8695 6066	0.8397 6228	0.8110 0850
43	0.8981 9701	0.8822 9031	0.8666 7175	0.8362 7779	0.8069 7363
44	0.8959 5712	0.8797 2445	0.8637 9245	0.8328 0776	0.8029 5884
45	0.8937 2281	0.8771 6605	0.8609 2270	0.8293 5212	0.7989 6402
46	0.8914 9407	0.8746 1509	0.8580 6249	0.8259 1083	0.7949 8907
47	0.8892 7090	0.8720 7155	0.8552 1179	0.8224 8381	0.7910 3390
48	0.8870 5326	0.8695 3540	0.8523 7055	0.8190 7102	0.7870 9841
49	0.8848 4116	0.8670 0663	0.8495 3876	0.8156 7238	0.7831 8250
50	0.8826 3457	0.8644 8522	0.8467 1637	0.8122 8785	0.7792 8607

$v^n = (1 + i)^{-n}$ (*Continued*)

n	¼ %	$\frac{7}{24}$ %	⅓ %	$\frac{5}{12}$ %	½ %
51	0.8804 3349	0.8619 7114	0.8439 0336	0.8089 1736	0.7754 0902
52	0.8782 3790	0.8594 6436	0.8410 9969	0.8055 6086	0.7715 5127
53	0.8760 4778	0.8569 6488	0.8383 0534	0.8022 1828	0.7677 1270
54	0.8738 6312	0.8544 7267	0.8355 2027	0.7988 8957	0.7638 9324
55	0.8716 8391	0.8519 8771	0.8327 4446	0.7955 7468	0.7600 9277
56	0.8695 1013	0.8495 0997	0.8299 7787	0.7922 7354	0.7563 1122
57	0.8673 4178	0.8470 3944	0.8272 2047	0.7889 8610	0.7525 4847
58	0.8651 7883	0.8445 7609	0.8244 7222	0.7857 1230	0.7488 0445
59	0.8630 2128	0.8421 1991	0.8217 3311	0.7824 5208	0.7450 7906
60	0.8608 6911	0.8396 7087	0.8190 0310	0.7792 0539	0.7413 7220
61	0.8587 2230	0.8372 2895	0.8162 8216	0.7759 7217	0.7376 8378
62	0.8565 8085	0.8347 9413	0.8135 7026	0.7727 5237	0.7340 1371
63	0.8544 4474	0.8323 6640	0.8108 6737	0.7695 4593	0.7303 6190
64	0.8523 1395	0.8299 4572	0.8081 7346	0.7663 5279	0.7267 2826
65	0.8501 8848	0.8275 3209	0.8054 8850	0.7631 7291	0.7231 1269
66	0.8480 6831	0.8251 2547	0.8028 1246	0.7600 0621	0.7195 1512
67	0.8459 5343	0.8227 2586	0.8001 4531	0.7568 5266	0.7159 3544
68	0.8438 4382	0.8203 3322	0.7974 8702	0.7537 1219	0.7123 7357
69	0.8417 3947	0.8179 4754	0.7948 3756	0.7505 8476	0.7088 2943
70	0.8396 4037	0.8155 6879	0.7921 9690	0.7474 7030	0.7053 0291
71	0.8375 4650	0.8131 9697	0.7895 6502	0.7443 6876	0.7017 9394
72	0.8354 5786	0.8108 3204	0.7869 4188	0.7412 8009	0.6983 0243
73	0.8333 7442	0.8084 7399	0.7843 2745	0.7382 0424	0.6948 2829
74	0.8312 9618	0.8061 2280	0.7817 2171	0.7351 4115	0.6913 7143
75	0.8292 2312	0.8037 7845	0.7791 2463	0.7320 9078	0.6879 3177
76	0.8271 5523	0.8014 4091	0.7765 3618	0.7290 5306	0.6845 0923
77	0.8250 9250	0.7991 1018	0.7739 5632	0.7260 2794	0.6811 0371
78	0.8230 3491	0.7967 8622	0.7713 8504	0.7230 1537	0.6777 1513
79	0.8209 8246	0.7944 6901	0.7688 2230	0.7200 1531	0.6743 4342
80	0.8189 3512	0.7921 5855	0.7662 6807	0.7170 2770	0.6709 8847
81	0.8168 9289	0.7898 5481	0.7637 2233	0.7140 5248	0.6676 5022
82	0.8148 5575	0.7875 5776	0.7611 8505	0.7110 8960	0.6643 2858
83	0.8128 2369	0.7852 6740	0.7586 5619	0.7081 3902	0.6610 2346
84	0.8107 9670	0.7829 8370	0.7561 3574	0.7052 0069	0.6577 3479
85	0.8087 7476	0.7807 0664	0.7536 2366	0.7022 7454	0.6544 6248
86	0.8067 5787	0.7784 3620	0.7511 1993	0.6993 6054	0.6512 0644
87	0.8047 4600	0.7761 7236	0.7486 2451	0.6964 5863	0.6479 6661
88	0.8027 3915	0.7739 1511	0.7461 3739	0.6935 6876	0.6447 4290
89	0.8007 3731	0.7716 6442	0.7436 5853	0.6906 9088	0.6415 3522
90	0.7987 4046	0.7694 2028	0.7411 8790	0.6878 2495	0.6383 4350
91	0.7967 4859	0.7671 8266	0.7387 2548	0.6849 7090	0.6351 6766
92	0.7947 6168	0.7649 5156	0.7362 7125	0.6821 2870	0.6320 0763
93	0.7927 7973	0.7627 2694	0.7338 2516	0.6792 9829	0.6288 6331
94	0.7908 0273	0.7605 0878	0.7313 8720	0.6764 7962	0.6257 3464
95	0.7888 3065	0.7582 9708	0.7289 5735	0.6736 7265	0.6226 2153
96	0.7868 6349	0.7560 9182	0.7265 3556	0.6708 7733	0.6195 2391
97	0.7849 0124	0.7538 9296	0.7241 2182	0.6680 9361	0.6164 4170
98	0.7829 4388	0.7517 0050	0.7217 1610	0.6653 2143	0.6133 7483
99	0.7809 9140	0.7495 1442	0.7193 1837	0.6625 6076	0.6103 2321
100	0.7790 4379	0.7473 3469	0.7169 2861	0.6598 1155	0.6072 8678

$v^n = (1 + i)^{-n}$ (*Continued*)

n	¼ %	7/24 %	⅓ %	5/12 %	½ %
101	0.7771 0104	0.7451 6131	0.7145 4679	0.6570 7374	0.6042 6545
102	0.7751 6313	0.7429 9424	0.7121 7288	0.6543 4730	0.6012 5915
103	0.7732 3006	0.7408 3348	0.7098 0686	0.6516 3216	0.5982 6781
104	0.7713 0180	0.7386 7899	0.7074 4869	0.6489 2829	0.5952 9136
105	0.7693 7836	0.7365 3078	0.7050 9837	0.6462 3565	0.5923 2971
106	0.7674 5971	0.7343 8881	0.7027 5585	0.6435 5417	0.5893 8279
107	0.7655 4584	0.7322 5307	0.7004 2111	0.6408 8382	0.5864 5054
108	0.7636 3675	0.7301 2355	0.6980 9413	0.6382 2455	0.5835 3288
109	0.7617 3242	0.7280 0021	0.6957 7488	0.6355 7632	0.5806 2973
110	0.7598 3284	0.7258 8305	0.6934 6334	0.6329 3907	0.5777 4102
111	0.7579 3799	0.7237 7205	0.6911 5947	0.6303 1277	0.5748 6669
112	0.7560 4787	0.7216 6719	0.6888 6326	0.6276 9736	0.5720 0666
113	0.7541 6247	0.7195 6845	0.6865 7468	0.6250 9281	0.5691 6085
114	0.7522 8176	0.7174 7581	0.6842 9370	0.6224 9906	0.5663 2921
115	0.7504 0575	0.7153 8926	0.6820 2030	0.6199 1608	0.5635 1165
116	0.7485 3441	0.7133 0878	0.6797 5445	0.6173 4381	0.5607 0811
117	0.7466 6774	0.7112 3434	0.6774 9613	0.6147 8222	0.5579 1852
118	0.7448 0573	0.7091 6594	0.6752 4531	0.6122 3126	0.5551 4280
119	0.7429 4836	0.7071 0356	0.6730 0197	0.6096 9088	0.5523 8090
120	0.7410 9562	0.7050 4717	0.6707 6608	0.6071 6104	0.5496 3273
121	0.7392 4750	0.7029 9676	0.6685 3763	0.6046 4170	0.5468 9824
122	0.7374 0399	0.7009 5232	0.6663 1657	0.6021 3281	0.5441 7736
123	0.7355 6508	0.6989 1382	0.6641 0289	0.5996 3434	0.5414 7001
124	0.7337 3075	0.6968 8125	0.6618 9657	0.5971 4623	0.5387 7612
125	0.7319 0100	0.6948 5459	0.6596 9758	0.5946 6844	0.5360 9565
126	0.7300 7581	0.6928 3382	0.6575 0589	0.5922 0094	0.5334 2850
127	0.7282 5517	0.6908 1893	0.6553 2149	0.5897 4367	0.5307 7463
128	0.7264 3907	0.6888 0991	0.6531 4434	0.5872 9660	0.5281 3396
129	0.7246 2750	0.6868 0672	0.6509 7443	0.5848 5969	0.5255 0643
130	0.7228 2045	0.6848 0936	0.6488 1172	0.5824 3288	0.5228 9197
131	0.7210 1791	0.6828 1781	0.6466 5620	0.5800 1615	0.5202 9052
132	0.7192 1986	0.6808 3205	0.6445 0784	0.5776 0944	0.5177 0201
133	0.7174 2629	0.6788 5206	0.6423 6662	0.5752 1273	0.5151 2637
134	0.7156 3720	0.6768 7783	0.6402 3251	0.5728 2595	0.5125 6356
135	0.7138 5257	0.6749 0935	0.6381 0549	0.5704 4908	0.5100 1349
136	0.7120 7239	0.6729 4659	0.6359 8554	0.5680 8207	0.5074 7611
137	0.7102 9664	0.6709 8954	0.6338 7263	0.5657 2488	0.5049 5135
138	0.7085 2533	0.6690 3817	0.6317 6674	0.5633 7748	0.5024 3916
139	0.7067 5843	0.6670 9249	0.6296 6785	0.5610 3981	0.4999 3946
140	0.7049 9595	0.6651 5246	0.6275 7593	0.5587 1185	0.4974 5220
141	0.7032 3785	0.6632 1807	0.6254 9096	0.5563 9354	0.4949 7731
142	0.7014 8414	0.6612 8931	0.6234 1292	0.5540 8485	0.4925 1474
143	0.6997 3480	0.6593 6616	0.6213 4178	0.5517 8574	0.4900 6442
144	0.6979 8983	0.6574 4860	0.6192 7752	0.5494 9618	0.4876 2628
145	0.6962 4921	0.6555 3662	0.6172 2012	0.5472 1611	0.4852 0028
146	0.6945 1292	0.6536 3020	0.6151 6955	0.5449 4550	0.4827 8635
147	0.6927 8097	0.6517 2932	0.6131 2580	0.5426 8432	0.4803 8443
148	0.6910 5334	0.6498 3397	0.6110 8884	0.5404 3252	0.4779 9446
149	0.6893 3001	0.6479 4414	0.6090 5864	0.5381 9006	0.4756 1637
150	0.6876 1098	0.6460 5980	0.6070 3519	0.5359 5690	0.4732 5012

$v^n = (1 + i)^{-n}$ (*Continued*)

n	$\frac{1}{4}$%	$\frac{7}{24}$%	$\frac{1}{3}$%	$\frac{5}{12}$%	$\frac{1}{2}$%
151	0.6858 9624	0.6441 8093	0.6050 1846	0.5337 3302	0.4708 9565
152	0.6841 8578	0.6423 0754	0.6030 0843	0.5315 1836	0.4685 5288
153	0.6824 7958	0.6404 3959	0.6010 0508	0.5293 1289	0.4662 2177
154	0.6807 7764	0.6385 7707	0.5990 0839	0.5271 1657	0.4639 0226
155	0.6790 7994	0.6367 1997	0.5970 1833	0.5249 2936	0.4615 9429
156	0.6773 8647	0.6348 6827	0.5950 3488	0.5227 5123	0.4592 9780
157	0.6756 9723	0.6330 2196	0.5930 5802	0.5205 8214	0.4570 1274
158	0.6740 1220	0.6311 8101	0.5910 8773	0.5184 2205	0.4547 3904
159	0.6723 3137	0.6293 4542	0.5891 2398	0.5162 7092	0.4524 7666
160	0.6706 5473	0.6275 1517	0.5871 6676	0.5141 2872	0.4502 2553
161	0.6689 8228	0.6256 9024	0.5852 1604	0.5119 9540	0.4479 8560
162	0.6673 1399	0.6238 7062	0.5832 7180	0.5098 7094	0.4457 5682
163	0.6656 4987	0.6220 5629	0.5813 3402	0.5077 5529	0.4435 3912
164	0.6639 8989	0.6202 4723	0.5794 0268	0.5056 4842	0.4413 3246
165	0.6623 3406	0.6184 4344	0.5774 7775	0.5035 5030	0.4391 3678
166	0.6606 8235	0.6166 4489	0.5755 5922	0.5014 6088	0.4369 5202
167	0.6590 3476	0.6148 5158	0.5736 4706	0.4993 8013	0.4347 7813
168	0.6573 9129	0.6130 6347	0.5717 4126	0.4973 0801	0.4326 1505
169	0.6557 5191	0.6112 8057	0.5698 4179	0.4952 4449	0.4304 6274
170	0.6541 1661	0.6095 0285	0.5679 4862	0.4931 8954	0.4283 2113
171	0.6524 8540	0.6077 3031	0.5660 6175	0.4911 4311	0.4261 9018
172	0.6508 5826	0.6059 6292	0.5641 8115	0.4891 0517	0.4240 6983
173	0.6492 3517	0.6042 0066	0.5623 0679	0.4870 7569	0.4219 6003
174	0.6476 1613	0.6024 4354	0.5604 3866	0.4850 5462	0.4198 6073
175	0.6460 0112	0.6006 9152	0.5585 7674	0.4830 4195	0.4177 7187
176	0.6443 9015	0.5989 4460	0.5567 2100	0.4810 3763	0.4156 9340
177	0.6427 8319	0.5972 0276	0.5548 7143	0.4790 4162	0.4136 2528
178	0.6411 8024	0.5954 6598	0.5530 2801	0.4770 5390	0.4115 6744
179	0.6395 8129	0.5937 3426	0.5511 9070	0.4750 7442	0.4095 1984
180	0.6379 8632	0.5920 0757	0.5493 5950	0.4731 0316	0.4074 8243
181	0.6363 9533	0.5902 8590	0.5475 3439	0.4711 4007	0.4054 5515
182	0.6348 0831	0.5885 6924	0.5457 1534	0.4691 8513	0.4034 3796
183	0.6332 2525	0.5868 5757	0.5439 0233	0.4672 3831	0.4014 3081
184	0.6316 4613	0.5851 5088	0.5420 9535	0.4652 9956	0.3994 3364
185	0.6300 7096	0.5834 4916	0.5402 9437	0.4633 6886	0.3974 4641
186	0.6284 9971	0.5817 5238	0.5384 9937	0.4614 4616	0.3954 6906
187	0.6269 3238	0.5800 6053	0.5367 1033	0.4595 3145	0.3935 0155
188	0.6253 6895	0.5783 7361	0.5349 2724	0.4576 2468	0.3915 4383
189	0.6238 0943	0.5766 9159	0.5331 5008	0.4557 2582	0.3895 9586
190	0.6222 5380	0.5750 1447	0.5313 7881	0.4538 3484	0.3876 5757
191	0.6207 0204	0.5733 4222	0.5296 1343	0.4519 5171	0.3857 2892
192	0.6191 5416	0.5716 7484	0.5278 5392	0.4500 7639	0.3838 0987
193	0.6176 1013	0.5700 1230	0.5261 0025	0.4482 0886	0.3819 0037
194	0.6160 6996	0.5683 5460	0.5243 5241	0.4463 4907	0.3800 0037
195	0.6145 3362	0.5667 0172	0.5226 1038	0.4444 9700	0.3781 0982
196	0.6130 0112	0.5650 5365	0.5208 7413	0.4426 5261	0.3762 2868
197	0.6114 7244	0.5634 1037	0.5191 4365	0.4408 1588	0.3743 5689
198	0.6099 4757	0.5617 7186	0.5174 1892	0.4389 8677	0.3724 9442
199	0.6084 2650	0.5601 3813	0.5156 9992	0.4371 6525	0.3706 4121
200	0.6069 0923	0.5585 0914	0.5139 8663	0.4353 5128	0.3687 9723

$v^n = (1 + i)^{-n}$ *(Continued)*

n	$\frac{7}{12}\%$	$\frac{2}{3}\%$	$\frac{3}{4}\%$	$\frac{7}{8}\%$	1%
1	0.9942 0050	0.9933 7748	0.9925 5583	0.9913 2590	0.9900 9901
2	0.9884 3463	0.9867 9882	0.9851 6708	0.9827 2704	0.9802 9605
3	0.9827 0220	0.9802 6373	0.9778 3333	0.9742 0276	0.9705 9015
4	0.9770 0301	0.9737 7192	0.9705 5417	0.9657 5243	0.9609 8034
5	0.9713 3688	0.9673 2310	0.9633 2920	0.9573 7539	0.9514 6569
6	0.9657 0361	0.9609 1699	0.9561 5802	0.9490 7102	0.9420 4524
7	0.9601 0301	0.9545 5330	0.9490 4022	0.9408 3868	0.9327 1805
8	0.9545 3489	0.9482 3175	0.9419 7540	0.9326 7775	0.9234 8322
9	0.9489 9906	0.9419 5207	0.9349 6318	0.9245 8761	0.9143 3982
10	0.9434 9534	0.9357 1398	0.9280 0315	0.9165 6765	0.9052 8695
11	0.9380 2354	0.9295 1720	0.9210 9494	0.9086 1724	0.8963 2372
12	0.9325 8347	0.9233 6145	0.9142 3815	0.9007 3581	0.8874 4923
13	0.9271 7495	0.9172 4648	0.9074 3241	0.8929 2273	0.8786 6260
14	0.9217 9779	0.9111 7200	0.9006 7733	0.8851 7743	0.8699 6297
15	0.9164 5182	0.9051 3775	0.8939 7254	0.8774 9931	0.8613 4947
16	0.9111 3686	0.8991 4346	0.8873 1766	0.8698 8779	0.8528 2126
17	0.9058 5272	0.8931 8886	0.8807 1231	0.8623 4230	0.8443 7749
18	0.9005 9922	0.8872 7371	0.8741 5614	0.8548 6225	0.8360 1731
19	0.8953 7619	0.8813 9772	0.8676 4878	0.8474 4709	0.8277 3992
20	0.8901 8346	0.8755 6065	0.8611 8985	0.8400 9625	0.8195 4447
21	0.8850 2084	0.8697 6224	0.8547 7901	0.8328 0917	0.8114 3017
22	0.8798 8815	0.8640 0222	0.8484 1589	0.8255 8530	0.8033 9621
23	0.8747 8524	0.8582 8035	0.8421 0014	0.8184 2409	0.7954 4179
24	0.8697 1192	0.8525 9638	0.8358 3140	0.8113 2499	0.7875 6613
25	0.8646 6802	0.8469 5004	0.8296 0933	0.8042 8748	0.7797 6844
26	0.8596 5338	0.8413 4110	0.8234 3358	0.7973 1101	0.7720 4796
27	0.8546 6782	0.8357 6931	0.8173 0380	0.7903 9505	0.7644 0392
28	0.8497 1117	0.8302 3441	0.8112 1966	0.7835 3908	0.7568 3557
29	0.8447 8327	0.8247 3617	0.8051 8080	0.7767 4258	0.7493 4215
30	0.8398 8394	0.8192 7434	0.7991 8690	0.7700 0504	0.7419 2292
31	0.8350 1303	0.8138 4868	0.7932 3762	0.7633 2594	0.7345 7715
32	0.8301 7037	0.8084 5896	0.7873 3262	0.7567 0477	0.7273 0411
33	0.8253 5580	0.8031 0492	0.7814 7158	0.7501 4104	0.7201 0307
34	0.8205 6914	0.7977 8635	0.7756 5418	0.7436 3424	0.7129 7334
35	0.8158 1025	0.7925 0299	0.7698 8008	0.7371 8388	0.7059 1420
36	0.8110 7896	0.7872 5463	0.7641 4896	0.7307 8947	0.6989 2495
37	0.8063 7510	0.7820 4102	0.7584 6051	0.7244 5053	0.6920 0490
38	0.8016 9853	0.7768 6194	0.7528 1440	0.7181 6657	0.6851 5337
39	0.7970 4907	0.7717 1716	0.7472 1032	0.7119 3712	0.6783 6967
40	0.7924 2659	0.7666 0645	0.7416 4796	0.7057 6171	0.6716 5314
41	0.7878 3091	0.7615 2959	0.7361 2701	0.6996 3986	0.6650 0311
42	0.7832 6188	0.7564 8635	0.7306 4716	0.6935 7111	0.6584 1892
43	0.7787 1935	0.7514 7650	0.7252 0809	0.6875 5500	0.6518 9992
44	0.7742 0316	0.7464 9984	0.7198 0952	0.6815 9108	0.6454 4546
45	0.7697 1317	0.7415 5613	0.7144 5114	0.6756 7889	0.6390 5492
46	0.7652 4922	0.7366 4516	0.7091 3264	0.6698 1798	0.6327 2764
47	0.7608 1115	0.7317 6672	0.7038 5374	0.6640 0792	0.6264 6301
48	0.7563 9883	0.7269 2058	0.6986 1414	0.6582 4824	0.6202 6041
49	0.7520 1209	0.7221 0654	0.6934 1353	0.6525 3853	0.6141 1921
50	0.7476 5079	0.7173 2437	0.6882 5165	0.6468 7835	0.6080 3882

$v^n = (1+i)^{-n}$ *(Continued)*

n	$\frac{7}{12}\%$	$\frac{2}{3}\%$	$\frac{3}{4}\%$	$\frac{7}{8}\%$	1%
51	0.7433 1479	0.7125 7388	0.6831 2819	0.6412 6726	0.6020 1864
52	0.7390 0393	0.7078 5485	0.6780 4286	0.6357 0484	0.5960 5806
53	0.7347 1808	0.7031 6707	0.6729 9540	0.6301 9067	0.5901 5649
54	0.7304 5708	0.6985 1033	0.6679 8551	0.6247 2433	0.5843 1336
55	0.7262 2079	0.6938 8444	0.6630 1291	0.6193 0541	0.5785 2808
56	0.7220 0907	0.6892 8918	0.6580 7733	0.6139 3349	0.5728 0008
57	0.7178 2178	0.6847 2435	0.6531 7849	0.6086 0817	0.5671 2879
58	0.7136 5877	0.6801 8975	0.6483 1612	0.6033 2904	0.5615 1365
59	0.7095 1990	0.6756 8518	0.6434 8995	0.5980 9571	0.5559 5411
60	0.7054 0504	0.6712 1044	0.6386 9970	0.5929 0776	0.5504 4962
61	0.7013 1404	0.6667 6534	0.6339 4511	0.5877 6482	0.5449 9962
62	0.6972 4677	0.6623 4968	0.6292 2592	0.5826 6649	0.5396 0358
63	0.6932 0308	0.6579 6326	0.6245 4185	0.5776 1238	0.5342 6097
64	0.6891 8285	0.6536 0588	0.6198 9266	0.5726 0211	0.5289 7126
65	0.6851 8593	0.6492 7737	0.6152 7807	0.5676 3530	0.5237 3392
66	0.6812 1219	0.6449 7752	0.6106 9784	0.5627 1158	0.5185 4844
67	0.6772 6150	0.6407 0614	0.6061 5170	0.5578 3056	0.5134 1429
68	0.6733 3372	0.6364 6306	0.6016 3940	0.5529 9188	0.5083 3099
69	0.6694 2872	0.6322 4807	0.5971 6070	0.5481 9517	0.5032 9801
70	0.6655 4637	0.6280 6100	0.5927 1533	0.5434 4007	0.4983 1486
71	0.6616 8653	0.6239 0165	0.5883 0306	0.5387 2622	0.4933 8105
72	0.6578 4908	0.6197 6985	0.5839 2363	0.5340 5325	0.4884 9609
73	0.6540 3388	0.6156 6542	0.5795 7681	0.5294 2082	0.4836 5949
74	0.6502 4081	0.6115 8816	0.5752 6234	0.5248 2857	0.4788 7078
75	0.6464 6973	0.6075 3791	0.5709 7999	0.5202 7615	0.4741 2949
76	0.6427 2053	0.6035 1448	0.5667 2952	0.5157 6322	0.4694 3514
77	0.6389 9307	0.5995 1769	0.5625 1069	0.5112 8944	0.4647 8726
78	0.6352 8723	0.5955 4738	0.5583 2326	0.5068 5447	0.4601 8541
79	0.6316 0288	0.5916 0336	0.5541 6701	0.5024 5796	0.4556 2912
80	0.6279 3990	0.5876 8545	0.5500 4170	0.4980 9959	0.4511 1794
81	0.6242 9816	0.5837 9350	0.5459 4710	0.4937 7902	0.4466 5142
82	0.6206 7754	0.5799 2731	0.5418 8297	0.4894 9593	0.4422 2913
83	0.6170 7792	0.5760 8674	0.5378 4911	0.4852 4999	0.4378 5063
84	0.6134 9917	0.5722 7159	0.5338 4527	0.4810 4089	0.4335 1547
85	0.6099 4118	0.5684 8171	0.5298 7123	0.4768 6829	0.4292 2324
86	0.6064 0382	0.5647 1693	0.5259 2678	0.4727 3188	0.4249 7350
87	0.6028 8698	0.5609 7709	0.5220 1169	0.4686 3136	0.4207 6585
88	0.5993 9054	0.5572 6201	0.5181 2575	0.4645 6640	0.4165 9985
89	0.5959 1437	0.5535 7153	0.5142 6873	0.4605 3671	0.4124 7510
90	0.5924 5836	0.5499 0549	0.5104 4043	0.4565 4197	0.4083 9119
91	0.5890 2240	0.5462 6374	0.5066 4063	0.4525 8187	0.4043 4771
92	0.5856 0636	0.5426 4610	0.5028 6911	0.4486 5613	0.4003 4427
93	0.5822 1014	0.5390 5241	0.4991 2567	0.4447 6444	0.3963 8046
94	0.5788 3361	0.5354 8253	0.4954 1009	0.4409 0651	0.3924 5590
95	0.5754 7666	0.5319 3629	0.4917 2217	0.4370 8204	0.3885 7020
96	0.5721 3918	0.5284 1353	0.4880 6171	0.4332 9075	0.3847 2297
97	0.5688 2106	0.5249 1410	0.4844 2850	0.4295 3234	0.3809 1383
98	0.5655 2218	0.5214 3785	0.4808 2233	0.4258 0654	0.3771 4241
99	0.5622 4243	0.5179 8462	0.4772 4301	0.4221 1305	0.3734 0832
100	0.5589 8171	0.5145 5426	0.4736 9033	0.4184 5159	0.3697 1121

$v^n = (1 + i)^{-n}$ *(Continued)*

n	$\frac{7}{12}\%$	$\frac{2}{3}\%$	$\frac{3}{4}\%$	$\frac{7}{8}\%$	1%
101	0.5557 3989	0.5111 4661	0.4701 6410	0.4148 2190	0.3660 5071
102	0.5525 1688	0.5077 6154	0.4666 6412	0.4112 2370	0.3624 2644
103	0.5493 1255	0.5043 9888	0.4631 9019	0.4076 5670	0.3588 3806
104	0.5461 2681	0.5010 5849	0.4597 4213	0.4041 2064	0.3552 8521
105	0.5429 5955	0.4977 4022	0.4563 1973	0.4006 1526	0.3517 6753
106	0.5398 1065	0.4944 4393	0.4529 2281	0.3971 4028	0.3482 8469
107	0.5366 8002	0.4911 6946	0.4495 5117	0.3936 9545	0.3448 3632
108	0.5335 6754	0.4879 1669	0.4462 0464	0.3902 8049	0.3414 2210
109	0.5304 7312	0.4846 8545	0.4428 8302	0.3868 9516	0.3380 4168
110	0.5273 9664	0.4814 7561	0.4395 8612	0.3835 3919	0.3346 9474
111	0.5243 3800	0.4782 8703	0.4363 1377	0.3802 1233	0.3313 8093
112	0.5212 9710	0.4751 1957	0.4330 6577	0.3769 1433	0.3280 9993
113	0.5182 7383	0.4719 7308	0.4298 4196	0.3736 4494	0.3248 5141
114	0.5152 6810	0.4688 4743	0.4266 4214	0.3704 0391	0.3216 3506
115	0.5122 7980	0.4657 4248	0.4234 6615	0.3671 9099	0.3184 5056
116	0.5093 0884	0.4626 5809	0.4203 1379	0.3640 0593	0.3152 9758
117	0.5063 5510	0.4595 9413	0.4171 8491	0.3608 4851	0.3121 7582
118	0.5034 1849	0.4565 5046	0.4140 7931	0.3577 1847	0.3090 8497
119	0.5004 9891	0.4535 2695	0.4109 9683	0.3546 1559	0.3060 2473
120	0.4975 9627	0.4505 2346	0.4079 3730	0.3515 3961	0.3029 9478
121	0.4947 1046	0.4475 3986	0.4049 0055	0.3484 9032	0.2999 9483
122	0.4918 4138	0.4445 7602	0.4018 8640	0.3454 6748	0.2970 2459
123	0.4889 8895	0.4416 3181	0.3988 9469	0.3424 7086	0.2940 8375
124	0.4861 5305	0.4387 0710	0.3959 2524	0.3395 0024	0.2911 7203
125	0.4833 3361	0.4358 0175	0.3929 7792	0.3365 5538	0.2882 8914
126	0.4805 3051	0.4329 1565	0.3900 5252	0.3336 3606	0.2854 3479
127	0.4777 4367	0.4300 4866	0.3871 4891	0.3307 4207	0.2826 0870
128	0.4749 7300	0.4272 0065	0.3842 6691	0.3278 7318	0.2798 1060
129	0.4722 1839	0.4243 7151	0.3814 0636	0.3250 2917	0.2770 4019
130	0.4694 7976	0.4215 6110	0.3785 6711	0.3222 0984	0.2742 9722
131	0.4667 5701	0.4187 6930	0.3757 4899	0.3194 1496	0.2715 8141
132	0.4640 5005	0.4159 9600	0.3729 5185	0.3166 4432	0.2688 9248
133	0.4613 5879	0.4132 4106	0.3701 7553	0.3138 9771	0.2662 3108
134	0.4586 8314	0.4105 0436	0.3674 1988	0.3111 7493	0.2635 9424
135	0.4560 2301	0.4077 8579	0.3646 8475	0.3084 7577	0.2609 8439
136	0.4533 7830	0.4050 8522	0.3619 6997	0.3058 0002	0.2584 0039
137	0.4507 4893	0.4024 0254	0.3592 7541	0.3031 4748	0.2558 4197
138	0.4481 3481	0.3997 3762	0.3566 0090	0.3005 1795	0.2533 0888
139	0.4455 3585	0.3970 9035	0.3539 4630	0.2979 1122	0.2508 0087
140	0.4429 5197	0.3944 6061	0.3513 1147	0.2953 2711	0.2483 1770
141	0.4403 8306	0.3918 4829	0.3486 9625	0.2927 6541	0.2458 5911
142	0.4378 2906	0.3892 5327	0.3461 0049	0.2902 2594	0.2434 2486
143	0.4352 8987	0.3866 7543	0.3435 2406	0.2877 0849	0.2410 1471
144	0.4327 6541	0.3841 1467	0.3409 6681	0.2852 1288	0.2386 2843
145	0.4302 5558	0.3815 7086	0.3384 2860	0.2827 3891	0.2362 6577
146	0.4277 6031	0.3790 4390	0.3359 0928	0.2802 8640	0.2339 2650
147	0.4252 7952	0.3765 3368	0.3334 0871	0.2778 5517	0.2316 1040
148	0.4228 1311	0.3740 4008	0.3309 2676	0.2754 4503	0.2293 1723
149	0.4203 6100	0.3715 6299	0.3284 6329	0.2730 5579	0.2270 4676
150	0.4179 2312	0.3691 0231	0.3260 1815	0.2706 8728	0.2247 9877

n	7/12 %	2/3 %	3/4 %	7/8 %	1 %
151	0.4154 9937	0.3666 5792	0.3235 9122	0.2683 3931	0.2225 7304
152	0.4130 8968	0.3642 2973	0.3211 8235	0.2660 1170	0.2203 6935
153	0.4106 9396	0.3618 1761	0.3187 9141	0.2637 0429	0.2181 8747
154	0.4083 1214	0.3594 2147	0.3164 1828	0.2614 1689	0.2160 2720
155	0.4059 4414	0.3570 4119	0.3140 6280	0.2591 4934	0.2138 8832
156	0.4035 8986	0.3546 7668	0.3117 2487	0.2569 0145	0.2117 7061
157	0.4012 4924	0.3523 2783	0.3094 0434	0.2546 7306	0.2096 7387
158	0.3989 2220	0.3499 9453	0.3071 0108	0.2524 6400	0.2075 9789
159	0.3966 0864	0.3476 7669	0.3048 1496	0.2502 7410	0.2055 4247
160	0.3943 0851	0.3453 7419	0.3025 4587	0.2481 0320	0.2035 0739
161	0.3920 2172	0.3430 8695	0.3002 9367	0.2459 5113	0.2014 9247
162	0.3897 4819	0.3408 1485	0.2980 5823	0.2438 1772	0.1994 9750
163	0.3874 8784	0.3385 5779	0.2958 3944	0.2417 0282	0.1975 2227
164	0.3852 4060	0.3363 1569	0.2936 3716	0.2396 0627	0.1955 6661
165	0.3830 0640	0.3340 8843	0.2914 5127	0.2375 2790	0.1936 3030
166	0.3807 8515	0.3318 7593	0.2892 8166	0.2354 6756	0.1917 1317
167	0.3785 7679	0.3296 7807	0.2871 2820	0.2334 2509	0.1898 1502
168	0.3763 8123	0.3274 9478	0.2849 9077	0.2314 0033	0.1879 3566
169	0.3741 9841	0.3253 2594	0.2828 6925	0.2293 9314	0.1860 7492
170	0.3720 2824	0.3231 7146	0.2807 6352	0.2274 0336	0.1842 3259
171	0.3698 7066	0.3210 3125	0.2786 7347	0.2254 3084	0.1824 0850
172	0.3677 2560	0.3189 0522	0.2765 9898	0.2234 7543	0.1806 0248
173	0.3655 9297	0.3167 9326	0.2745 3993	0.2215 3699	0.1788 1434
174	0.3634 7272	0.3146 9529	0.2724 9621	0.2196 1535	0.1770 4390
175	0.3613 6475	0.3126 1122	0.2704 6770	0.2177 1039	0.1752 9099
176	0.3592 6902	0.3105 4094	0.2684 5429	0.2158 2194	0.1735 5543
177	0.3571 8544	0.3084 8438	0.2664 5587	0.2139 4988	0.1718 3706
178	0.3551 1394	0.3064 4144	0.2644 7233	0.2120 9406	0.1701 3571
179	0.3530 5445	0.3044 1203	0.2625 0355	0.2102 5433	0.1684 5119
180	0.3510 0691	0.3023 9605	0.2605 4943	0.2084 3057	0.1667 8336
181	0.3489 7125	0.3003 9343	0.2586 0986	0.2066 2262	0.1651 3204
182	0.3469 4739	0.2984 0407	0.2566 8472	0.2048 3035	0.1634 9707
183	0.3449 3527	0.2964 2788	0.2547 7392	0.2030 5363	0.1618 7829
184	0.3429 3481	0.2944 6478	0.2528 7734	0.2012 9233	0.1602 7553
185	0.3409 4596	0.2925 1469	0.2509 9488	0.1995 4630	0.1586 8864
186	0.3389 6864	0.2905 7750	0.2491 2643	0.1978 1541	0.1571 1747
187	0.3370 0279	0.2886 5315	0.2472 7189	0.1960 9954	0.1555 6185
188	0.3350 4835	0.2867 4154	0.2454 3116	0.1943 9855	0.1540 2163
189	0.3331 0523	0.2848 4259	0.2436 0413	0.1927 1232	0.1524 9667
190	0.3311 7339	0.2829 5621	0.2417 9070	0.1910 4071	0.1509 8680
191	0.3292 5275	0.2810 8233	0.2399 9077	0.1893 8361	0.1494 9188
192	0.3273 4324	0.2792 2086	0.2382 0423	0.1877 4087	0.1480 1176
193	0.3254 4482	0.2773 7171	0.2364 3100	0.1861 1239	0.1465 4630
194	0.3235 5740	0.2755 3482	0.2346 7097	0.1844 9803	0.1450 9535
195	0.3216 8093	0.2737 1008	0.2329 2404	0.1828 9768	0.1436 5876
196	0.3198 1534	0.2718 9743	0.2311 9011	0.1813 1121	0.1422 3640
197	0.3179 6057	0.2700 9679	0.2294 6909	0.1797 3849	0.1408 2811
198	0.3161 1655	0.2683 0807	0.2277 6089	0.1781 7942	0.1394 3378
199	0.3142 8323	0.2665 3119	0.2260 6540	0.1766 3388	0.1380 5324
200	0.3124 6055	0.2647 6608	0.2243 8253	0.1751 0174	0.1366 8638

$v^n = (1 + i)^{-n}$ (*Continued*)

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
1	0.9888 7515	0.9876 5432	0.9864 3650	0.9852 2167	0.9828 0098
2	0.9778 7407	0.9754 6106	0.9730 5696	0.9706 6175	0.9658 9777
3	0.9669 9537	0.9634 1833	0.9598 5890	0.9563 1699	0.9492 8528
4	0.9562 3770	0.9515 2428	0.9468 3986	0.9421 8423	0.9329 5851
5	0.9455 9970	0.9397 7706	0.9339 9739	0.9282 6033	0.9169 1254
6	0.9350 8005	0.9281 7488	0.9213 2912	0.9145 4219	0.9011 4254
7	0.9246 7743	0.9167 1593	0.9088 3267	0.9010 2679	0.8856 4378
8	0.9143 9054	0.9053 9845	0.8965 0571	0.8877 1112	0.8704 1157
9	0.9042 1808	0.8942 2069	0.8843 4596	0.8745 9224	0.8554 4135
10	0.8941 5880	0.8831 8093	0.8723 5113	0.8616 6723	0.8407 2860
11	0.8842 1142	0.8722 7746	0.8605 1899	0.8489 3323	0.8262 6889
12	0.8743 7470	0.8615 0860	0.8488 4734	0.8363 8742	0.8120 5788
13	0.8646 4742	0.8508 7269	0.8373 3400	0.8240 2702	0.7980 9128
14	0.8550 2835	0.8403 6809	0.8259 7682	0.8118 4928	0.7843 6490
15	0.8455 1629	0.8299 9318	0.8147 7368	0.7998 5150	0.7708 7459
16	0.8361 1005	0.8197 4635	0.8037 2250	0.7880 3104	0.7576 1631
17	0.8268 0846	0.8096 2602	0.7928 2120	0.7763 8526	0.7445 8605
18	0.8176 1034	0.7996 3064	0.7820 6777	0.7649 1159	0.7317 7990
19	0.8085 1455	0.7897 5866	0.7714 6020	0.7536 0747	0.7191 9401
20	0.7995 1995	0.7800 0855	0.7609 9649	0.7424 7042	0.7068 2458
21	0.7906 2542	0.7703 7881	0.7506 7472	0.7314 9795	0.6946 6789
22	0.7818 2983	0.7608 6796	0.7404 9294	0.7206 8763	0.6827 2028
23	0.7731 3210	0.7514 7453	0.7304 4926	0.7100 3708	0.6709 7817
24	0.7645 3112	0.7421 9707	0.7205 4181	0.6995 4392	0.6594 3800
25	0.7560 2583	0.7330 3414	0.7107 6874	0.6892 0583	0.6480 9632
26	0.7476 1516	0.7239 8434	0.7011 2823	0.6790 2052	0.6369 4970
27	0.7392 9806	0.7150 4626	0.6916 1847	0.6689 8574	0.6259 9479
28	0.7310 7348	0.7062 1853	0.6822 3771	0.6590 9925	0.6152 2829
29	0.7229 4040	0.6974 9978	0.6729 8417	0.6493 5887	0.6046 4697
30	0.7148 9780	0.6888 8867	0.6638 5615	0.6397 6243	0.5942 4764
31	0.7069 4467	0.6803 8387	0.6548 5194	0.6303 0781	0.5840 2716
32	0.6990 8002	0.6719 8407	0.6459 6985	0.6209 9292	0.5739 8247
33	0.6913 0287	0.6636 8797	0.6372 0824	0.6118 1568	0.5641 1053
34	0.6836 1223	0.6554 9429	0.6285 6546	0.6027 7407	0.5544 0839
35	0.6760 0715	0.6474 0177	0.6200 3991	0.5938 6608	0.5448 7311
36	0.6684 8667	0.6394 0916	0.6116 3000	0.5850 8974	0.5355 0183
37	0.6610 4986	0.6315 1522	0.6033 3416	0.5764 4309	0.5262 9172
38	0.6536 9578	0.6237 1873	0.5951 5083	0.5679 2423	0.5172 4002
39	0.6464 2352	0.6160 1850	0.5870 7850	0.5595 3126	0.5083 4400
40	0.6392 3216	0.6084 1334	0.5791 1566	0.5512 6232	0.4996 0098
41	0.6321 2080	0.6009 0206	0.5712 6083	0.5431 1559	0.4910 0834
42	0.6250 8855	0.5934 8352	0.5635 1253	0.5350 8925	0.4825 6348
43	0.6181 3454	0.5861 5656	0.5558 6933	0.5271 8153	0.4742 6386
44	0.6112 5789	0.5789 2006	0.5483 2979	0.5193 9067	0.4661 0699
45	0.6044 5774	0.5717 7290	0.5408 9252	0.5117 1494	0.4580 9040
46	0.5977 3324	0.5647 1397	0.5335 5612	0.5041 5265	0.4502 1170
47	0.5910 8355	0.5577 4219	0.5263 1923	0.4967 0212	0.4424 6850
48	0.5845 0784	0.5508 5649	0.5191 8050	0.4893 6170	0.4348 5848
49	0.5780 0528	0.5440 5579	0.5121 3860	0.4821 2975	0.4273 7934
50	0.5715 7506	0.5373 3905	0.5051 9220	0.4750 0468	0.4200 2883

$v^n = (1+i)^{-n}$ (*Continued*)

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
51	0.5652 1637	0.5307 0524	0.4983 4003	0.4679 8491	0.4128 0475
52	0.5589 2843	0.5241 5332	0.4915 8079	0.4610 6887	0.4057 0492
53	0.5527 1044	0.5176 8229	0.4849 1323	0.4542 5505	0.3987 2719
54	0.5465 6162	0.5112 9115	0.4783 3611	0.4475 4192	0.3918 6947
55	0.5404 8120	0.5049 7892	0.4718 4820	0.4409 2800	0.3851 2970
56	0.5344 6843	0.4987 4461	0.4654 4829	0.4344 1182	0.3785 0585
57	0.5285 2256	0.4925 8727	0.4591 3518	0.4279 9194	0.3719 9592
58	0.5226 4282	0.4865 0594	0.4529 0770	0.4216 6694	0.3655 9796
59	0.5168 2850	0.4804 9970	0.4467 6468	0.4154 3541	0.3593 1003
60	0.5110 7887	0.4745 6760	0.4407 0499	0.4092 9597	0.3531 3025
61	0.5053 9319	0.4687 0874	0.4347 2749	0.4032 4726	0.3470 5676
62	0.4997 7077	0.4629 2222	0.4288 3106	0.3972 8794	0.3410 8772
63	0.4942 1090	0.4572 0713	0.4230 1461	0.3914 1669	0.3352 2135
64	0.4887 1288	0.4515 6259	0.4172 7705	0.3856 3221	0.3294 5587
65	0.4832 7602	0.4459 8775	0.4116 1731	0.3799 3321	0.3237 8956
66	0.4778 9965	0.4404 8173	0.4060 3434	0.3743 1843	0.3182 2069
67	0.4725 8309	0.4350 4368	0.4005 2709	0.3687 8663	0.3127 4761
68	0.4673 2568	0.4296 7277	0.3950 9454	0.3633 3658	0.3073 6866
69	0.4621 2675	0.4243 6817	0.3897 3568	0.3579 6708	0.3020 8222
70	0.4569 8566	0.4191 2905	0.3844 4949	0.3526 7692	0.2968 8670
71	0.4519 0177	0.4139 5462	0.3792 3501	0.3474 6495	0.2917 8054
72	0.4468 7443	0.4088 4407	0.3740 9126	0.3423 3000	0.2867 6221
73	0.4419 0302	0.4037 9661	0.3690 1727	0.3372 7093	0.2818 3018
74	0.4369 8692	0.3988 1147	0.3640 1210	0.3322 8663	0.2769 8298
75	0.4321 2551	0.3938 8787	0.3590 7483	0.3273 7599	0.2722 1914
76	0.4273 1818	0.3890 2506	0.3542 0451	0.3225 3793	0.2675 3724
77	0.4225 6433	0.3842 2228	0.3494 0026	0.3177 7136	0.2629 3586
78	0.4178 6337	0.3794 7879	0.3446 6117	0.3130 7523	0.2584 1362
79	0.4132 1470	0.3747 9387	0.3399 8636	0.3084 4850	0.2539 6916
80	0.4086 1775	0.3701 6679	0.3353 7495	0.3038 9015	0.2496 0114
81	0.4040 7194	0.3655 9683	0.3308 2609	0.2993 9916	0.2453 0825
82	0.3995 7670	0.3610 8329	0.3263 3893	0.2949 7454	0.2410 8919
83	0.3951 3148	0.3566 2547	0.3219 1263	0.2906 1531	0.2369 4269
84	0.3907 3570	0.3522 2268	0.3175 4637	0.2863 2050	0.2328 6751
85	0.3863 8882	0.3478 7426	0.3132 3933	0.2820 8917	0.2288 6242
86	0.3820 9031	0.3435 7951	0.3089 9071	0.2779 2036	0.2249 2621
87	0.3778 3961	0.3393 3779	0.3047 9971	0.2738 1316	0.2210 5770
88	0.3736 3621	0.3351 4843	0.3006 6556	0.2697 6666	0.2172 5572
89	0.3694 7956	0.3310 1080	0.2965 8748	0.2657 7997	0.2135 1914
90	0.3653 6916	0.3269 2425	0.2925 6472	0.2618 5218	0.2098 4682
91	0.3613 0448	0.3228 8814	0.2885 9652	0.2579 8245	0.2062 3766
92	0.3572 8503	0.3189 0187	0.2846 8214	0.2541 6990	0.2026 9057
93	0.3533 1029	0.3149 6481	0.2808 2085	0.2504 1369	0.1992 0450
94	0.3493 7976	0.3110 7636	0.2770 1194	0.2467 1300	0.1957 7837
95	0.3454 9297	0.3072 3591	0.2732 5468	0.2430 6699	0.1924 1118
96	0.3416 4941	0.3034 4287	0.2695 4839	0.2394 7487	0.1891 0190
97	0.3378 4861	0.2996 9666	0.2658 9237	0.2359 3583	0.1858 4953
98	0.3340 9010	0.2959 9670	0.2622 8594	0.2324 4909	0.1826 5310
99	0.3303 7340	0.2923 4242	0.2587 2843	0.2290 1389	0.1795 1165
100	0.3266 9805	0.2887 3326	0.2552 1916	0.2256 2944	0.1764 2422

$v^n = (1+i)^{-n}$ (*Continued*)

n	2%	2¼%	2½%	2¾%	3%
1	0.9803 9216	0.9779 9511	0.9756 0976	0.9732 3601	0.9708 7379
2	0.9611 6878	0.9564 7444	0.9518 1440	0.9471 8833	0.9425 9591
3	0.9423 2233	0.9354 2732	0.9285 9941	0.9218 3779	0.9151 4166
4	0.9238 4543	0.9148 4335	0.9059 5064	0.8971 6573	0.8884 8705
5	0.9057 3081	0.8947 1232	0.8838 5429	0.8731 5400	0.8626 0878
6	0.8879 7138	0.8750 2427	0.8622 9687	0.8497 8491	0.8374 8426
7	0.8705 6018	0.8557 6946	0.8412 6524	0.8270 4128	0.8130 9151
8	0.8534 9037	0.8369 3835	0.8207 4657	0.8049 0635	0.7894 0923
9	0.8367 5527	0.8185 2161	0.8007 2836	0.7833 6385	0.7664 1673
10	0.8203 4830	0.8005 1013	0.7811 9840	0.7623 9791	0.7440 9391
11	0.8042 6304	0.7828 9499	0.7621 4478	0.7419 9310	0.7224 2128
12	0.7884 9318	0.7656 6748	0.7435 5589	0.7221 3440	0.7013 7988
13	0.7730 3253	0.7488 1905	0.7254 2038	0.7028 0720	0.6809 5134
14	0.7578 7502	0.7323 4137	0.7077 2720	0.6839 9728	0.6611 1781
15	0.7430 1473	0.7162 2628	0.6904 6556	0.6656 9078	0.6418 6195
16	0.7284 4581	0.7004 6580	0.6736 2493	0.6478 7424	0.6231 6694
17	0.7141 6256	0.6850 5212	0.6571 9506	0.6305 3454	0.6050 1645
18	0.7001 5937	0.6699 7763	0.6411 6591	0.6136 5892	0.5873 9461
19	0.6864 3076	0.6552 3484	0.6255 2772	0.5972 3496	0.5702 8603
20	0.6729 7133	0.6408 1647	0.6102 7094	0.5812 5057	0.5536 7575
21	0.6597 7582	0.6267 1538	0.5953 8629	0.5656 9398	0.5375 4928
22	0.6468 3904	0.6129 2457	0.5808 6467	0.5505 5375	0.5218 9250
23	0.6341 5592	0.5994 3724	0.5666 9724	0.5358 1874	0.5066 9175
24	0.6217 2149	0.5862 4668	0.5528 7535	0.5214 7809	0.4919 3374
25	0.6095 3087	0.5733 4639	0.5393 9059	0.5075 2126	0.4776 0557
26	0.5975 7928	0.5607 2997	0.5262 3472	0.4939 3796	0.4636 9473
27	0.5858 6204	0.5483 9117	0.5133 9973	0.4807 1821	0.4501 8906
28	0.5743 7455	0.5363 2388	0.5008 7778	0.4678 5227	0.4370 7675
29	0.5631 1231	0.5245 2213	0.4886 6125	0.4553 3068	0.4243 4636
30	0.5520 7089	0.5129 8008	0.4767 4269	0.4431 4421	0.4119 8676
31	0.5412 4597	0.5016 9201	0.4651 1481	0.4312 8391	0.3999 8715
32	0.5306 3330	0.4906 5233	0.4537 7055	0.4197 4103	0.3883 3703
33	0.5202 2873	0.4798 5558	0.4427 0298	0.4085 0708	0.3770 2625
34	0.5100 2817	0.4692 9641	0.4319 0534	0.3975 7380	0.3660 4490
35	0.5000 2761	0.4589 6960	0.4213 7107	0.3869 3314	0.3553 8340
36	0.4902 2315	0.4488 7002	0.4110 9372	0.3765 7727	0.3450 3243
37	0.4806 1093	0.4389 9268	0.4010 6705	0.3664 9856	0.3349 8294
38	0.4711 8719	0.4293 3270	0.3912 8492	0.3566 8959	0.3252 2615
39	0.4619 4822	0.4198 8528	0.3817 4139	0.3471 4316	0.3157 5355
40	0.4528 9042	0.4106 4575	0.3724 3062	0.3378 5222	0.3065 5684
41	0.4440 1021	0.4016 0954	0.3633 4695	0.3288 0995	0.2976 2800
42	0.4353 0413	0.3927 7216	0.3544 8483	0.3200 0968	0.2889 5922
43	0.4267 6875	0.3841 2925	0.3458 3886	0.3114 4495	0.2805 4294
44	0.4184 0074	0.3756 7653	0.3374 0376	0.3031 0944	0.2723 7178
45	0.4101 9680	0.3674 0981	0.3291 7440	0.2949 9702	0.2644 3862
46	0.4021 5373	0.3593 2500	0.3211 4576	0.2871 0172	0.2567 3653
47	0.3942 6836	0.3514 1809	0.3133 1294	0.2794 1773	0.2492 5876
48	0.3865 3761	0.3436 8518	0.3056 7116	0.2719 3940	0.2419 9880
49	0.3789 5844	0.3361 2242	0.2982 1576	0.2646 6122	0.2349 5029
50	0.3715 2788	0.3287 2608	0.2909 4221	0.2575 7783	0.2281 0708

$v^n = (1 + i)^{-n}$ (*Continued*)

n	2%	2¼%	2½%	2¾%	3%
51	0.3642 4302	0.3214 9250	0.2838 4606	0.2506 8402	0.2214 6318
52	0.3571 0100	0.3144 1810	0.2769 2298	0.2439 7471	0.2150 1280
53	0.3500 9902	0.3074 9936	0.2701 6876	0.2374 4497	0.2087 5029
54	0.3432 3433	0.3007 3287	0.2635 7928	0.2310 9000	0.2026 7019
55	0.3365 0425	0.2941 1528	0.2571 5052	0.2249 0511	0.1967 6717
56	0.3299 0613	0.2876 4330	0.2508 7855	0.2188 8575	0.1910 3609
57	0.3234 3738	0.2813 1374	0.2447 5956	0.2130 2749	0.1854 7193
58	0.3170 9547	0.2751 2347	0.2387 8982	0.2073 2603	0.1800 6984
59	0.3108 7791	0.2690 6940	0.2329 6568	0.2017 7716	0.1748 2508
60	0.3047 8227	0.2631 4856	0.2272 8359	0.1963 7679	0.1697 3309
61	0.2988 0614	0.2573 5801	0.2217 4009	0.1911 2097	0.1647 8941
62	0.2929 4720	0.2516 9487	0.2163 3179	0.1860 0581	0.1599 8972
63	0.2872 0314	0.2461 5635	0.2110 5541	0.1810 2755	0.1553 2982
64	0.2815 7170	0.2407 3971	0.2059 0771	0.1761 8253	0.1508 0565
65	0.2760 5069	0.2354 4226	0.2008 8557	0.1714 6718	0.1464 1325
66	0.2706 3793	0.2302 6138	0.1959 8593	0.1668 7804	0.1421 4879
67	0.2653 3130	0.2251 9450	0.1912 0578	0.1624 1172	0.1380 0853
68	0.2601 2873	0.2202 3912	0.1865 4223	0.1580 6493	0.1339 8887
69	0.2550 2817	0.2153 9278	0.1819 9241	0.1538 3448	0.1300 8628
70	0.2500 2761	0.2106 5309	0.1775 5358	0.1497 1726	0.1262 9736
71	0.2451 2511	0.2060 1769	0.1732 2300	0.1457 1023	0.1226 1880
72	0.2403 1874	0.2014 8429	0.1689 9805	0.1418 1044	0.1190 4737
73	0.2356 0661	0.1970 5065	0.1648 7615	0.1380 1503	0.1155 7998
74	0.2309 8687	0.1927 1458	0.1608 5478	0.1343 2119	0.1122 1357
75	0.2264 5771	0.1884 7391	0.1569 3149	0.1307 2622	0.1089 4521
76	0.2220 1737	0.1843 2657	0.1531 0389	0.1272 2747	0.1057 7205
77	0.2176 6408	0.1802 7048	0.1493 6965	0.1238 2235	0.1026 9131
78	0.2133 9616	0.1763 0365	0.1457 2649	0.1205 0837	0.0997 0030
79	0.2092 1192	0.1724 2411	0.1421 7218	0.1172 8309	0.0967 9641
80	0.2051 0973	0.1686 2993	0.1387 0457	0.1141 4412	0.0939 7710
81	0.2010 8797	0.1649 1925	0.1353 2153	0.1110 8917	0.0912 3990
82	0.1971 4507	0.1612 9022	0.1320 2101	0.1081 1598	0.0885 8243
83	0.1932 7948	0.1577 4105	0.1288 0098	0.1052 2237	0.0860 0236
84	0.1894 8968	0.1542 6997	0.1256 5949	0.1024 0620	0.0834 9743
85	0.1857 7420	0.1508 7528	0.1225 9463	0.0996 6540	0.0810 6547
86	0.1821 3157	0.1475 5528	0.1196 0452	0.0969 9795	0.0787 0434
87	0.1785 6036	0.1443 0835	0.1166 8733	0.0944 0190	0.0764 1198
88	0.1750 5918	0.1411 3286	0.1138 4130	0.0918 7533	0.0741 8639
89	0.1716 2665	0.1380 2724	0.1110 6468	0.0894 1638	0.0720 2562
90	0.1682 6142	0.1349 8997	0.1083 5579	0.0870 2324	0.0699 2779
91	0.1649 6217	0.1320 1953	0.1057 1296	0.0846 9415	0.0678 9105
92	0.1617 2762	0.1291 1445	0.1031 3460	0.0824 2740	0.0659 1364
93	0.1585 5649	0.1262 7331	0.1006 1912	0.0802 2131	0.0639 9383
94	0.1554 4754	0.1234 9468	0.0981 6500	0.0780 7427	0.0621 2993
95	0.1523 9955	0.1207 7719	0.0957 7073	0.0759 8469	0.0603 2032
96	0.1494 1132	0.1181 1950	0.0934 3486	0.0739 5104	0.0585 6342
97	0.1464 8169	0.1155 2029	0.0911 5596	0.0719 7181	0.0568 5769
98	0.1436 0950	0.1129 7828	0.0889 3264	0.0700 4556	0.0552 0164
99	0.1407 9363	0.1104 9221	0.0867 6355	0.0681 7086	0.0535 9383
100	0.1380 3297	0.1080 6084	0.0846 4737	0.0663 4634	0.0520 3284

$v^n = (1 + i)^{-n}$ (*Continued*)

n	3½ %	4 %	4½ %	5 %	5½ %
1	0.9661 8357	0.9615 3846	0.9569 3780	0.9523 8095	0.9478 6730
2	0.9335 1070	0.9245 5621	0.9157 2995	0.9070 2948	0.8984 5242
3	0.9019 4271	0.8889 9636	0.8762 9660	0.8638 3760	0.8516 1366
4	0.8714 4223	0.8548 0419	0.8385 6134	0.8227 0247	0.8072 1674
5	0.8419 7317	0.8219 2711	0.8024 5105	0.7835 2617	0.7651 3435
6	0.8135 0064	0.7903 1453	0.7678 9574	0.7462 1540	0.7252 4583
7	0.7859 9096	0.7599 1781	0.7348 2846	0.7106 8133	0.6874 3681
8	0.7594 1156	0.7306 9021	0.7031 8513	0.6768 3936	0.6515 9887
9	0.7337 3097	0.7025 8674	0.6729 0443	0.6446 0892	0.6176 2926
10	0.7089 1881	0.6755 6417	0.6439 2768	0.6139 1325	0.5854 3058
11	0.6849 4571	0.6495 8093	0.6161 9874	0.5846 7929	0.5549 1050
12	0.6617 8330	0.6245 9705	0.5896 6386	0.5568 3742	0.5259 8152
13	0.6394 0415	0.6005 7409	0.5642 7164	0.5303 2135	0.4985 6068
14	0.6177 8179	0.5774 7508	0.5399 7286	0.5050 6795	0.4725 6937
15	0.5968 9062	0.5552 6450	0.5167 2044	0.4810 1710	0.4479 3305
16	0.5767 0591	0.5339 0818	0.4944 6932	0.4581 1152	0.4245 8109
17	0.5572 0378	0.5133 7325	0.4731 7639	0.4362 9669	0.4024 4653
18	0.5383 6114	0.4936 2812	0.4528 0037	0.4155 2065	0.3814 6590
19	0.5201 5569	0.4746 4242	0.4333 0179	0.3957 3396	0.3615 7906
20	0.5025 6588	0.4563 8695	0.4146 4286	0.3768 8948	0.3427 2896
21	0.4855 7090	0.4388 3360	0.3967 8743	0.3589 4236	0.3248 6158
22	0.4691 5063	0.4219 5539	0.3797 0089	0.3418 4987	0.3079 2567
23	0.4532 8563	0.4057 2633	0.3633 5013	0.3255 7131	0.2918 7267
24	0.4379 5713	0.3901 2147	0.3477 0347	0.3100 6791	0.2766 5656
25	0.4231 4699	0.3751 1680	0.3327 3060	0.2953 0277	0.2622 3370
26	0.4088 3767	0.3606 8923	0.3184 0248	0.2812 4073	0.2485 6275
27	0.3950 1224	0.3468 1657	0.3046 9137	0.2678 4832	0.2356 0450
28	0.3816 5434	0.3334 7747	0.2915 7069	0.2550 9364	0.2233 2181
29	0.3687 4815	0.3206 5141	0.2790 1502	0.2429 4632	0.2116 7944
30	0.3562 7841	0.3083 1867	0.2670 0002	0.2313 7745	0.2006 4402
31	0.3442 3035	0.2964 6026	0.2555 0241	0.2203 5947	0.1901 8390
32	0.3325 8971	0.2850 5794	0.2444 9991	0.2098 6617	0.1802 6910
33	0.3213 4271	0.2740 9417	0.2339 7121	0.1998 7254	0.1708 7119
34	0.3104 7605	0.2635 5209	0.2238 9589	0.1903 5480	0.1619 6321
35	0.2999 7686	0.2534 1547	0.2142 5444	0.1812 9029	0.1535 1963
36	0.2898 3272	0.2436 6872	0.2050 2817	0.1726 5741	0.1455 1624
37	0.2800 3161	0.2342 9685	0.1961 9921	0.1644 3563	0.1379 3008
38	0.2705 6194	0.2252 8543	0.1877 5044	0.1566 0536	0.1307 3941
39	0.2614 1250	0.2166 2061	0.1796 6549	0.1491 4797	0.1239 2362
40	0.2525 7247	0.2082 8904	0.1719 2870	0.1420 4568	0.1174 6314
41	0.2440 3137	0.2002 7793	0.1645 2507	0.1352 8160	0.1113 3947
42	0.2357 7910	0.1925 7493	0.1574 4026	0.1288 3962	0.1055 3504
43	0.2278 0590	0.1851 6820	0.1506 6054	0.1227 0440	0.1000 3322
44	0.2201 0231	0.1780 4635	0.1441 7276	0.1168 6133	0.0948 1822
45	0.2126 5924	0.1711 9841	0.1379 6437	0.1112 9651	0.0898 7509
46	0.2054 6787	0.1646 1386	0.1320 2332	0.1059 9668	0.0851 8965
47	0.1985 1968	0.1582 8256	0.1263 3810	0.1009 4921	0.0807 4849
48	0.1918 0645	0.1521 9476	0.1208 9771	0.0961 4211	0.0765 3885
49	0.1853 2024	0.1463 4112	0.1156 9158	0.0915 6391	0.0725 4867
50	0.1790 5337	0.1407 1262	0.1107 0965	0.0872 0373	0.0687 6652

n	3½ %	4 %	4½ %	5 %	5½ %
51	0.1729 9843	0.1353 0059	0.1059 4225	0.0830 5117	0.0651 8153
52	0.1671 4824	0.1300 9672	0.1013 8014	0.0790 9635	0.0617 8344
53	0.1614 9589	0.1250 9300	0.0970 1449	0.0753 2986	0.0585 6250
54	0.1560 3467	0.1202 8173	0.0928 3683	0.0717 4272	0.0555 0948
55	0.1507 5814	0.1156 5551	0.0888 3907	0.0683 2640	0.0526 1562
56	0.1456 6004	0.1112 0722	0.0850 1347	0.0650 7276	0.0498 7263
57	0.1407 3433	0.1069 3002	0.0813 5260	0.0619 7406	0.0472 7263
58	0.1359 7520	0.1028 1733	0.0778 4938	0.0590 2291	0.0448 0818
59	0.1313 7701	0.0988 6282	0.0744 9701	0.0562 1230	0.0424 7221
60	0.1269 3431	0.0950 6040	0.0712 8901	0.0535 3552	0.0402 5802
61	0.1226 4184	0.0914 0423	0.0682 1915	0.0509 8621	0.0381 5926
62	0.1184 9453	0.0878 8868	0.0652 8148	0.0485 5830	0.0361 6992
63	0.1144 8747	0.0845 0835	0.0624 7032	0.0462 4600	0.0342 8428
64	0.1106 1591	0.0812 5803	0.0597 8021	0.0440 4381	0.0324 9695
65	0.1068 7528	0.0781 3272	0.0572 0594	0.0419 4648	0.0308 0279
56	0.1032 6114	0.0751 2762	0.0547 4253	0.0399 4903	0.0291 9696
67	0.0997 6922	0.0722 3809	0.0523 8519	0.0380 4670	0.0276 7485
68	0.0963 9538	0.0694 5970	0.0501 2937	0.0362 3495	0.0262 3208
69	0.0931 3563	0.0667 8818	0.0479 7069	0.0345 0948	0.0248 6453
70	0.0899 8612	0.0642 1940	0.0459 0497	0.0328 6617	0.0235 6828
71	0.0869 4311	0.0617 4942	0.0439 2820	0.0313 0111	0.0223 3960
72	0.0840 0300	0.0593 7445	0.0420 3655	0.0298 1058	0.0211 7498
73	0.0811 6232	0.0570 9081	0.0402 2637	0.0283 9103	0.0200 7107
74	0.0784 1770	0.0548 9501	0.0384 9413	0.0270 3908	0.0190 2471
75	0.0757 6590	0.0527 8367	0.0368 3649	0.0257 5150	0.0180 3290
76	0.0732 0376	0.0507 5353	0.0352 5023	0.0245 2524	0.0170 9279
77	0.0707 2827	0.0488 0147	0.0337 3228	0.0233 5737	0.0162 0170
78	0.0683 3650	0.0469 2449	0.0322 7969	0.0222 4512	0.0153 5706
79	0.0660 2560	0.0451 1970	0.0308 8965	0.0211 8582	0.0145 5646
80	0.0637 9285	0.0433 8433	0.0295 5948	0.0201 7698	0.0137 9759
81	0.0616 3561	0.0417 1570	0.0282 8658	0.0192 1617	0.0130 7828
82	0.0595 5131	0.0401 1125	0.0270 6850	0.0183 0111	0.0123 9648
83	0.0575 3750	0.0385 6851	0.0259 0287	0.0174 2963	0.0117 5022
84	0.0555 9178	0.0370 8510	0.0247 8744	0.0165 9965	0.0111 3765
85	0.0537 1187	0.0356 5875	0.0237 2003	0.0158 0919	0.0105 5701
86	0.0518 9553	0.0342 8726	0.0226 9860	0.0150 5637	0.0100 0664
87	0.0501 4060	0.0329 6852	0.0217 2115	0.0143 3940	0.0094 8497
88	0.0484 4503	0.0317 0050	0.0207 8579	0.0136 5657	0.0089 9049
89	0.0468 0679	0.0304 8125	0.0198 9070	0.0130 0626	0.0085 2180
90	0.0452 2395	0.0293 0890	0.0190 3417	0.0123 8691	0.0080 7753
91	0.0436 9464	0.0281 8163	0.0182 1451	0.0117 9706	0.0076 5643
92	0.0422 1704	0.0270 9772	0.0174 3016	0.0112 3530	0.0072 5728
93	0.0407 8941	0.0260 5550	0.0166 7958	0.0107 0028	0.0068 7894
94	0.0394 1006	0.0250 5337	0.0159 6132	0.0101 9074	0.0065 2032
95	0.0380 7735	0.0240 8978	0.0152 7399	0.0097 0547	0.0061 8040
96	0.0367 8971	0.0231 6325	0.0146 1626	0.0092 4331	0.0058 5820
97	0.0355 4562	0.0222 7235	0.0139 8685	0.0088 0315	0.0055 5279
98	0.0343 4359	0.0214 1572	0.0133 8454	0.0083 8395	0.0052 6331
99	0.0331 8221	0.0205 9204	0.0128 0817	0.0079 8471	0.0049 8892
100	0.0320 6011	0.0198 0004	0.0122 5663	0.0076 0449	0.0047 2883

$v^n = (1 + i)^{-n}$ (*Continued*)

n	6%	6½%	7%	7½%	8%
1	0.9433 9623	0.9389 6714	0.9345 7944	0.9302 3256	0.9259 2593
2	0.8899 9644	0.8816 5928	0.8734 3873	0.8653 3261	0.8573 3882
3	0.8396 1928	0.8278 4909	0.8162 9788	0.8049 6057	0.7938 3224
4	0.7920 9366	0.7773 2309	0.7628 9521	0.7488 0053	0.7350 2985
5	0.7472 5817	0.7298 8084	0.7129 8618	0.6965 5863	0.6805 8320
6	0.7049 6054	0.6853 3412	0.6663 4222	0.6479 6152	0.6301 6963
7	0.6650 5711	0.6435 0621	0.6227 4974	0.6027 5490	0.5834 9040
8	0.6274 1237	0.6042 3119	0.5820 0910	0.5607 0223	0.5402 6888
9	0.5918 9846	0.5673 5323	0.5439 3374	0.5215 8347	0.5002 4897
10	0.5583 9478	0.5327 2604	0.5083 4929	0.4851 9393	0.4631 9349
11	0.5267 8753	0.5002 1224	0.4750 9280	0.4513 4319	0.4288 8286
12	0.4969 6936	0.4696 8285	0.4440 1196	0.4198 5413	0.3971 1376
13	0.4688 3902	0.4410 1676	0.4149 6445	0.3905 6198	0.3676 9792
14	0.4423 0096	0.4141 0025	0.3878 1724	0.3633 1347	0.3404 6104
15	0.4172 6506	0.3888 2652	0.3624 4602	0.3379 6602	0.3152 4170
16	0.3936 4628	0.3650 9533	0.3387 3460	0.3143 8699	0.2918 9047
17	0.3713 6442	0.3428 1251	0.3165 7439	0.2924 5302	0.2702 6895
18	0.3503 4379	0.3218 8969	0.2958 6392	0.2720 4932	0.2502 4903
19	0.3305 1301	0.3022 4384	0.2765 0833	0.2530 6913	0.2317 1206
20	0.3118 0473	0.2837 9703	0.2584 1900	0.2354 1315	0.2145 4821
21	0.2941 5540	0.2664 7608	0.2415 1309	0.2189 8897	0.1986 5575
22	0.2775 0510	0.2502 1228	0.2257 1317	0.2037 1067	0.1839 4051
23	0.2617 9726	0.2349 4111	0.2109 4688	0.1894 9830	0.1703 1528
24	0.2469 7855	0.2206 0198	0.1971 4662	0.1762 7749	0.1576 9934
25	0.2329 9863	0.2071 3801	0.1842 4918	0.1639 7906	0.1460 1790
26	0.2198 1003	0.1944 9579	0.1721 9549	0.1525 3866	0.1352 0176
27	0.2073 6795	0.1826 2515	0.1609 3037	0.1418 9643	0.1251 8682
28	0.1956 3014	0.1714 7902	0.1504 0221	0.1319 9668	0.1159 1372
29	0.1845 5674	0.1610 1316	0.1405 6282	0.1227 8761	0.1073 2752
30	0.1741 1013	0.1511 8607	0.1313 6712	0.1142 2103	0.0993 7733
31	0.1642 5484	0.1419 5875	0.1227 7301	0.1062 5212	0.0920 1605
32	0.1549 5740	0.1332 9460	0.1147 4113	0.0988 3918	0.0852 0005
33	0.1461 8622	0.1251 5925	0.1072 3470	0.0919 4343	0.0788 8893
34	0.1379 1153	0.1175 2042	0.1002 1934	0.0855 2877	0.0730 4531
35	0.1301 0522	0.1103 4781	0.0936 6294	0.0795 6164	0.0676 3454
36	0.1227 4077	0.1036 1297	0.0875 3546	0.0740 1083	0.0626 2458
37	0.1157 9318	0.0972 8917	0.0818 0884	0.0688 4729	0.0579 8572
38	0.1092 3885	0.0913 5134	0.0764 5686	0.0640 4399	0.0536 9048
39	0.1030 5552	0.0857 7590	0.0714 5501	0.0595 7580	0.0497 1341
40	0.0972 2219	0.0805 4075	0.0667 8038	0.0554 1935	0.0460 3093
41	0.0917 1905	0.0756 2512	0.0624 1157	0.0515 5288	0.0426 2123
42	0.0865 2740	0.0710 0950	0.0583 2857	0.0479 5617	0.0394 6411
43	0.0816 2962	0.0666 7559	0.0545 1268	0.0446 1039	0.0365 4084
44	0.0770 0908	0.0626 0619	0.0509 4643	0.0414 9804	0.0338 3411
45	0.0726 5007	0.0587 8515	0.0476 1349	0.0386 0283	0.0313 2788
46	0.0685 3781	0.0551 9733	0.0444 9859	0.0359 0961	0.0290 0730
47	0.0646 5831	0.0518 2848	0.0415 8747	0.0334 0428	0.0268 5861
48	0.0609 9840	0.0486 6524	0.0388 6679	0.0310 7375	0.0248 6908
49	0.0575 4566	0.0456 9506	0.0363 2410	0.0289 0582	0.0230 2693
50	0.0542 8836	0.0429 0616	0.0339 4776	0.0268 8913	0.0213 2123

$v^n = (1 + i)^{-n}$ (*Continued*)

n	8½%	9%	9½%	10%	10½%
1	0.9216 5899	0.9174 3119	0.9132 4201	0.9090 9091	0.9049 7738
2	0.8494 5529	0.8416 7999	0.8340 1097	0.8264 4628	0.8189 8405
3	0.7829 0810	0.7721 8348	0.7616 5385	0.7513 1480	0.7411 6204
4	0.7215 7428	0.7084 2521	0.6955 7429	0.6830 1346	0.6707 3487
5	0.6650 4542	0.6499 3139	0.6352 2767	0.6209 2132	0.6069 9989
6	0.6129 4509	0.5962 6733	0.5801 1659	0.5644 7393	0.5493 2116
7	0.5649 2635	0.5470 3424	0.5297 8684	0.5131 5812	0.4971 2323
8	0.5206 6945	0.5018 6628	0.4838 2360	0.4665 0738	0.4498 8527
9	0.4798 7968	0.4604 2778	0.4418 4803	0.4240 9762	0.4071 3599
10	0.4422 8542	0.4224 1081	0.4035 1419	0.3855 4329	0.3684 4886
11	0.4076 3633	0.3875 3285	0.3685 0611	0.3504 9390	0.3334 3788
12	0.3757 0168	0.3555 3473	0.3365 3526	0.3186 3082	0.3017 5374
13	0.3462 6883	0.3261 7865	0.3073 3813	0.2896 6438	0.2730 8031
14	0.3191 4178	0.2992 4647	0.2806 7410	0.2633 3125	0.2471 3150
15	0.2941 3989	0.2745 3804	0.2563 2337	0.2393 9205	0.2236 4842
16	0.2710 9667	0.2518 6976	0.2340 8527	0.2176 2914	0.2023 9676
17	0.2498 5869	0.2310 7318	0.2137 7651	0.1978 4467	0.1831 6449
18	0.2302 8450	0.2119 9374	0.1952 2969	0.1798 5879	0.1657 5972
19	0.2122 4378	0.1944 8967	0.1782 9195	0.1635 0799	0.1500 0879
20	0.1956 1639	0.1784 3089	0.1628 2370	0.1486 4363	0.1357 5456
21	0.1802 9160	0.1636 9806	0.1486 9744	0.1351 3057	0.1228 5481
22	0.1661 6738	0.1501 8171	0.1357 9675	0.1228 4597	0.1111 8082
23	0.1531 4965	0.1377 8139	0.1240 1530	0.1116 7816	0.1006 1613
24	0.1411 5176	0.1264 0494	0.1132 5598	0.1015 2560	0.0910 5532
25	0.1300 9378	0.1159 6784	0.1034 3012	0.0922 9600	0.0824 0301
26	0.1199 0210	0.1063 9251	0.0944 5673	0.0839 0545	0.0745 7286
27	0.1105 0885	0.0976 0781	0.0862 6185	0.0762 7768	0.0674 8675
28	0.1018 5148	0.0895 4845	0.0787 7795	0.0693 4335	0.0610 7398
29	0.0938 7233	0.0821 5454	0.0719 4333	0.0630 3941	0.0552 7057
30	0.0865 1828	0.0753 7114	0.0657 0167	0.0573 0855	0.0500 1861
31	0.0797 4035	0.0691 4783	0.0600 0153	0.0520 9868	0.0452 6571
32	0.0734 9341	0.0634 3838	0.0547 9592	0.0473 6244	0.0409 6445
33	0.0677 3586	0.0582 0035	0.0500 4193	0.0430 5676	0.0370 7190
34	0.0624 2936	0.0533 9481	0.0457 0039	0.0391 4251	0.0335 4923
35	0.0575 3858	0.0489 8607	0.0417 3552	0.0355 8410	0.0303 6129
36	0.0530 3095	0.0449 4135	0.0381 1463	0.0323 4918	0.0274 7628
37	0.0488 7645	0.0412 3059	0.0348 0788	0.0294 0835	0.0248 6542
38	0.0450 4742	0.0378 2623	0.0317 8802	0.0267 3486	0.0225 0264
39	0.0415 1836	0.0347 0296	0.0290 3015	0.0243 0442	0.0203 6438
40	0.0382 6577	0.0318 3758	0.0265 1156	0.0220 9493	0.0184 2930
41	0.0352 6799	0.0292 0879	0.0242 1147	0.0200 8630	0.0166 7810
42	0.0325 0506	0.0267 9706	0.0221 1093	0.0182 6027	0.0150 9330
43	0.0299 5858	0.0245 8446	0.0201 9263	0.0166 0025	0.0136 5910
44	0.0276 1160	0.0225 5455	0.0184 4076	0.0150 9113	0.0123 6118
45	0.0254 4848	0.0206 9224	0.0168 4087	0.0137 1921	0.0111 8658
46	0.0234 5482	0.0189 8371	0.0153 7979	0.0124 7201	0.0101 2361
47	0.0216 1734	0.0174 1625	0.0140 4547	0.0113 3819	0.0091 6163
48	0.0199 2382	0.0159 7821	0.0128 2692	0.0103 0745	0.0082 9107
49	0.0183 6297	0.0146 5891	0.0117 1408	0.0093 7041	0.0075 0323
50	0.0169 2439	0.0134 4854	0.0106 9779	0.0085 1855	0.0067 9026

* Source : Taken from, MATHEMATICS OF FINANCE,

Third Ed. : Hummel and Seeback, 1971, pp. 276 - 291.

Appendix D

Annuity Table

Values of $s_{\overline{n}|i}$

n	$\frac{1}{4}$%	$\frac{7}{24}$%	$\frac{1}{3}$%	$\frac{5}{12}$%	$\frac{1}{2}$%
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0025 0000	2.0029 1667	2.0033 3333	2.0041 6667	2.0050 0000
3	3.0075 0625	3.0087 5851	3.0100 1111	3.0125 1736	3.0150 2500
4	4.0150 2502	4.0175 3405	4.0200 4448	4.0250 6952	4.0301 0013
5	5.0250 6258	5.0292 5186	5.0334 4463	5.0418 4064	5.0502 5063
6	6.0376 2523	6.0439 2051	6.0502 2278	6.0628 4831	6.0755 0188
7	7.0527 1930	7.0615 4861	7.0703 9019	7.0881 1018	7.1058 7939
8	8.0703 5110	8.0821 4480	8.0939 5816	8.1176 4397	8.1414 0879
9	9.0905 2697	9.1057 1772	9.1209 3802	9.1514 6749	9.1821 1583
10	10.1132 5329	10.1322 7606	10.1513 4114	10.1895 9860	10.2280 2641
11	11.1385 3642	11.1618 2853	11.1851 7895	11.2320 5526	11.2791 6654
12	12.1663 8277	12.1943 8387	12.2224 6288	12.2788 5549	12.3355 6237
13	13.1967 9872	13.2299 5082	13.2632 0442	13.3300 1739	13.3972 4018
14	14.2297 9072	14.2685 3818	14.3074 1510	14.3855 5913	14.4642 2639
15	15.2653 6520	15.3101 5475	15.3551 0648	15.4454 9896	15.5365 4752
16	16.3035 2861	16.3548 0936	16.4062 9017	16.5098 5520	16.6142 3026
17	17.3442 8743	17.4025 1089	17.4609 7781	17.5786 4627	17.6973 0141
18	18.3876 4815	18.4532 6822	18.5191 8107	18.6518 9063	18.7857 8791
19	19.4336 1727	19.5070 9025	19.5809 1167	19.7296 0684	19.8797 1685
20	20.4822 0131	20.5639 8593	20.6461 8137	20.8118 1353	20.9791 1544
21	21.5334 0682	21.6239 6422	21.7150 0198	21.8985 2942	22.0840 1101
22	22.5872 4033	22.6870 3412	22.7873 8532	22.9897 7330	23.1944 3107
23	23.6437 0843	23.7532 0463	23.8633 4327	24.0855 6402	24.3104 0322
24	24.7028 1770	24.8224 8481	24.9428 8775	25.1859 2053	25.4319 5524
25	25.7645 7475	25.8948 8373	26.0260 3071	26.2908 6187	26.5591 1502
26	26.8289 8619	26.9704 1047	27.1127 8414	27.4004 0713	27.6919 1059
27	27.8960 5865	28.0490 7417	28.2031 6009	28.5145 7549	28.8303 7015
28	28.9657 9880	29.1308 8397	29.2971 7062	29.6333 8622	29.9745 2200
29	30.0382 1330	30.2158 4904	30.3948 2786	30.7568 5866	31.1243 9461
30	31.1133 0883	31.3039 7860	31.4961 4395	31.8850 1224	32.2800 1658
31	32.1910 9210	32.3952 8188	32.6011 3110	33.0178 6646	33.4414 1666
32	33.2715 6983	33.4897 6811	33.7098 0154	34.1554 4090	34.6086 2375
33	34.3547 4876	34.5874 4660	34.8221 6754	35.2977 5524	35.7816 6686
34	35.4406 3563	35.6883 2666	35.9382 4143	36.4448 2922	36.9605 7520
35	36.5292 3722	36.7924 1761	37.0580 3557	37.5966 8268	38.1453 7807
36	37.6205 6031	37.8997 2883	38.1815 6236	38.7533 3552	39.3361 0496
37	38.7146 1171	39.0102 6970	39.3088 3423	39.9148 0775	40.5327 8549
38	39.8113 9824	40.1240 4966	40.4398 6368	41.0811 1945	41.7354 4942
39	40.9109 2673	41.2410 7814	41.5746 6322	42.2522 9078	42.9441 2666
40	42.0132 0405	42.3613 6461	42.7132 4543	43.4283 4199	44.1588 4730
41	43.1182 3706	43.4849 1859	43.8556 2292	44.6092 9342	45.3796 4153
42	44.2260 3265	44.6117 4961	45.0018 0833	45.7951 6547	46.6065 3974
43	45.3365 9774	45.7418 6721	46.1518 1436	46.9859 7866	47.8395 7244
44	46.4499 3923	46.8752 8099	47.3056 5374	48.1817 5357	49.0787 7030
45	47.5660 6408	48.0120 0056	48.4633 3925	49.3825 1088	50.3241 6415
46	48.6849 7924	49.1520 3556	49.6248 8371	50.5882 7134	51.5757 8497
47	49.8066 9169	50.2953 9566	50.7902 9999	51.7990 5581	52.8336 6390
48	50.9312 0842	51.4420 9057	51.9596 0099	53.0148 8521	54.0978 3222
49	52.0585 3644	52.5921 3000	53.1327 9966	54.2357 8056	55.3683 2138
50	53.1886 8278	53.7455 2371	54.3099 0899	55.4617 6298	56.6451 6299

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \quad (Continued)$$

n	¼ %	7/24 %	⅓ %	5/12 %	½ %
51	54.3216 5449	54.9022 8149	55.4909 4202	56.6928 5366	57.9283 8880
52	55.4574 5862	56.0624 1314	56.6759 1183	57.9290 7388	59.2180 3075
53	56.5961 0227	57.2259 2851	57.8648 3154	59.1704 4502	60.5141 2090
54	57.7375 9252	58.3928 3747	59.0577 1431	60.4169 8854	61.8166 9150
55	58.8819 3650	59.5631 4991	60.2545 7336	61.6687 2600	63.1257 7496
56	60.0291 4135	60.7368 7577	61.4554 2194	62.9256 7902	64.4414 0384
57	61.1792 1420	61.9140 2499	62.6602 7334	64.1878 6935	65.7636 1086
58	62.3321 6223	63.0946 0756	63.8691 4092	65.4553 1881	67.0924 2891
59	63.4879 9264	64.2786 3350	65.0820 3806	66.7280 4930	68.4278 9105
60	64.6467 1262	65.4661 1285	66.2989 7818	68.0060 8284	69.7700 3051
61	65.8083 2940	66.6570 5568	67.5199 7478	69.2894 4152	71.1188 8066
62	66.9728 5023	67.8514 7209	68.7450 4136	70.5781 4753	72.4744 7507
63	68.1402 8235	69.0493 7222	69.9741 9150	71.8722 2314	73.8368 4744
64	69.3106 3306	70.2507 6622	71.2074 3880	73.1716 9074	75.2060 3168
65	70.4839 0964	71.4556 6429	72.4447 9693	74.4765 7278	76.5820 6184
66	71.6601 1942	72.6640 7664	73.6862 7959	75.7868 9183	77.9649 7215
67	72.8392 6971	73.8760 1353	74.9319 0052	77.1026 7055	79.3547 9701
68	74.0213 6789	75.0914 8524	76.1816 7352	78.4239 3168	80.7515 7099
69	75.2064 2131	76.3105 0207	77.4356 1243	79.7506 9806	82.1553 2885
70	76.3944 3736	77.5330 7437	78.6937 3114	81.0829 9264	83.5661 0549
71	77.5854 2345	78.7592 1250	79.9560 4358	82.4208 3844	84.9839 3602
72	78.7793 8701	79.9889 2687	81.2225 6372	83.7642 5860	86.4088 5570
73	79.9763 3548	81.2222 2791	82.4933 0560	85.1132 7634	87.8408 9998
74	81.1762 7632	82.4591 2607	83.7682 8329	86.4679 1499	89.2801 0448
75	82.3792 1701	83.6996 3186	85.0475 1090	87.8281 9797	90.7265 0500
76	83.5851 6505	84.9437 5578	86.3310 0260	89.1941 4880	92.1801 3752
77	84.7941 2797	86.1915 0840	87.6187 7261	90.5657 9108	93.6410 3821
78	86.0061 1329	87.4429 0030	88.9108 3519	91.9431 4855	95.1092 4340
79	87.2211 2857	88.6979 4210	90.2072 0464	93.3262 4500	96.5847 8962
80	88.4391 8139	89.9566 4443	91.5078 9532	94.7151 0435	98.0677 1357
81	89.6602 7934	91.2190 1797	92.8129 2164	96.1097 5062	99.5580 5214
82	90.8844 3004	92.4850 7344	94.1222 9804	97.5102 0792	101.0558 4240
83	92.1116 4112	93.7548 2157	95.4360 3904	98.9165 0045	102.5611 2161
84	93.3419 2022	95.0282 7314	96.7541 5917	100.3286 5253	104.0739 2722
85	94.5752 7502	96.3054 3893	98.0766 7303	101.7466 8859	105.5942 9685
86	95.8117 1321	97.5863 2980	99.4035 9527	103.1706 3312	107.1222 6834
87	97.0512 4249	98.8709 5659	100.7349 4059	104.6005 1076	108.6578 7968
88	98.2938 7060	100.1593 3022	102.0707 2373	106.0363 4622	110.2011 6908
89	99.5396 0527	101.4514 6160	103.4109 5947	107.4781 6433	111.7521 7492
90	100.7884 5429	102.7473 6169	104.7556 6267	108.9259 9002	113.3109 3580
91	102.0404 2542	104.0470 4150	106.1048 4821	110.3798 4831	114.8774 9048
92	103.2955 2649	105.3505 1203	107.4585 3104	111.8397 6434	116.4518 7793
93	104.5537 6530	106.6577 8436	108.8167 2614	113.3057 6336	118.0341 3732
94	105.8151 4972	107.9688 6957	110.1794 4856	114.7778 7071	119.6243 0800
95	107.0796 8759	109.2837 7877	111.5467 1339	116.2561 1184	121.2224 2954
96	108.3473 8681	110.6025 2312	112.9185 3577	117.7405 1230	122.8285 4169
97	109.6182 5528	111.9251 1382	114.2949 3089	119.2310 9777	124.4426 8440
98	110.8923 0091	113.2515 6206	115.6759 1399	120.7278 9401	126.0648 9782
99	112.1695 3167	114.5818 7912	117.0615 0037	122.2309 2690	127.6952 2231
100	113.4499 5550	115.9160 7627	118.4517 0537	123.7402 2243	129.3336 9842

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \text{ (Continued)}$$

n	¼ %	7/24 %	⅓ %	5/12 %	½ %
101	114.7335 8038	117.2541 6482	119.8465 4439	125.2558 0669	130.9803 6692
102	116.0204 1434	118.5961 5614	121.2460 3287	126.7777 0589	132.6352 6875
103	117.3104 6537	119.9420 6159	122.6501 8632	128.3059 4633	134.2984 4509
104	118.6037 4153	121.2918 9261	124.0590 2027	129.8405 5444	135.9699 3732
105	119.9002 5089	122.6456 6063	125.4725 5034	131.3815 5675	137.6497 8701
106	121.2000 0152	124.0033 7714	126.8907 9217	132.9289 7990	139.3380 3594
107	122.5030 0152	125.3650 5365	128.3137 6148	134.4828 5065	141.0347 2612
108	123.8092 5902	126.7307 0173	129.7414 7402	136.0431 9586	142.7398 9975
109	125.1187 8217	128.1003 3294	131.1739 4560	137.6100 4251	144.4535 9925
110	126.4315 7913	129.4739 5891	132.6111 9208	139.1834 1769	146.1758 6725
111	127.7476 5807	130.8515 9129	134.0532 2939	140.7633 4859	147.9067 4658
112	129.0670 2722	132.2332 4177	135.5000 7349	142.3498 6255	149.6462 8032
113	130.3896 9479	133.6189 2205	136.9517 4040	143.9429 8697	151.3945 1172
114	131.7156 6902	135.0086 4391	138.4082 4620	145.5427 4942	153.1514 8428
115	133.0449 5820	136.4024 1912	139.8696 0702	147.1491 7754	154.9172 4170
116	134.3775 7059	137.8002 5951	141.3358 3904	148.7622 9911	156.6918 2791
117	135.7135 1452	139.2021 7693	142.8069 5851	150.3821 4203	158.4752 8704
118	137.0527 9830	140.6081 8328	144.2829 8170	152.0087 3429	160.2676 6348
119	138.3954 3030	142.0182 9048	145.7639 2498	153.6421 0401	162.0690 0180
120	139.7414 1888	143.4325 1050	147.2498 0473	155.2822 7945	163.8793 4681
121	141.0907 7242	144.8508 5532	148.7406 3741	156.9292 8894	165.6987 4354
122	142.4434 9935	146.2733 3698	150.2364 3953	158.5831 6098	167.5272 3726
123	143.7996 0810	147.6999 6755	151.7372 2766	160.2439 2415	169.3648 7344
124	145.1591 0712	149.1307 5912	153.2430 1842	161.9116 0717	171.2116 9781
125	146.5220 0489	150.5657 2383	154.7538 2848	163.5862 3887	173.0677 5630
126	147.8883 0990	152.0048 7386	156.2696 7458	165.2678 4819	174.9330 9508
127	149.2580 3068	153.4482 2141	157.7905 7349	166.9564 6423	176.8077 6056
128	150.6311 7575	154.8957 7872	159.3165 4207	168.6521 1616	178.6917 9936
129	152.0077 5369	156.3475 5808	160.8475 9721	170.3548 3331	180.5852 5836
130	153.3877 7308	157.8035 7179	162.3837 5587	172.0646 4512	182.4881 8465
131	154.7712 4251	159.2638 3221	163.9250 3506	173.7815 8114	184.4006 2557
132	156.1581 7062	160.7283 5172	165.4714 5184	175.5056 7106	186.3226 2870
133	157.5485 6604	162.1971 4274	167.0230 2335	177.2369 4469	188.2542 4184
134	158.9424 3746	163.6702 1774	168.5797 6676	178.9754 3196	190.1955 1305
135	160.3397 9355	165.1475 8921	170.1416 9931	180.7211 6293	192.1464 9062
136	161.7406 4304	166.6292 6968	171.7088 3831	182.4741 6777	194.1072 2307
137	163.1449 9464	168.1152 7172	173.2812 0111	184.2344 7680	196.0777 5919
138	164.5528 5713	169.6056 0793	174.8588 0511	186.0021 2046	198.0581 4798
139	165.9642 3927	171.1002 9095	176.4416 6779	187.7771 2929	200.0484 3872
140	167.3791 4987	172.5993 3346	178.0298 0669	189.5595 3400	202.0486 8092
141	168.7975 9775	174.1027 4819	179.6232 3937	191.3493 6539	204.0589 2432
142	170.2195 9174	175.6105 4787	181.2219 8351	193.1466 5441	206.0792 1894
143	171.6451 4072	177.1227 4530	182.8260 5678	194.9514 3214	208.1096 1504
144	173.0742 5357	178.6393 5331	184.4354 7697	196.7637 2977	210.1501 6311
145	174.5069 3921	180.1603 8475	186.0502 6190	198.5835 7865	212.2009 1393
146	175.9432 0655	181.6858 5254	187.6704 2944	200.4110 1023	214.2619 1850
147	177.3830 6457	183.2157 6961	189.2959 9753	202.2460 5610	216.3332 2809
148	178.8265 2223	184.7501 4894	190.9269 8419	204.0887 4800	218.4148 9423
149	180.2735 8854	186.2890 0354	192.5634 0747	205.9391 1778	220.5069 6870
150	181.7242 7251	187.8323 4647	194.2052 8550	207.7971 9744	222.6095 0354

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \text{ (Continued)}$$

n	$\frac{1}{4}\%$	$\frac{7}{24}\%$	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$
151	183.1785 8319	189.3801 9081	195.8526 3645	209.6630 1910	224.7225 5106
152	184.6365 2965	190.9325 4970	197.5054 7857	211.5366 1501	226.8461 6382
153	186.0981 2097	192.4894 3631	199.1638 3017	213.4180 1757	228.9803 9464
154	187.5633 6627	194.0508 6383	200.8277 0960	215.3072 5931	231.1252 9661
155	189.0322 7469	195.6168 4551	202.4971 3530	217.2043 7289	233.2809 2309
156	190.5048 5538	197.1873 9465	204.1721 2575	219.1093 9111	235.4473 2771
157	191.9811 1752	198.7625 2455	205.8526 9950	221.0223 4691	237.6245 6435
158	193.4610 7031	200.3422 4858	207.5388 7517	222.9432 7336	239.8126 8717
159	194.9447 2298	201.9265 8014	209.2306 7142	224.8722 0366	242.0117 5060
160	196.4320 8479	203.5155 3266	210.9281 0699	226.8091 7118	244.2218 0936
161	197.9231 6500	205.1091 1963	212.6312 0068	228.7542 0939	246.4429 1840
162	199.4179 7292	206.7073 5456	214.3399 7135	230.7073 5193	248.6751 3300
163	200.9165 1785	208.3102 5102	216.0544 3792	232.6686 3256	250.9185 0866
164	202.4188 0914	209.9178 2258	217.7746 1938	234.6380 8520	253.1731 0121
165	203.9248 5617	211.5300 8290	219.5005 3478	236.6157 4389	255.4389 6671
166	205.4346 6831	213.1470 4564	221.2322 0323	238.6016 4282	257.7161 6154
167	206.9482 5498	214.7687 2452	222.9696 4390	240.5958 1633	260.0047 4235
168	208.4656 2562	216.3951 3330	224.7128 7605	242.5982 9890	262.3047 6606
169	209.9867 8968	218.0262 8577	226.4619 1897	244.6091 2514	264.6162 8989
170	211.5117 5665	219.6621 9577	228.2167 9203	246.6283 2983	266.9393 7134
171	213.0405 3605	221.3028 7718	229.9775 1467	248.6559 4787	269.2740 6820
172	214.5731 3739	222.9483 4390	231.7441 0639	250.6920 1432	271.6204 3854
173	216.1095 7023	224.5986 0991	233.5165 8674	252.7365 6438	273.9785 4073
174	217.6498 4415	226.2536 8919	235.2949 7537	254.7896 3340	276.3484 3344
175	219.1939 6876	227.9135 9578	237.0792 9195	256.8512 5687	278.7301 7561
176	220.7419 5369	229.5783 4377	238.8695 5626	258.9214 7044	281.1238 2648
177	222.2938 0857	231.2479 4727	240.6657 8811	261.0003 0990	283.5294 4562
178	223.8495 4309	232.9224 2045	242.4680 0741	263.0878 1120	285.9470 9284
179	225.4091 6695	234.6017 7751	244.2762 3410	265.1840 1041	288.3768 2831
180	226.9726 8987	236.2860 3269	246.0904 8821	267.2889 4379	290.8187 1245
181	228.5401 2159	237.9752 0029	247.9107 8984	269.4026 4772	293.2728 0601
182	230.1114 7190	239.6692 9462	249.7371 5914	271.5251 5875	295.7391 7004
183	231.6867 5058	241.3683 3007	251.5696 1634	273.6565 1358	298.2178 6589
184	233.2659 6745	243.0723 2103	253.4081 8172	275.7967 4905	300.7089 5522
185	234.8491 3237	244.7812 8196	255.2528 7566	277.9459 0217	303.2125 0000
186	236.4362 5520	246.4952 2737	257.1037 1858	280.1040 1010	305.7285 6250
187	238.0273 4584	248.2141 7178	258.9607 3098	282.2711 1014	308.2572 0531
188	239.6224 1420	249.9381 2978	260.8239 3341	284.4472 3977	310.7984 9134
189	241.2214 7024	251.6671 1600	262.6933 4652	286.6324 3660	313.3524 8379
190	242.8245 2392	253.4011 4508	264.5689 9101	288.8267 3842	315.9192 4621
191	244.4315 8523	255.1402 3176	266.4508 8765	291.0301 8316	318.4988 4244
192	246.0426 6419	256.8843 9077	268.3390 5727	293.2428 0892	321.0913 3666
193	247.6577 7085	258.6336 3691	270.2335 2080	295.4646 5396	323.6967 9334
194	249.2769 1528	260.3879 8501	272.1342 9920	297.6957 5669	326.3152 7731
195	250.9001 0756	262.1474 4997	274.0414 1353	299.9361 5567	328.9468 5369
196	252.5273 5783	263.9120 4670	275.9548 8491	302.1858 8965	331.5915 8796
197	254.1586 7623	265.6817 9017	277.8747 3453	304.4449 9753	334.2495 4590
198	255.7940 7292	267.4566 9539	279.8009 8364	306.7135 1835	336.9207 9363
199	257.4335 5810	269.2367 7742	281.7336 5359	308.9914 9134	339.6053 9760
200	259.0771 4200	271.0220 5135	283.6727 6577	311.2789 5589	342.3034 2459

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \text{ (Continued)}$$

n	$\frac{5}{12}\%$	$\frac{2}{3}\%$	$\frac{3}{4}\%$	$\frac{7}{8}\%$	1%
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0058 3333	2.0066 6667	2.0075 0000	2.0087 5000	2.0100 0000
3	3.0175 3403	3.0200 4444	3.0225 5625	3.0263 2656	3.0301 0000
4	4.0351 3631	4.0401 7807	4.0452 2542	4.0528 0692	4.0604 0100
5	5.0586 7460	5.0671 1259	5.0755 6461	5.0882 6898	5.1010 0501
6	6.0881 8354	6.1008 9335	6.1136 3135	6.1327 9133	6.1520 1506
7	7.1236 9794	7.1415 6597	7.1594 8358	7.1864 5326	7.2135 3521
8	8.1652 5285	8.1891 7641	8.2131 7971	8.2493 3472	8.2856 7056
9	9.2128 8349	9.2437 7092	9.2747 7856	9.3215 1640	9.3685 2727
10	10.2656 2531	10.3053 9606	10.3443 3940	10.4030 7967	10.4622 1254
11	11.3265 1396	11.3740 9870	11.4219 2194	11.4941 0662	11.5668 3467
12	12.3925 8529	12.4499 2602	12.5075 8636	12.5946 8005	12.6825 0301
13	13.4648 7537	13.5329 2553	13.6013 9325	13.7048 8350	13.8093 2804
14	14.5434 2048	14.6231 4503	14.7034 0370	14.8248 0123	14.9474 2132
15	15.6282 5710	15.7206 3267	15.8136 7923	15.9545 1824	16.0968 9554
16	16.7194 2193	16.8254 3688	16.9322 8183	17.0941 2028	17.2578 6449
17	17.8169 5189	17.9376 0646	18.0592 7394	18.2436 9383	18.4304 4314
18	18.9208 8411	19.0571 9051	19.1947 1849	19.4033 2615	19.6147 4757
19	20.0312 5593	20.1842 3844	20.3386 7888	20.5731 0526	20.8108 9504
20	21.1481 0493	21.3188 0003	21.4912 1897	21.7531 1993	22.0190 0399
21	22.2714 6887	22.4609 2536	22.6524 0312	22.9434 5973	23.2391 9403
22	23.4013 8577	23.6106 6487	23.8222 9614	24.1442 1500	24.4715 8598
23	24.5378 9386	24.7680 6930	25.0009 6336	25.3554 7688	25.7163 0183
24	25.6810 3157	25.9331 8976	26.1884 7059	26.5773 3730	26.9734 6485
25	26.8308 3759	27.1060 7769	27.3848 8412	27.8098 8900	28.2431 9950
26	27.9873 5081	28.2867 8488	28.5902 7075	29.0532 2553	29.5256 3150
27	29.1506 1035	29.4753 6344	29.8046 9778	30.3074 4126	30.8208 8781
28	30.3206 5558	30.6718 6587	31.0282 3301	31.5726 3137	32.1290 9669
29	31.4975 2607	31.8763 4497	32.2609 4476	32.8488 9189	33.4503 8766
30	32.6812 6164	33.0888 5394	33.5029 0184	34.1363 1970	34.7848 9153
31	33.8719 0233	34.3094 4630	34.7541 7361	35.4350 1249	36.1327 4045
32	35.0694 8843	35.5381 7594	36.0148 2991	36.7450 6885	37.4940 6785
33	36.2740 6045	36.7750 9711	37.2849 4113	38.0665 8820	38.8690 0853
34	37.4856 5913	38.0202 6443	38.5645 7819	39.3996 7085	40.2576 9862
35	38.7043 2548	39.2737 3286	39.8538 1253	40.7444 1797	41.6602 7560
36	39.9301 0071	40.5355 5774	41.1527 1612	42.1009 3163	43.0768 7836
37	41.1630 2630	41.8057 9479	42.4613 6149	43.4693 1478	44.5076 4714
38	42.4031 4395	43.0845 0009	43.7798 2170	44.8496 7128	45.9527 2361
39	43.6504 9562	44.3717 3009	45.1081 7037	46.2421 0591	47.4122 5085
40	44.9051 2352	45.6675 4163	46.4464 8164	47.6467 2433	48.8863 7336
41	46.1670 7007	46.9719 9191	47.7948 3026	49.0636 3317	50.3752 3709
42	47.4363 7798	48.2851 3852	49.1532 9148	50.4929 3996	51.8789 8946
43	48.7130 9018	49.6070 3944	50.5219 4117	51.9347 5319	53.3977 7936
44	49.9972 4988	50.9377 5304	51.9008 5573	53.3891 8228	54.9317 5715
45	51.2889 0050	52.2773 3806	53.2901 1215	54.8563 3762	56.4810 7472
46	52.5880 8575	53.6258 5365	54.6897 8799	56.3363 3058	58.0458 8547
47	53.8948 4959	54.9833 5934	56.0999 6140	57.8292 7347	59.6263 4432
48	55.2092 3621	56.3499 1507	57.5207 1111	59.3352 7961	61.2226 0777
49	56.5312 9009	57.7255 8117	58.9521 1644	60.8544 6331	62.8348 3385
50	57.8610 5595	59.1104 1837	60.3942 5732	62.3869 3986	64.4631 8218

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \quad (Continued)$$

n	$\frac{7}{12}\%$	$\frac{2}{3}\%$	$\frac{3}{4}\%$	$\frac{7}{8}\%$	1%
51	59.1985 7877	60.5044 8783	61.8472 1424	63.9328 2559	66.1078 1401
52	60.5439 0381	61.9078 5108	63.3110 6835	65.4922 3781	67.7688 9215
53	61.8970 7659	63.3205 7009	64.7859 0136	67.0652 9489	69.4465 8107
54	63.2581 4287	64.7427 0722	66.2717 9562	68.6521 1622	71.1410 4688
55	64.6271 4870	66.1743 2527	67.7688 3409	70.2528 2224	72.8524 5735
56	66.0041 4040	67.6154 8744	69.2771 0035	71.8675 3443	74.5809 8192
57	67.3891 6455	69.0662 5736	70.7966 7860	73.4963 7536	76.3267 9174
58	68.7822 6801	70.5266 9907	72.3276 5369	75.1394 6864	78.0900 5966
59	70.1834 9791	71.9968 7706	73.8701 1109	76.7969 3900	79.8709 6025
60	71.5929 0165	73.4768 5625	75.4241 3693	78.4689 1221	81.6696 6986
61	73.0105 2691	74.9667 0195	76.9898 1795	80.1555 1519	83.4863 6655
62	74.4364 2165	76.4664 7997	78.5672 4159	81.8568 7595	85.3212 3022
63	75.8706 3411	77.9762 5650	80.1564 9590	83.5731 2362	87.1744 4252
64	77.3132 1281	79.4960 9821	81.7576 6962	85.3043 8845	89.0461 8695
65	78.7642 0655	81.0260 7220	83.3708 5214	87.0508 0185	90.9366 4882
66	80.2236 6442	82.5662 4601	84.9961 3353	88.8124 9636	92.8460 1531
67	81.6916 3580	84.1166 8765	86.6336 0453	90.5896 0571	94.7744 7546
68	83.1681 7034	85.6774 6557	88.2833 5657	92.3822 6476	96.7222 2021
69	84.6533 1800	87.2486 4867	89.9454 8174	94.1906 0957	98.6894 4242
70	86.1471 2902	88.8303 0633	91.6200 7285	96.0147 7741	100.6763 3684
71	87.6496 5394	90.4225 0837	93.3072 2340	97.8549 0671	102.6831 0021
72	89.1609 4359	92.0253 2510	95.0070 2758	99.7111 3714	104.7099 3121
73	90.6810 4909	93.6388 2726	96.7195 8028	101.5836 0959	106.7570 3052
74	92.2100 2188	95.2630 8611	98.4449 7714	103.4724 6618	108.8246 0083
75	93.7479 1367	96.8981 7335	100.1833 1446	105.3778 5025	110.9128 4684
76	95.2947 7650	98.5441 6118	101.9346 8932	107.2999 0644	113.0219 7530
77	96.8506 6270	100.2011 2225	103.6991 9949	109.2387 8063	115.1521 9506
78	98.4156 2490	101.8691 2973	105.4769 4349	111.1946 1996	117.3037 1701
79	99.9897 1604	103.5482 5726	107.2680 2056	113.1675 7288	119.4767 5418
80	101.5729 8939	105.2385 7898	109.0725 3072	115.1577 8914	121.6715 2172
81	103.1654 9849	106.9401 6950	110.8905 7470	117.1654 1980	123.8882 3694
82	104.7672 9723	108.6531 0397	112.7222 5401	119.1906 1722	126.1271 1931
83	106.3784 3980	110.3774 5799	114.5676 7091	121.2335 3512	128.3883 9050
84	107.9989 8070	112.1133 0771	116.4269 2845	123.2943 2855	130.6722 7440
85	109.6289 7475	113.8607 2977	118.3001 3041	125.3731 5393	132.9789 9715
86	111.2684 7710	115.6198 0130	120.1873 8139	127.4701 6903	135.3087 8712
87	112.9175 4322	117.3905 9997	122.0887 8675	129.5855 3301	137.6618 7499
88	114.5762 2889	119.1732 0397	124.0044 5265	131.7194 0642	140.0384 9374
89	116.2445 9022	120.9676 9200	125.9344 8604	133.8719 5123	142.4388 7868
90	117.9226 8367	122.7741 4328	127.8789 9469	136.0433 3080	144.8632 6746
91	119.6105 6599	124.5926 3757	129.8380 8715	138.2337 0994	147.3119 0014
92	121.3082 9429	126.4232 5515	131.8118 7280	140.4432 5491	149.7850 1914
93	123.0159 2601	128.2660 7685	133.8004 6185	142.6721 3339	152.2828 6933
94	124.7335 1891	130.1211 8403	135.8039 6531	144.9205 1455	154.8056 9803
95	126.4611 3110	131.9886 5859	137.8224 9505	147.1885 6906	157.3537 5501
96	128.1988 2103	133.8685 8298	139.8561 6377	149.4764 6903	159.9272 9256
97	129.9466 4749	135.7610 4020	141.9050 8499	151.7843 8814	162.5265 6548
98	131.7046 6960	137.6661 1380	143.9693 7313	154.1125 0153	165.1518 3114
99	133.4729 4684	139.5838 8790	146.0491 4343	156.4609 8592	167.8033 4945
100	135.2515 3903	141.5144 4715	148.1445 1201	158.8300 1955	170.4813 8294

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \text{ (Continued)}$$

n	$\frac{7}{12}\%$	$\frac{2}{3}\%$	$\frac{3}{4}\%$	$\frac{7}{8}\%$	1%
101	137.0405 0634	143.4578 7680	150.2555 9585	161.2197 8222	173.1861 9677
102	138.8399 0929	145.4142 6264	152.3825 1281	163.6304 5532	175.9180 5874
103	140.6498 0877	147.3836 9106	154.5253 8166	166.0622 2180	178.6772 3933
104	142.4702 6598	149.3662 4900	156.6843 2202	168.5152 6624	181.4640 1172
105	144.3013 4253	151.3620 2399	158.8594 5444	170.9897 7482	184.2786 5184
106	146.1431 0037	153.3711 0415	161.0509 0035	173.4859 3535	187.1214 3836
107	147.9956 0178	155.3935 7818	163.2587 8210	176.0039 3728	189.9926 5274
108	149.8589 0946	157.4295 3537	165.4832 2296	178.5439 7174	192.8925 7927
109	151.7330 8643	159.4790 6560	167.7243 4714	181.1062 3149	195.8215 0506
110	153.6181 9610	161.5422 5937	169.9822 7974	183.6909 1101	198.7797 2011
111	155.5143 0225	163.6192 0777	172.2571 4684	186.2982 0648	201.7675 1731
112	157.4214 6901	165.7100 0249	174.5490 7544	188.9283 1579	204.7851 9248
113	159.3397 6091	167.8147 3584	176.8581 9351	191.5814 3855	207.8330 4441
114	161.2692 4285	169.9335 0074	179.1846 2996	194.2577 7614	210.9113 7485
115	163.2099 8010	172.0663 9075	181.5285 1468	196.9575 3168	214.0204 8860
116	165.1620 3832	174.2135 0002	183.8899 7854	199.6809 1009	217.1606 9349
117	167.1254 8354	176.3749 2335	186.2691 5338	202.4281 1805	220.3323 0042
118	169.1003 8220	178.5507 5618	188.6661 7203	205.1993 6408	223.5356 2343
119	171.0868 0109	180.7410 9455	191.0811 6832	207.9948 5852	226.7709 7966
120	173.0848 0743	182.9460 3518	193.5142 7708	210.8148 1353	230.0386 8946
121	175.0944 6881	185.1656 7542	195.9656 3416	213.6594 4315	233.3390 7635
122	177.1158 5321	187.4001 1325	198.4353 7642	216.5289 6328	236.6724 6712
123	179.1490 2902	189.6494 4734	200.9236 4174	219.4235 9170	240.0391 9179
124	181.1940 6502	191.9137 7699	203.4305 6905	222.3435 4813	243.4395 8370
125	183.2510 3040	194.1932 0217	205.9562 9832	225.2890 5418	246.8739 7954
126	185.3199 9475	196.4878 2352	208.5009 7056	228.2603 3340	250.3427 1934
127	187.4010 2805	198.7977 4234	211.0647 2784	231.2576 1132	253.8461 4653
128	189.4942 0071	201.1230 6062	213.6477 1330	234.2811 1542	257.3846 0800
129	191.5995 8355	203.4638 8103	216.2500 7115	237.3310 7518	260.9584 5408
130	193.7172 4779	205.8203 0690	218.8719 4668	240.4077 2209	264.5680 3862
131	195.8472 6507	208.1924 4228	221.5134 8628	243.5112 8965	268.2137 1900
132	197.9897 0745	210.5803 9190	224.1748 3743	246.6420 1344	271.8958 5619
133	200.1446 4741	212.9842 6117	226.8561 4871	249.8001 3106	275.6148 1475
134	202.3121 5785	215.4041 5625	229.5575 6982	252.9858 8220	279.3709 6290
135	204.4923 1210	217.8401 8396	232.2792 5160	256.1995 0867	283.1646 7253
136	206.6851 8393	220.2924 5185	235.0213 4598	259.4412 5437	286.9963 1926
137	208.8908 4750	222.7610 6820	237.7840 0608	262.7113 6535	290.8662 8245
138	211.1093 7744	225.2461 4198	240.5673 8612	266.0100 8980	294.7749 4527
139	213.3408 4881	227.7477 8293	243.3716 4152	269.3376 7808	298.7226 9473
140	215.5853 3710	230.2661 0148	246.1969 2883	272.6943 8276	302.7099 2167
141	217.8429 1823	232.8012 0883	249.0434 0580	276.0804 5861	306.7370 2089
142	220.1136 6858	235.3532 1689	251.9112 3134	279.4961 6263	310.8043 9110
143	222.3976 6498	237.9222 3833	254.8005 6558	282.9417 5405	314.9124 3501
144	224.6949 8470	240.5083 8659	257.7115 6982	286.4174 9440	319.0615 5936
145	227.0057 0544	243.1117 7583	260.6444 0659	289.9236 4747	323.2521 7495
146	229.3299 0539	245.7325 2100	263.5992 3964	293.4604 7939	327.4846 9670
147	231.6676 6317	248.3707 3781	266.5762 3394	297.0282 5858	331.7595 4367
148	234.0190 5787	251.0265 4273	269.5755 5569	300.6272 5585	336.0771 3911
149	236.3841 6904	253.7000 5301	272.5973 7236	304.2577 4433	340.4379 1050
150	238.7630 7670	256.3913 8670	275.6418 5265	307.9199 9960	344.8422 8960

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \text{ (Continued)}$$

n	7/12 %	2/3 %	3/4 %	7/8 %	1 %
151	241.1558 6131	259.1006 6261	278.7091 6655	311.6142 9959	349.2907 1250
152	243.5626 0383	261.8280 0036	281.7994 8530	315.3409 2472	353.7836 1962
153	245.9833 8569	264.5735 2036	284.9129 8144	319.1001 5781	358.3214 5582
154	248.4182 8877	267.3373 4383	288.0498 2880	322.8922 8419	362.9046 7038
155	250.8673 9546	270.1195 9279	291.2102 0251	326.7175 9167	367.5337 1708
156	253.3307 8860	272.9203 9008	294.3942 7903	330.5763 7060	372.2090 5425
157	255.8085 5153	275.7398 5935	297.6022 3613	334.4689 1384	376.9311 4480
158	258.3007 6808	278.5781 2507	300.8342 5290	338.3955 1684	381.7004 5624
159	260.8075 2256	281.4353 1257	304.0905 0979	342.3564 7761	386.5174 6081
160	263.3288 9978	284.3115 4799	307.3711 8862	346.3520 9679	391.3826 3541
161	265.8649 8503	287.2069 5831	310.6764 7253	350.3826 7764	396.2964 6177
162	268.4158 6411	290.1216 7137	314.0065 4608	354.4485 2607	401.2594 2639
163	270.9816 2331	293.0558 1584	317.3615 9517	358.5499 5067	406.2720 2065
164	273.5623 4945	296.0095 2128	320.7418 0714	362.6872 6274	411.3347 4086
165	276.1581 2982	298.9829 1809	324.1473 7069	366.8607 7629	416.4480 8826
166	278.7690 5225	301.9761 3754	327.5784 7597	371.0708 0808	421.6125 6915
167	281.3952 0505	304.9893 1179	331.0353 1454	375.3176 7765	426.8286 9484
168	284.0366 7708	308.0225 7387	334.5180 7940	379.6017 0733	432.0969 8179
169	286.6935 5770	311.0760 5770	338.0269 6499	383.9232 2227	437.4179 5161
170	289.3659 3678	314.1498 9808	341.5621 6723	388.2825 5046	442.7921 3112
171	292.0539 0475	317.2442 3074	345.1238 8349	392.6800 2278	448.2200 5243
172	294.7575 5252	320.3591 9228	348.7123 1261	397.1159 7298	453.7022 5296
173	297.4769 7158	323.4949 2022	352.3276 5496	401.5907 3774	459.2392 7549
174	300.2122 5392	326.6515 5303	355.9701 1237	406.1046 5670	464.8316 6824
175	302.9634 9206	329.8292 3005	359.6398 8821	410.6580 7245	470.4799 8492
176	305.7307 7910	333.0280 9158	363.3371 8737	415.2513 3058	476.1847 8477
177	308.5142 0864	336.2482 7886	367.0622 1628	419.8847 7972	481.9465 3262
178	311.3138 7486	339.4899 3405	370.8151 8290	424.5587 7154	487.7660 9895
179	314.1298 7247	342.7532 0028	374.5962 9677	429.2736 6080	493.6437 5994
180	316.9622 9672	346.0382 2161	378.4057 6900	434.0298 0533	499.5801 9754
181	319.8112 4345	349.3451 4309	382.2438 1226	438.8275 6612	505.5759 9951
182	322.6768 0904	352.6741 1071	386.1106 4086	443.6673 0733	511.6317 5951
183	325.5590 9042	356.0252 7145	390.0064 7066	448.5493 9627	517.7480 7710
184	328.4581 8512	359.3987 7326	393.9315 1919	453.4742 0348	523.9255 5787
185	331.3741 9120	362.7947 6508	397.8860 0559	458.4421 0276	530.1648 1345
186	334.3072 0731	366.2133 9685	401.8701 5063	463.4534 7116	536.4664 6159
187	337.2573 3269	369.6548 1949	405.8841 7676	468.5086 8904	542.8311 2620
188	340.2246 6713	373.1191 8496	409.9283 0808	473.6081 4007	549.2594 3746
189	343.2093 1102	376.6066 4619	414.0027 7039	478.7522 1129	555.7520 3184
190	346.2113 6534	380.1173 5716	418.1077 9117	483.9412 9314	562.3095 5216
191	349.2309 3163	383.6514 7288	422.2435 9961	489.1757 7946	568.9326 4768
192	352.2681 1207	387.2091 4936	426.4104 2660	494.4560 6753	575.6219 7415
193	355.3230 0939	390.7905 4369	430.6085 0480	499.7825 5812	582.3781 9390
194	358.3957 2694	394.3958 1398	434.8380 6859	505.1556 5550	589.2019 7584
195	361.4863 6868	398.0251 1941	439.0993 5410	510.5757 6749	596.0939 9559
196	364.5950 3917	401.6786 2021	443.3925 9926	516.0433 0545	603.0549 3555
197	367.7218 4356	405.3564 7767	447.7180 4375	521.5586 8437	610.0854 8490
198	370.8668 8765	409.0588 5419	452.0759 2908	527.1223 2286	617.1863 3975
199	374.0302 7783	412.7859 1322	456.4664 9855	532.7346 4319	624.3582 0315
200	377.2121 2112	416.5378 1931	460.8899 9729	538.3960 7131	631.6017 8518

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \text{ (Continued)}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0112 5000	2.0125 0000	2.0137 5000	2.0150 0000	2.0175 0000
3	3.0338 7656	3.0376 5625	3.0414 3906	3.0452 2500	3.0528 0625
4	4.0680 0767	4.0756 2695	4.0832 5885	4.0909 0338	4.1062 3036
5	5.1137 7276	5.1265 7229	5.1394 0366	5.1522 6693	5.1780 8939
6	6.1713 0270	6.1906 5444	6.2100 7046	6.2295 5093	6.2687 0596
7	7.2407 2986	7.2680 3762	7.2954 5893	7.3229 9419	7.3784 0831
8	8.3221 8807	8.3588 8809	8.3957 7149	8.4328 3911	8.5075 3045
9	9.4158 1269	9.4633 7420	9.5112 1335	9.5593 3169	9.6564 1224
10	10.5217 4058	10.5816 6637	10.6419 9253	10.7027 2167	10.8253 9945
11	11.6401 1016	11.7139 3720	11.7883 1993	11.8632 6249	12.0148 4394
12	12.7710 6140	12.8603 6142	12.9504 0933	13.0412 1143	13.2251 0371
13	13.9147 3584	14.0211 1594	14.1284 7745	14.2368 2960	14.4565 4303
14	15.0712 7662	15.1963 7988	15.3227 4402	15.4503 8205	15.7095 3253
15	16.2408 2848	16.3863 3463	16.5334 3175	16.6821 3778	16.9844 4935
16	17.4235 3780	17.5911 6382	17.7607 6644	17.9323 6984	18.2816 7721
17	18.6195 5260	18.8110 5336	19.0049 7697	19.2013 5539	19.6016 0656
18	19.8290 2257	20.0461 9153	20.2662 9541	20.4893 7572	20.9446 3468
19	21.0520 9907	21.2967 6893	21.5449 5697	21.7967 1636	22.3111 6578
20	22.2889 3519	22.5629 7854	22.8412 0013	23.1236 6710	23.7016 1119
21	23.5396 8571	23.8450 1577	24.1552 6663	24.4705 2211	25.1163 8938
22	24.8045 0717	25.1430 7847	25.4874 0155	25.8375 7994	26.5559 2620
23	26.0835 5788	26.4573 6695	26.8378 5332	27.2251 4364	28.0206 5490
24	27.3769 9790	27.7880 8403	28.2068 7380	28.6335 2080	29.5110 1637
25	28.6849 8913	29.1354 3508	29.5947 1832	30.0630 2361	31.0274 5915
26	30.0076 9526	30.4996 2802	31.0016 4569	31.5139 6896	32.5704 3969
27	31.3452 8183	31.8808 7337	32.4279 1832	32.9866 7850	34.1404 2238
28	32.6979 1625	33.2793 8429	33.8738 0220	34.4814 7867	35.7378 7977
29	34.0657 6781	34.6953 7659	35.3395 6698	35.9987 0085	37.3632 9267
30	35.4490 0769	36.1290 6880	36.8254 8602	37.5386 8137	39.0171 5029
31	36.8478 0903	37.5806 8216	38.3318 3646	39.1017 6159	40.6999 5042
32	38.2623 4688	39.0504 4069	39.8588 9921	40.6882 8801	42.4121 9955
33	39.6927 9829	40.5385 7120	41.4069 5907	42.2986 1233	44.1544 1305
34	41.1393 4227	42.0453 0334	42.9763 0476	43.9330 9152	45.9271 1527
35	42.6021 5987	43.5708 6963	44.5672 2895	45.5920 8789	47.7308 3979
36	44.0814 3417	45.1155 0550	46.1800 2835	47.2759 6921	49.5661 2949
37	45.5773 5030	46.6794 4932	47.8150 0374	48.9851 0874	51.4335 3675
38	47.0900 9549	48.2629 4243	49.4724 6004	50.7198 8538	53.3336 2365
39	48.6198 5906	49.8662 2921	51.1527 0636	52.4806 8366	55.2669 6206
40	50.1668 3248	51.4895 5708	52.8560 5608	54.2678 9391	57.2341 3390
41	51.7312 0934	53.1331 7654	54.5828 2685	56.0819 1232	59.2357 3124
42	53.3131 8545	54.7973 4125	56.3333 4072	57.9231 4100	61.2723 5654
43	54.9129 5879	56.4823 0801	58.1079 2415	59.7919 8812	63.3446 2278
44	56.5307 2057	58.1883 3687	59.9069 0811	61.6888 6794	65.4531 5367
45	58.1667 0028	59.9156 9108	61.7306 2810	63.6142 0096	67.5985 8386
46	59.8210 7566	61.6646 3721	63.5794 2423	65.5684 1398	69.7815 5908
47	61.4940 6276	63.4354 4518	65.4536 4131	67.5519 4018	72.0027 3637
48	63.1858 7097	65.2283 8824	67.3536 2888	69.5652 1929	74.2627 8425
49	64.8967 1201	67.0437 4310	69.2797 4128	71.6086 9758	76.5623 8298
50	66.6268 0002	68.8817 8989	71.2323 3772	73.6828 2804	78.9022 2468

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \text{ (Continued)}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
51	68.3763 5152	70.7428 1226	73.2117 8237	75.7880 7046	81.2830 1361
52	70.1455 8548	72.6270 9741	75.2184 4437	77.9248 9152	83.7054 6635
53	71.9347 2332	74.5349 3613	77.2526 9798	80.0937 6489	86.1703 1201
54	73.7439 8895	76.4666 2283	79.3149 2258	82.2951 7136	88.6782 9247
55	75.5736 0883	78.4224 5562	81.4055 0277	84.5295 9893	91.2301 6259
56	77.4238 1193	80.4027 3631	83.5248 2843	86.7975 4292	93.8266 9043
57	79.2948 2981	82.4077 7052	85.6732 9482	89.0995 0606	96.4686 5752
58	81.1868 9665	84.4378 6765	87.8513 0262	91.4359 9865	99.1568 5902
59	83.1002 4923	86.4933 4099	90.0592 5804	93.8075 3863	101.8921 0405
60	85.0351 2704	88.5745 0776	92.2975 7283	96.2146 5171	104.6752 1588
61	86.9917 7222	90.6816 8910	94.5666 6446	98.6578 7149	107.5070 3215
62	88.9704 2966	92.8152 1022	96.8669 5610	101.1377 3956	110.3884 0522
63	90.9713 4699	94.9754 0034	99.1988 7674	103.6548 0565	113.3202 0231
64	92.9947 7464	97.1625 9285	101.5628 6130	106.2096 2774	116.3033 0585
65	95.0409 6586	99.3771 2526	103.9593 5064	108.8027 7215	119.3386 1370
66	97.1101 7672	101.6193 3933	106.3887 9171	111.4348 1374	122.4270 3944
67	99.2026 6621	103.8895 8107	108.8516 3760	114.1063 3594	125.5695 1263
68	101.3186 9621	106.1882 0083	111.3483 4761	116.8179 3098	128.7669 7910
69	103.4584 3154	108.5155 5334	113.8793 8739	119.5701 9995	132.0204 0124
70	105.6224 4002	110.8719 9776	116.4452 2897	122.3637 5295	135.3307 5826
71	107.8106 9247	113.2578 9773	119.0463 5087	125.1992 0924	138.6990 4653
72	110.0235 6276	115.6736 2145	121.6832 3819	128.0771 9738	142.1262 7984
73	112.2613 2784	118.1195 4172	124.3563 8272	130.9983 5534	145.6134 8974
74	114.5242 6778	120.5960 3599	127.0662 8298	133.9633 3067	149.1617 2581
75	116.8126 6579	123.1034 8644	129.8134 4437	136.9727 8063	152.7720 5601
76	119.1268 0828	125.6422 8002	132.5983 7923	140.0273 7234	156.4455 6699
77	121.4669 8487	128.2128 0852	135.4216 0695	143.1277 8292	160.1833 6441
78	123.8334 8845	130.8154 6863	138.2836 5404	146.2746 9967	163.9865 7329
79	126.2266 1520	133.4506 6199	141.1850 5429	149.4688 2016	167.8563 3832
80	128.6466 6462	136.1187 9526	144.1263 4878	152.7108 5247	171.7938 2424
81	131.0939 3960	138.8202 8020	147.1080 8608	156.0015 1525	175.8002 1617
82	133.5687 4642	141.5555 3370	150.1308 2226	159.3415 3798	179.8767 1995
83	136.0713 9481	144.3249 7787	153.1951 2107	162.7316 6105	184.0245 6255
84	138.6021 9801	147.1290 4010	156.3015 5398	166.1726 3597	188.2449 9239
85	141.1614 7273	149.9681 5310	159.4507 0035	169.6652 2551	192.5392 7976
86	143.7495 3930	152.8427 5501	162.6431 4748	173.2102 0389	196.9087 1716
87	146.3667 2162	155.7532 8945	165.8794 9076	176.8083 5695	201.3546 1971
88	149.0133 4724	158.7002 0557	169.1603 3375	180.4604 8230	205.8783 2555
89	151.6897 4739	161.6839 5814	172.4862 8834	184.1673 8954	210.4811 9625
90	154.3962 5705	164.7050 0762	175.8579 7481	187.9299 0038	215.1646 1718
91	157.1332 1494	167.7638 2021	179.2760 2196	191.7488 4889	219.9299 9798
92	159.9009 6361	170.8608 6796	182.7410 6726	195.6250 8162	224.7787 7295
93	162.6998 4945	173.9966 2881	186.2537 5694	199.5594 5784	229.7124 0148
94	165.5302 2276	177.1715 8667	189.8147 4610	203.5528 4971	234.7323 6850
95	168.3924 3776	180.3862 3151	193.4246 9886	207.6061 4246	239.8401 8495
96	171.2868 5269	183.6410 5940	197.0842 8847	211.7202 3459	245.0373 8819
97	174.2138 2978	186.9365 7264	200.7941 9743	215.8960 3811	250.3255 4248
98	177.1737 3537	190.2732 7980	204.5551 1765	220.1344 7868	255.7062 3947
99	180.1669 3989	193.6516 9580	208.3677 5051	224.4364 9586	261.1810 9866
100	183.1938 1796	197.0723 4200	212.2328 0708	228.8030 4330	266.7517 6789

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \text{ (Continued)}$$

n	2%	2¼%	2½%	2¾%	3%
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0200 0000	2.0225 0000	2.0250 0000	2.0275 0000	2.0300 0000
3	3.0604 0000	3.0680 0625	3.0756 2500	3.0832 5625	3.0909 0000
4	4.1216 0800	4.1370 3639	4.1525 1563	4.1680 4580	4.1836 2700
5	5.2040 4016	5.2301 1971	5.2563 2852	5.2826 6706	5.3091 3581
6	6.3081 2096	6.3477 9740	6.3877 3673	6.4279 4040	6.4684 0988
7	7.4342 8338	7.4906 2284	7.5474 3015	7.6047 0876	7.6624 6218
8	8.5829 6905	8.6591 6186	8.7361 1590	8.8138 3825	8.8923 3605
9	9.7546 2843	9.8539 9300	9.9545 1880	10.0562 1880	10.1591 0613
10	10.9497 2100	11.0757 0784	11.2033 8177	11.3327 6482	11.4638 7931
11	12.1687 1542	12.3249 1127	12.4834 6631	12.6444 1585	12.8077 9569
12	13.4120 8973	13.6022 2177	13.7955 5297	13.9921 3729	14.1920 2956
13	14.6803 3152	14.9082 7176	15.1404 4179	15.3769 2107	15.6177 9045
14	15.9739 3815	16.2437 0788	16.5189 5284	16.7997 8639	17.0863 2416
15	17.2934 1692	17.6091 9130	17.9319 2666	18.2617 8052	18.5989 1389
16	18.6392 8525	19.0053 9811	19.3802 2483	19.7639 7948	20.1568 8130
17	20.0120 7096	20.4330 1957	20.8647 3045	21.3074 8892	21.7615 8774
18	21.4123 1238	21.8927 6251	22.3863 4871	22.8934 4487	23.4144 3537
19	22.8405 5863	23.3853 4966	23.9460 0743	24.5230 1460	25.1168 6844
20	24.2973 6980	24.9115 2003	25.5446 5761	26.1973 9750	26.8703 7449
21	25.7833 1719	26.4720 2923	27.1832 7405	27.9178 2593	28.6764 8572
22	27.2989 8354	28.0676 4989	28.8628 5590	29.6855 6615	30.5367 8030
23	28.8449 6321	29.6991 7201	30.5844 2730	31.5019 1921	32.4528 8370
24	30.4218 6247	31.3674 0338	32.3490 3798	33.3682 2199	34.4264 7022
25	32.0302 9972	33.0731 6996	34.1577 6393	35.2858 4810	36.4592 6432
26	33.6709 0572	34.8173 1628	36.0117 0803	37.2562 0892	38.5530 4225
27	35.3443 2383	36.6007 0590	37.9120 0073	39.2807 5467	40.7096 3352
28	37.0512 1031	38.4242 2178	39.8598 0075	41.3609 7542	42.9309 2252
29	38.7922 3451	40.2887 6677	41.8562 9577	43.4984 0224	45.2188 5020
30	40.5680 7921	42.1952 6402	43.9027 0316	45.6946 0830	47.5754 1571
31	42.3794 4079	44.1446 5746	46.0002 7074	47.9512 1003	50.0026 7818
32	44.2270 2961	46.1379 1226	48.1502 7751	50.2698 6831	52.5027 5852
33	46.1115 7020	48.1760 1528	50.3540 3445	52.6522 8969	55.0778 4128
34	48.0338 0160	50.2599 7563	52.6128 8531	55.1002 2765	57.7301 7652
35	49.9944 7763	52.3908 2508	54.9282 0744	57.6154 8391	60.4620 8181
36	51.9943 6719	54.5696 1864	57.3014 1263	60.1999 0972	63.2759 4427
37	54.0342 5453	56.7974 3506	59.7339 4794	62.8554 0724	66.1742 2259
38	56.1149 3962	59.0753 7735	62.2272 9664	65.5839 3094	69.1594 4927
39	58.2372 3841	61.4045 7334	64.7829 7906	68.3874 8904	72.2342 3275
40	60.4019 8318	63.7861 7624	67.4025 5354	71.2681 4499	75.4012 5973
41	62.6100 2284	66.2213 6521	70.0876 1737	74.2280 1898	78.6632 9753
42	64.8622 2330	68.7113 4592	72.8398 0781	77.2692 8950	82.0231 9645
43	67.1594 6777	71.2573 5121	75.6608 0300	80.3941 9496	85.4838 9234
44	69.5026 5712	73.8606 4161	78.5523 2308	83.6050 3532	89.0484 0911
45	71.8927 1027	76.5225 0605	81.5161 3116	86.9041 7379	92.7198 6139
46	74.3305 6447	79.2442 6243	84.5540 3443	90.2940 3857	96.5014 5723
47	76.8171 7576	82.0272 5834	87.6678 8530	93.7771 2463	100.3965 0095
48	79.3535 1927	84.8728 7165	90.8595 8243	97.3559 9556	104.4083 9598
49	81.9405 8966	87.7825 1126	94.1310 7199	101.0332 8544	108.5406 4785
50	84.5794 0145	90.7576 1776	97.4843 4879	104.8117 0079	112.7968 6729

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \quad (Continued)$$

n	2%	2¼%	2½%	2¾%	3%
51	87.2709 8948	93.7996 6416	100.9214 5751	108.6940 2256	117.1807 7331
52	90.0164 0927	96.9101 5661	104.4444 9395	112.6831 0818	121.6961 9651
53	92.8167 3746	100.0906 3513	108.0556 0629	116.7818 9365	126.3470 8240
54	95.6730 7221	103.3426 7442	111.7569 9645	120.9933 9573	131.1374 9488
55	98.5865 3365	106.6678 8460	115.5509 2136	125.3207 1411	136.0716 1972
56	101.5582 6432	110.0679 1200	119.4396 9440	129.7670 3375	141.1537 6831
57	104.5894 2961	113.5444 4002	123.4256 8676	134.3356 2718	146.3883 8136
58	107.6812 1820	117.0991 8992	127.5113 2893	139.0298 5692	151.7800 3280
59	110.8348 4257	120.7339 2169	131.6991 1215	143.8531 7799	157.3334 3379
60	114.0515 3942	124.4504 3493	135.9915 8995	148.8091 4038	163.0534 3680
61	117.3325 7021	128.2505 6972	140.3913 7970	153.9013 9174	168.9450 3991
62	120.6792 2161	132.1362 0754	144.9011 6419	159.1336 8002	175.0133 9110
63	124.0928 0604	136.1092 7221	149.5236 9330	164.5098 5622	181.2637 9284
64	127.5746 6216	140.1717 3083	154.2617 8563	170.0338 7726	187.7017 0662
65	131.1261 5541	144.3255 9477	159.1183 3027	175.7098 0889	194.3327 5782
66	134.7486 7852	148.5729 2066	164.0962 8853	181.5418 2863	201.1627 4055
67	138.4436 5209	152.9158 1137	169.1986 9574	187.5342 2892	208.1976 2277
68	142.2125 2513	157.3564 1713	174.4286 6314	193.6914 2022	215.4435 5145
69	146.0567 7563	161.8969 3651	179.7893 7971	200.0179 3427	222.9068 5800
70	149.9779 1114	166.5396 1758	185.2841 1421	206.5184 2746	230.5940 6374
71	153.9774 6937	171.2867 5898	190.9162 1706	213.1976 8422	238.5118 8565
72	158.0570 1875	176.1407 1106	196.6891 2249	220.0606 2054	246.6672 4222
73	162.2181 5913	181.1038 7705	202.6063 5055	227.1122 8760	255.0672 5949
74	166.4625 2231	186.1787 1429	208.6715 0931	234.3578 7551	263.7192 7727
75	170.7917 7276	191.3677 3536	214.8882 9705	241.8027 1709	272.6308 5559
76	175.2076 0821	196.6735 0941	221.2605 0447	249.4522 9181	281.8097 8126
77	179.7117 6038	202.0986 6337	227.7920 1709	257.3122 2983	291.2640 7469
78	184.3059 9558	207.6458 8329	234.4868 1751	265.3883 1615	301.0019 9693
79	188.9921 1549	213.3179 1567	241.3489 8795	273.6864 9485	311.0320 5684
80	193.7719 5780	219.1175 6877	248.3827 1265	282.2128 7345	321.3630 1855
81	198.6473 9696	225.0477 1407	255.5922 8047	290.9737 2747	332.0039 0910
82	203.6203 4490	231.1112 8763	262.9820 8748	299.9755 0498	342.9640 2638
83	208.6927 5180	237.3112 9160	270.5566 3966	309.2248 3137	354.2529 4717
84	213.8666 0683	243.6507 9567	278.3205 5566	318.7285 1423	365.8805 3558
85	219.1439 3897	250.1329 3857	286.2785 6955	328.4935 4837	377.8569 5165
86	224.5268 1775	256.7609 2969	294.4355 3379	338.5271 2095	390.1926 6020
87	230.0173 5411	263.5380 5060	302.7964 2213	348.8366 1678	402.8984 4001
88	235.6177 0119	270.4676 5674	311.3663 3268	359.4296 2374	415.9853 9321
89	241.3300 5521	277.5531 7902	320.1504 9100	370.3139 3839	429.4649 5500
90	247.1566 5632	284.7981 2555	329.1542 5328	381.4975 7170	443.3489 0365
91	253.0997 8944	292.2060 8337	338.3831 0961	392.9887 5492	457.6493 7076
92	259.1617 8523	299.7807 2025	347.8426 8735	404.7959 4568	472.3788 5189
93	265.3450 2094	307.5257 8645	357.5387 5453	416.9278 3418	487.5502 1744
94	271.6519 2135	315.4451 1665	367.4772 2339	429.3933 4962	503.1767 2397
95	278.0849 5978	323.5426 3177	377.6641 5398	442.2016 6674	519.2720 2568
96	284.6466 5898	331.8223 4099	388.1057 5783	455.3622 1257	535.8501 8645
97	291.3395 9216	340.2883 4366	398.8084 0177	468.8846 7342	552.9256 9205
98	298.1663 8400	348.9448 3139	409.7786 1182	482.7790 0194	570.5134 6281
99	305.1297 1168	357.7960 9010	421.0230 7711	497.0554 2449	588.6288 6669
100	312.2323 0591	366.8465 0213	432.5486 5404	511.7244 4867	607.2877 3270

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \text{ (Continued)}$$

n	3½%	4%	4½%	5%	5½%
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0350 0000	2.0400 0000	2.0450 0000	2.0500 0000	2.0550 0000
3	3.1062 2500	3.1216 0000	3.1370 2500	3.1525 0000	3.1680 2500
4	4.2149 4288	4.2464 6400	4.2781 9113	4.3101 2500	4.3422 6638
5	5.3624 6588	5.4163 2256	5.4707 0973	5.5256 3125	5.5810 9103
6	6.5501 5218	6.6329 7546	6.7168 9166	6.8019 1281	6.8880 5103
7	7.7794 0751	7.8982 9448	8.0191 5179	8.1420 0845	8.2668 9384
8	9.0516 8677	9.2142 2626	9.3800 1362	9.5491 0888	9.7215 7300
9	10.3684 9581	10.5827 9531	10.8021 1423	11.0265 6432	11.2562 5951
10	11.7313 9316	12.0061 0712	12.2882 0937	12.5778 9254	12.8753 5379
11	13.1419 9192	13.4863 5141	13.8411 7879	14.2067 8716	14.5834 9825
12	14.6019 6164	15.0258 0546	15.4640 3184	15.9171 2652	16.3855 9065
13	16.1130 3030	16.6268 3768	17.1599 1327	17.7129 8285	18.2867 9814
14	17.6769 8636	18.2919 1119	18.9321 0937	19.5986 3199	20.2925 7203
15	19.2956 8088	20.0235 8764	20.7840 5429	21.5785 6359	22.4086 6350
16	20.9710 2971	21.8245 3114	22.7193 3673	23.6574 9177	24.6411 3999
17	22.7050 1575	23.6975 1239	24.7417 0689	25.8403 6636	26.9964 0269
18	24.4996 9130	25.6454 1288	26.8550 8370	28.1323 8467	29.4812 0483
19	26.3571 8050	27.6712 2940	29.0635 6246	30.5390 0391	32.1026 7110
20	28.2796 8181	29.7780 7858	31.3714 2277	33.0659 5410	34.8683 1801
21	30.2694 7068	31.9692 0172	33.7831 3680	35.7192 5181	37.7860 7550
22	32.3289 0215	34.2479 6979	36.3033 7795	38.5052 1440	40.8643 0965
23	34.4604 1373	36.6178 8858	38.9370 2996	41.4304 7512	44.1118 4669
24	36.6665 2821	39.0826 0412	41.6891 9631	44.5019 9887	47.5379 9825
25	38.9498 5669	41.6459 0829	44.5652 1015	47.7270 9882	51.1525 8816
26	41.3131 0168	44.3117 4462	47.5706 4460	51.1134 5376	54.9659 8051
27	43.7590 6024	47.0842 1440	50.7113 2361	54.6691 2645	58.9891 0943
28	46.2906 2734	49.9675 8298	53.9933 3317	58.4025 8277	63.2335 1045
29	48.9107 9930	52.9662 8630	57.4230 3316	62.3227 1191	67.7113 5353
30	51.6226 7728	56.0849 3775	61.0070 6966	66.4388 4750	72.4354 7797
31	54.4294 7098	59.3283 3526	64.7523 8779	70.7607 8988	77.4194 2926
32	57.3345 0247	62.7014 6867	68.6662 4524	75.2988 2937	82.6774 9787
33	60.3412 1005	66.2095 2742	72.7562 2628	80.0637 7084	88.2247 6025
34	63.4531 5240	69.8579 0851	77.0302 5646	85.0669 5938	94.0771 2207
35	66.6740 1274	73.6522 2486	81.4966 1800	90.3203 0735	100.2513 6378
36	70.0076 0318	77.5983 1385	86.1639 6581	95.8363 2272	106.7651 8879
37	73.4578 6930	81.7022 4640	91.0413 4427	101.6281 3886	113.6372 7417
38	77.0288 9472	85.9703 3626	96.1382 0476	107.7095 4580	120.8873 2425
39	80.7249 0604	90.4091 4971	101.4644 2398	114.0950 2309	128.5361 2708
40	84.5502 7775	95.0255 1570	107.0303 2306	120.7997 7424	136.6056 1407
41	88.5095 3747	99.8265 3633	112.8466 8760	127.8397 6295	145.1189 2285
42	92.6073 7128	104.8195 9778	118.9247 8854	135.2317 5110	154.1004 6360
43	96.8486 2928	110.0123 8169	125.2764 0402	142.9933 3866	163.5759 8910
44	101.2383 3130	115.4128 7696	131.9138 4220	151.1430 0559	173.5726 6850
45	105.7816 7290	121.0293 9204	138.8499 6510	159.7001 5587	184.1191 6527
46	110.4840 3145	126.8705 6772	146.0982 1353	168.6851 6366	195.2457 1936
47	115.3509 7255	132.9453 9043	153.6726 3314	178.1194 2185	206.9842 3392
48	120.3882 5659	139.2632 0604	161.5879 0163	188.0253 9294	219.3683 6679
49	125.6018 4557	145.8337 3429	169.8593 5720	198.4266 6259	232.4336 2696
50	130.9979 1016	152.6670 8366	178.5030 2828	209.3479 9572	246.2174 7645

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \text{ (Continued)}$$

n	3½ %	4 %	4½ %	5 %	5½ %
51	136.5828 3702	159.7737 6700	187.5356 6455	220.8153 9550	260.7594 3765
52	142.3632 3631	167.1647 1768	196.9747 6946	232.8561 6528	276.1012 0672
53	148.3459 4958	174.8513 0639	206.8386 3408	245.4989 7354	292.2867 7309
54	154.5380 5782	182.8453 5865	217.1463 7262	258.7739 2222	309.3625 4561
55	160.9468 8984	191.1591 7299	227.9179 5938	272.7126 1833	327.3774 8562
56	167.5800 3099	199.8055 3991	239.1742 6756	287.3482 4924	346.3832 4733
57	174.4453 3207	208.7977 6151	250.9371 0960	302.7156 6171	366.4343 2593
58	181.5509 1869	218.1496 7197	263.2292 7953	318.8514 4479	387.5882 1386
59	188.9052 0085	227.8756 5885	276.0745 9711	335.7940 1703	409.9055 6562
60	196.5168 8288	237.9906 8520	289.4979 5398	353.5837 1788	433.4503 7173
61	204.3949 7378	248.5103 1261	303.5253 6190	372.2629 0378	458.2901 4217
62	212.5487 9786	259.4507 2511	318.1840 0319	391.8760 4897	484.4960 9999
63	220.9880 0579	270.8287 5412	333.5022 8333	412.4698 5141	512.1433 8549
64	229.7225 8599	282.6619 0428	349.5098 8608	434.0933 4398	541.3112 7170
65	238.7628 7650	294.9683 8045	366.2378 3096	456.7980 1118	572.0833 9164
66	248.1195 7718	307.7671 1567	383.7185 3335	480.6379 1174	604.5479 7818
67	257.8037 6238	321.0778 0030	401.9858 6735	505.6698 0733	638.7981 1698
68	267.8268 9406	334.9209 1231	421.0752 3138	531.9532 9770	674.9320 1341
69	278.2008 3535	349.3177 4880	441.0236 1679	559.5509 6258	713.0532 7415
70	288.9378 6459	364.2904 5876	461.8696 7955	588.5285 1071	753.2712 0423
71	300.0506 8985	379.8620 7711	483.6538 1513	618.9549 3625	795.7011 2046
72	311.5524 6400	396.0565 6019	506.4182 3681	650.9026 8306	840.4646 8209
73	323.4568 0024	412.8988 2260	530.2070 5747	684.4478 1721	887.6902 3960
74	335.7777 8824	430.4147 7550	555.0663 7505	719.6702 0807	937.5132 0278
75	348.5300 1083	448.6313 6652	581.0443 6193	756.6537 1848	990.0764 2893
76	361.7285 6121	467.5766 2118	608.1913 5822	795.4864 0440	1045.5306 3252
77	375.3890 6085	487.2796 8603	636.5599 6934	836.2607 2462	1104.0348 1731
78	389.5276 7798	507.7708 7347	666.2051 6796	879.0737 6085	1165.7567 3226
79	404.1611 4671	529.0817 0841	697.1844 0052	924.0274 4889	1230.8733 5254
80	419.3067 8685	551.2449 7675	729.5576 9854	971.2288 2134	1299.5713 8693
81	434.9825 2439	574.2947 7582	763.3877 9497	1020.7902 6240	1372.0478 1321
82	451.2069 1274	598.2665 6685	798.7402 4575	1072.8297 7552	1448.5104 4294
83	467.9991 5469	623.1972 2952	835.6835 5680	1127.4712 6430	1529.1785 1730
84	485.3791 2510	649.1251 1870	874.2893 1686	1184.8448 2752	1614.2833 3575
85	503.3673 9448	676.0901 2345	914.6323 3612	1245.0870 6889	1704.0689 1921
86	521.9852 5329	704.1337 2839	956.7907 9125	1308.3414 2234	1798.7927 0977
87	541.2547 3715	733.2990 7753	1000.8463 7685	1374.7584 9345	1898.7263 0881
88	561.1986 5295	763.6310 4063	1046.8844 6381	1444.4964 1812	2004.1562 5579
89	581.8406 0581	795.1762 8225	1094.9942 6468	1517.7212 3903	2115.3848 4986
90	603.2050 2701	827.9833 3354	1145.2690 0659	1594.6073 0098	2232.7310 1660
91	625.3172 0295	862.1026 6688	1197.8061 1189	1675.3376 6603	2356.5312 2252
92	648.2033 0506	897.5867 7356	1252.7073 8692	1760.1045 4933	2487.1404 3976
93	671.8904 2073	934.4902 4450	1310.0792 1933	1849.1097 7680	2624.9331 6394
94	696.4065 8546	972.8698 5428	1370.0327 8420	1942.5652 6564	2770.3044 8796
95	721.7808 1595	1012.7846 4845	1432.6842 5949	2040.6935 2892	2923.6712 3480
96	748.0431 4451	1054.2960 3439	1498.1550 5117	2143.7282 0537	3085.4731 5271
97	775.2246 5457	1097.4678 7577	1566.5720 2847	2251.9146 1564	3256.1741 7611
98	803.3575 1748	1142.3665 9080	1638.0677 6976	2365.5103 4642	3436.2637 5580
99	832.4750 3059	1189.0612 5443	1712.7808 1939	2484.7858 6374	3626.2582 6237
100	862.6116 5666	1237.6237 0461	1790.8559 5627	2610.0251 5693	3826.7024 6680

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \quad (Continued)$$

n	6 %	6½ %	7 %	7½ %	8 %
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0600 0000	2.0650 0000	2.0700 0000	2.0750 0000	2.0800 0000
3	3.1836 0000	3.1992 2500	3.2149 0000	3.2306 2500	3.2464 0000
4	4.3746 1600	4.4071 7463	4.4399 4300	4.4729 2188	4.5061 1200
5	5.6370 9296	5.6936 4098	5.7507 3901	5.8083 9102	5.8666 0096
6	6.9753 1854	7.0637 2764	7.1532 9074	7.2440 2034	7.3359 2904
7	8.3938 3765	8.5228 6994	8.6540 2109	8.7873 2187	8.9228 0336
8	9.8974 6791	10.0768 5648	10.2598 0257	10.4463 7101	10.6366 2763
9	11.4913 1598	11.7318 5215	11.9779 8875	12.2298 4883	12.4875 5784
10	13.1807 9494	13.4944 2254	13.8164 4796	14.1470 8750	14.4865 6247
11	14.9716 4264	15.3715 6001	15.7835 9932	16.2081 1906	16.6454 8746
12	16.8699 4120	17.3707 1141	17.8884 5127	18.4237 2799	18.9771 2646
13	18.8821 3767	19.4998 0765	20.1406 4286	20.8055 0759	21.4952 9658
14	21.0150 6593	21.7672 9515	22.5504 8786	23.3659 2066	24.2149 2030
15	23.2759 6988	24.1821 6933	25.1290 2201	26.1183 6470	27.1521 1393
16	25.6725 2808	26.7540 1034	27.8880 5355	29.0772 4206	30.3242 8304
17	28.2128 7976	29.4930 2101	30.8402 1730	32.2580 3521	33.7502 2569
18	30.9056 5255	32.4100 6738	33.9990 3251	35.6773 8785	37.4502 4374
19	33.7599 9170	35.5167 2176	37.3789 6479	39.3531 9194	41.4462 6324
20	36.7855 9120	38.8253 0867	40.9954 9232	43.3046 8134	45.7619 6430
21	39.9927 2668	42.3489 5373	44.8651 7678	47.5525 3244	50.4229 2144
22	43.3922 9028	46.1016 3573	49.0057 3916	52.1189 7237	55.4567 5516
23	46.9958 2769	50.0982 4205	53.4361 4090	57.0278 9530	60.8932 9557
24	50.8155 7735	54.3546 2778	58.1766 7076	62.3049 8744	66.7647 5922
25	54.8645 1200	58.8876 7859	63.2490 3772	67.9778 6150	73.1059 3995
26	59.1563 8272	63.7153 7769	68.6764 7036	74.0762 0112	79.9544 1515
27	63.7057 6568	68.8568 7725	74.4838 2328	80.6319 1620	87.3507 6836
28	68.5281 1162	74.3325 7427	80.6976 9091	87.6793 0991	95.3388 2983
29	73.6397 9832	80.1641 9159	87.3465 2927	95.2552 5816	103.9659 3622
30	79.0581 8622	86.3748 6405	94.4607 8632	103.3994 0252	113.2832 1111
31	84.8016 7739	92.9892 3021	102.0730 4137	112.1543 5771	123.3458 6800
32	90.8897 7803	100.0335 3017	110.2181 5426	121.5659 3454	134.2135 3744
33	97.3431 6471	107.5357 0963	118.9334 2506	131.6833 7963	145.9506 2044
34	104.1837 5460	115.5255 3076	128.2587 6481	142.5596 3310	158.6266 7007
35	111.4347 7987	124.0346 9026	138.2368 7835	154.2516 0558	172.3168 0368
36	119.1208 6666	133.0969 4513	148.9134 5984	166.8204 7600	187.1021 4797
37	127.2681 1866	142.7482 4656	160.3374 0202	180.3320 1170	203.0703 1981
38	135.9042 0578	153.0268 8259	172.5610 2017	194.8569 1258	220.3159 4540
39	145.0584 5813	163.9736 2996	185.6402 9158	210.4711 8102	238.9412 2103
40	154.7619 6562	175.6319 1590	199.6351 1199	227.2565 1960	259.0565 1871
41	165.0476 8356	188.0479 9044	214.6095 6983	245.3007 5857	280.7810 4021
42	175.9505 4457	201.2711 0981	230.6322 3972	264.6983 1546	304.2435 2342
43	187.5075 7724	215.3537 3195	247.7764 9650	285.5506 8912	329.5830 0530
44	199.7580 3188	230.3517 2453	266.1208 5125	307.9669 9080	356.9496 4572
45	212.7435 1379	246.3245 8662	285.7493 1084	332.0645 1511	386.5056 1738
46	226.5081 2462	263.3356 8475	306.7517 6260	357.9693 5375	418.4260 6677
47	241.0986 1210	281.4525 0426	329.2243 8598	385.8170 5528	452.9001 5211
48	256.5645 2882	300.7469 1704	353.2700 9300	415.7533 3442	490.1321 6428
49	272.9584 0055	321.2954 6665	378.9989 9951	447.9348 3451	530.3427 3742
50	290.3359 0458	343.1796 7198	406.5289 2947	482.5299 4709	573.7701 5642

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \text{ (Continued)}$$

n	8½%	9%	9½%	10%	10½%
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0850 0000	2.0900 0000	2.0950 0000	2.1000 0000	2.1050 0000
3	3.2622 2500	3.2781 0000	3.2940 2500	3.3100 0000	3.3260 2500
4	4.5395 1413	4.5731 2900	4.6069 5738	4.6410 0000	4.6752 5763
5	5.9253 7283	5.9847 1061	6.0446 1833	6.1051 0000	6.1661 5968
6	7.4290 2952	7.5233 3456	7.6188 5707	7.7156 1000	7.8136 0644
7	9.0604 9702	9.2004 3468	9.3426 4849	9.4871 7100	9.6340 3512
8	10.8306 3927	11.0284 7380	11.2302 0009	11.4358 8810	11.6456 0881
9	12.7512 4361	13.0210 3644	13.2970 6910	13.5794 7691	13.8683 9773
10	14.8350 9932	15.1929 2972	15.5602 9067	15.9374 2460	16.3245 7949
11	17.0960 8276	17.5602 9339	18.0385 1828	18.5311 6706	19.0386 6034
12	19.5492 4979	20.1407 1980	20.7521 7752	21.3842 8377	22.0377 1967
13	22.2109 3603	22.9533 8458	23.7236 3438	24.5227 1214	25.3516 8024
14	25.0988 6559	26.0191 8919	26.9773 7965	27.9749 8336	29.0136 0666
15	28.2322 6916	29.3609 1622	30.5402 3072	31.7724 8169	33.0600 3536
16	31.6320 1204	33.0033 9868	34.4415 5263	35.9497 2986	37.5313 3908
17	35.3207 3306	36.9737 0456	38.7135 0013	40.5447 0285	42.4721 2968
18	39.3229 9538	41.3013 3797	43.3912 8265	45.5991 7313	47.9317 0330
19	43.6654 4998	46.0184 5839	48.5134 5450	51.1590 9045	53.9645 3214
20	48.3770 1323	51.1601 1964	54.1222 3267	57.2749 9949	60.6308 0802
21	53.4890 5936	56.7645 3041	60.2638 4478	64.0024 9944	67.9970 4286
22	59.0356 2940	62.8733 3815	66.9889 1003	71.4027 4939	76.1367 3236
23	65.0536 5790	69.5319 3858	74.3528 5649	79.5430 2433	85.1310 8926
24	71.5832 1882	76.7898 1305	82.4163 7785	88.4973 2676	95.0698 5363
25	78.6677 9242	84.7008 9623	91.2459 3375	98.3470 5943	106.0521 8826
26	86.3545 5478	93.3239 7689	100.9142 9745	109.1817 6538	118.1876 6803
27	94.6946 9193	102.7231 3481	111.5011 5571	121.0999 4192	131.5973 7317
28	103.7437 4075	112.9682 1694	123.0937 6551	134.2099 3611	146.4150 9736
29	113.5619 5871	124.1353 5646	135.7876 7323	148.6309 2972	162.7886 8258
30	124.2147 2520	136.3075 3855	149.6875 0218	164.4940 2269	180.8814 9425
31	135.7729 7684	149.5752 1702	164.9078 1489	181.9434 2496	200.8740 5114
32	148.3136 7987	164.0369 8655	181.5740 5731	201.1377 6745	222.9658 2651
33	161.9203 4266	179.8003 1534	199.8235 9275	222.2515 4420	247.3772 3830
34	176.6835 7179	196.9823 4372	219.8068 3406	245.4766 9862	274.3518 4832
35	192.7016 7539	215.7107 5465	241.6884 8330	271.0243 6848	304.1587 9239
36	210.0813 1780	236.1247 2257	265.6488 8921	299.1268 0533	337.0954 6560
37	228.9382 2981	258.3759 4760	291.8855 3369	330.0394 8586	373.4904 8948
38	249.3979 7935	282.6297 8288	320.6146 5939	364.0434 3445	413.7069 9088
39	271.5968 0759	309.0664 6334	352.0730 5203	401.4477 7789	458.1462 2492
40	295.6825 3624	337.8824 4504	386.5199 9197	442.5925 5568	507.2515 7854
41	321.8155 5182	369.2918 6510	424.2393 9121	487.8518 1125	561.5129 9428
42	350.1698 7372	403.5281 3296	465.5421 3337	537.6369 9237	621.4718 5868
43	380.9343 1299	440.8456 6492	510.7686 3604	592.4006 9161	687.7264 0385
44	414.3137 2959	481.5217 7477	560.2916 5647	652.6407 6077	760.9376 7625
45	450.5303 9661	525.8587 3450	614.5193 6383	718.9048 3685	841.8361 3225
46	489.8254 8032	574.1860 2060	673.8987 0340	791.7953 2054	931.2289 2614
47	532.4606 4615	626.8627 6245	738.9190 8022	871.9748 5259	1030.0079 6339
48	578.7198 0107	684.2804 1107	810.1163 9284	960.1723 3785	1139.1587 9954
49	628.9109 8416	746.8656 4807	888.0774 5016	1057.1895 7163	1259.7704 7349
50	683.3684 1782	815.0835 5640	973.4448 0793	1163.9085 2880	1393.0463 7321

* Source : Taken from, MATHEMATICS OF FINANCE, Third Ed. : Hummel and Seebeck, 1971, pp. 292 - 307.

Appendix E

Annuity Table

Values of $a_{\overline{n}|i}$

n	¼ %	7/24 %	⅓ %	5/12 %	½ %
1	0.9975 0623	0.9970 9182	0.9966 7774	0.9958 5062	0.9950 2488
2	1.9925 2492	1.9912 8390	1.9900 4426	1.9875 6908	1.9850 9938
3	2.9850 6227	2.9825 8470	2.9801 1056	2.9751 7253	2.9702 4814
4	3.9751 2446	3.9710 0261	3.9668 8760	3.9586 7804	3.9504 9566
5	4.9627 1766	4.9565 4602	4.9503 8631	4.9381 0261	4.9258 6633
6	5.9478 4804	5.9392 2328	5.9306 1759	5.9134 6318	5.8963 8441
7	6.9305 2174	6.9190 4274	6.9075 9228	6.8847 7661	6.8620 7404
8	7.9107 4487	7.8960 1270	7.8813 2121	7.8520 5970	7.8229 5924
9	8.8885 2357	8.8701 4146	8.8518 1516	8.8153 2916	8.7790 6392
10	9.8638 6391	9.8414 3726	9.8190 8487	9.7746 0165	9.7304 1186
11	10.8367 7198	10.8099 0836	10.7831 4107	10.7298 9376	10.6770 2673
12	11.8072 5384	11.7755 6297	11.7439 9442	11.6812 2200	11.6189 3207
13	12.7753 1555	12.7384 0928	12.7016 5557	12.6286 0283	12.5561 5131
14	13.7409 6314	13.6984 5545	13.6561 3512	13.5720 5261	13.4887 0777
15	14.7042 0264	14.6557 0963	14.6074 4364	14.5115 8766	14.4166 2465
16	15.6650 4004	15.6101 7994	15.5555 9167	15.4472 2422	15.3399 2502
17	16.6234 8133	16.5618 7447	16.5005 8970	16.3789 7848	16.2586 3186
18	17.5795 3250	17.5108 0130	17.4424 4821	17.3068 6654	17.1727 6802
19	18.5331 9950	18.4569 6848	18.3811 7762	18.2309 0443	18.0823 5624
20	19.4844 8828	19.4003 8402	19.3167 8832	19.1511 0815	18.9874 1915
21	20.4334 0477	20.3410 5594	20.2492 9069	20.0674 9359	19.8879 7925
22	21.3799 5488	21.2789 9222	21.1786 9504	20.9800 7661	20.7840 5896
23	22.3241 4452	22.2142 0080	22.1050 1167	21.8888 7297	21.6756 8055
24	23.2659 7957	23.1466 8962	23.0282 5083	22.7938 9839	22.5628 6622
25	24.2054 6591	24.0764 6659	23.9484 2275	23.6951 6853	23.4456 3803
26	25.1426 0939	25.0035 3960	24.8655 3763	24.5926 9895	24.3240 1794
27	26.0774 1585	25.9279 1651	25.7796 0561	25.4865 0517	25.1980 2780
28	27.0098 9112	26.8496 0516	26.6906 3682	26.3766 0266	26.0676 8936
29	27.9400 4102	27.7686 1337	27.5986 4135	27.2630 0680	26.9330 2423
30	28.8678 7134	28.6849 4894	28.5036 2925	28.1457 3291	27.7940 5397
31	29.7933 8787	29.5986 1963	29.4056 1055	29.0247 9626	28.6507 9997
32	30.7165 9638	30.5096 3320	30.3045 9523	29.9002 1205	29.5032 8355
33	31.6375 0262	31.4179 9738	31.2005 9325	30.7719 9540	30.3515 2592
34	32.5561 1234	32.3237 1986	32.0936 1454	31.6401 6139	31.1955 4818
35	33.4724 3126	33.2268 0834	32.9836 6898	32.5047 2504	32.0353 7132
36	34.3864 6510	34.1272 7046	33.8707 6642	33.3657 0128	32.8710 1624
37	35.2982 1955	35.0251 1388	34.7549 1670	34.2231 0501	33.7025 0372
38	36.2077 0030	35.9203 4621	35.6361 2960	35.0769 5105	34.5298 5445
39	37.1149 1302	36.8129 7503	36.5144 1488	35.9272 5416	35.3530 8900
40	38.0198 6336	37.7030 0792	37.3897 8228	36.7740 2904	36.1722 2786
41	38.9225 5697	38.5904 5244	38.2622 4147	37.6172 9033	36.9872 9141
42	39.8229 9947	39.4753 1610	39.1318 0213	38.4570 5261	37.7982 9991
43	40.7211 9648	40.3576 0641	39.9984 7389	39.2933 3040	38.6052 7354
44	41.6171 5359	41.2373 3086	40.8622 6633	40.1261 3816	39.4082 3238
45	42.5108 7640	42.1144 9691	41.7231 8903	40.9554 9028	40.2071 9640
46	43.4023 7047	42.9891 1200	42.5812 5153	41.7814 0111	41.0021 8547
47	44.2916 4137	43.8611 8355	43.4364 6332	42.6038 8492	41.7932 1937
48	45.1786 9463	44.7307 1895	44.2888 3387	43.4229 5594	42.5803 1778
49	46.0635 3580	45.5977 2559	45.1383 7263	44.2386 2832	43.3635 0028
50	46.9461 7037	46.4622 1081	45.9850 8900	45.0509 1617	44.1427 8635

$$a_{\overline{n}|i} = \frac{1 - (1 + i)^{-n}}{i} \text{ (Continued)}$$

n	¼ %	7/24 %	⅓ %	5/12 %	½ %
51	47.8266 0386	47.3241 8194	46.8289 9236	45.8598 3353	44.9181 9537
52	48.7048 4176	48.1836 4631	47.6700 9205	46.6653 9439	45.6897 4664
53	49.5808 8953	49.0406 1119	48.5083 9739	47.4676 1267	46.4574 5934
54	50.4547 5265	49.8950 8386	49.3439 1767	48.2665 0224	47.2213 5258
55	51.3264 3656	50.7470 7157	50.1766 6213	49.0620 7692	47.9814 4535
56	52.1959 4669	51.5965 8154	51.0066 3999	49.8543 5046	48.7377 5657
57	53.0632 8847	52.4436 2098	51.8338 6046	50.6433 3656	49.4903 0505
58	53.9284 6730	53.2881 9707	52.6583 3268	51.4290 4885	50.2391 0950
59	54.7914 8858	54.1303 1698	53.4800 6580	52.2115 0093	50.9841 8855
60	55.6523 5769	54.9699 8785	54.2990 6890	52.9907 0632	51.7255 6075
61	56.5110 7999	55.8072 1680	55.1153 5106	53.7666 7850	52.4632 4453
62	57.3676 6083	56.6420 1094	55.9289 2133	54.5394 3087	53.1972 5824
63	58.2221 0557	57.4743 7734	56.7397 8870	55.3089 7680	53.9276 2014
64	59.0744 1952	58.3043 2306	57.5479 6216	56.0753 2959	54.6543 4839
65	59.9246 0800	59.1318 5515	58.3534 5065	56.8385 0250	55.3774 6109
66	60.7726 7631	59.9569 8062	59.1562 6311	57.5985 0871	56.0969 7621
67	61.6186 2974	60.7797 0648	59.9564 0842	58.3553 6137	56.8129 1165
68	62.4624 7355	61.6000 3970	60.7538 9543	59.1090 7357	57.5252 8522
69	63.3042 1302	62.4179 8723	61.5487 3299	59.8596 5832	58.2341 1465
70	64.1438 5339	63.2335 5603	62.3409 2989	60.6071 2862	58.9394 1756
71	64.9813 9989	64.0467 5300	63.1304 9491	61.3514 9738	59.6412 1151
72	65.8168 5774	64.8575 8504	63.9174 3678	62.0927 7748	60.3395 1394
73	66.6502 3216	65.6660 5904	64.7017 6424	62.8309 8172	61.0343 4222
74	67.4815 2834	66.4721 8184	65.4834 8595	63.5661 2287	61.7257 1366
75	68.3107 5146	67.2759 6029	66.2626 1058	64.2982 1365	62.4136 4543
76	69.1379 0670	68.0774 0120	67.0391 4676	65.0272 6670	63.0981 5466
77	69.9629 9920	68.8765 1138	67.8131 0308	65.7532 9464	63.7792 5836
78	70.7860 3411	69.6732 9759	68.5844 8812	66.4763 1002	64.4569 7349
79	71.6070 1657	70.4677 6661	69.3533 1042	67.1963 2533	65.1313 1691
80	72.4259 5169	71.2599 2516	70.1195 7849	67.9133 5303	65.8023 0538
81	73.2428 4458	72.0497 7997	70.8833 0082	68.6274 0550	66.4699 5561
82	74.0577 0033	72.8373 3773	71.6444 8587	69.3384 9511	67.1342 8419
83	74.8705 2402	73.6226 0513	72.4031 4206	70.0466 3413	67.7953 0765
84	75.6813 2072	74.4055 8883	73.1592 7780	70.7518 3482	68.4530 4244
85	76.4900 9548	75.1862 9547	73.9129 0146	71.4541 0936	69.1075 0491
86	77.2968 5335	75.9647 3167	74.6640 2139	72.1534 6991	69.7587 1135
87	78.1015 9935	76.7409 0403	75.4126 4591	72.8499 2854	70.4066 7796
88	78.9043 3850	77.5148 1914	76.1587 8330	73.5434 9730	71.0514 2086
89	79.7050 7581	78.2864 8357	76.9024 4182	74.2341 8818	71.6929 5608
90	80.5038 1627	79.0559 0385	77.6436 2972	74.9220 1313	72.3312 9958
91	81.3005 6486	79.8230 8651	78.3823 5521	75.6069 8403	72.9664 6725
92	82.0953 2654	80.5880 3807	79.1186 2645	76.2891 1272	73.5984 7487
93	82.8881 0628	81.3507 6500	79.8524 5161	76.9684 1101	74.2273 3818
94	83.6789 0900	82.1112 7379	80.5838 3882	77.6448 9063	74.8530 7282
95	84.4677 3966	82.8695 7087	81.3127 9616	78.3185 6329	75.4756 9434
96	85.2546 0315	83.6256 6269	82.0393 3172	78.9894 4062	76.0952 1825
97	86.0395 0439	84.3795 5565	82.7634 5355	79.6575 3422	76.7116 5995
98	86.8224 4827	85.1312 5616	83.4851 6965	80.3228 5566	77.3250 3478
99	87.6034 3967	85.8807 7057	84.2044 8802	80.9854 1642	77.9353 5799
100	88.3824 8346	86.6281 0527	84.9214 1663	81.6452 2797	78.5426 4477

$$a_{\overline{n}|i} = \frac{1 - (1 + i)^{-n}}{i} \text{ (Continued)}$$

n	$\frac{1}{4}\%$	$\frac{7}{24}\%$	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$
101	89.1595 8450	87.3732 6657	85.6359 6342	82.3023 0172	79.1469 1021
102	89.9347 4763	88.1162 6081	86.3481 3630	82.9566 4901	79.7481 6937
103	90.7079 7768	88.8570 9429	87.0579 4315	83.6082 8117	80.3464 3718
104	91.4792 7948	89.5957 7328	87.7653 9185	84.2572 0947	80.9417 2854
105	92.2486 5784	90.3323 0406	88.4704 9021	84.9034 4511	81.5340 5825
106	93.0161 1755	91.0666 9287	89.1732 4606	85.5469 9928	82.1234 4104
107	93.7816 6339	91.7989 4595	89.8736 6717	86.1878 8310	82.7098 9158
108	94.5453 0014	92.5290 6950	90.5717 6130	86.8261 0765	83.2934 2446
109	95.3070 3256	93.2570 6971	91.2675 3618	87.4616 8397	83.8740 5419
110	96.0668 6539	93.9829 5276	91.9609 9951	88.0946 2304	84.4517 9522
111	96.8248 0338	94.7067 2482	92.6521 5898	88.7249 3581	85.0266 6191
112	97.5808 5126	95.4283 9201	93.3410 2224	89.3526 3317	85.5986 6856
113	98.3350 1372	96.1479 6046	94.0275 9692	89.9777 2598	86.1678 2942
114	99.0872 9548	96.8654 3627	94.7118 9062	90.6002 2504	86.7341 5862
115	99.8377 0123	97.5808 2553	95.3939 1092	91.2201 4112	87.2976 7027
116	100.5862 3564	98.2941 3430	96.0736 6536	91.8374 8493	87.8583 7838
117	101.3329 0338	99.0053 6864	96.7511 6149	92.4522 6715	88.4162 9690
118	102.0777 0911	99.7145 3458	97.4264 0680	93.0644 9841	88.9714 3970
119	102.8206 5747	100.4216 3814	98.0994 0877	93.6741 8929	89.5238 2059
120	103.5617 5308	101.1266 8531	98.7701 7486	94.2813 5033	90.0734 5333
121	104.3010 0058	101.8296 8207	99.4387 1248	94.8859 9203	90.6203 5157
122	105.0384 0457	102.5306 3438	100.1050 2905	95.4881 2484	91.1645 2892
123	105.7739 6965	103.2295 4820	100.7691 3195	96.0877 5918	91.7059 9893
124	106.5077 0040	103.9264 2945	101.4310 2852	96.6849 0541	92.2447 7505
125	107.2396 0139	104.6212 8404	102.0907 2610	97.2795 7385	92.7808 7070
126	107.9696 7720	105.3141 1786	102.7482 3199	97.8717 7479	93.3142 9921
127	108.6979 3237	106.0049 3679	103.4035 5348	98.4615 1846	93.8450 7384
128	109.4243 7144	106.6937 4670	104.0566 9782	99.0488 1506	94.3732 0780
129	110.1489 9894	107.3805 5342	104.7076 7225	99.6336 7475	94.8987 1423
130	110.8718 1939	108.0653 6278	105.3564 8397	100.2161 0764	95.4216 0619
131	111.5928 3730	108.7481 8058	106.0031 4016	100.7961 2379	95.9418 9671
132	112.3120 5716	109.4290 1263	106.6476 4800	101.3737 3323	96.4595 9872
133	113.0294 8345	110.1078 6469	107.2900 1462	101.9489 4596	96.9747 2509
134	113.7451 2065	110.7847 4253	107.9302 4713	102.5217 7191	97.4872 8865
135	114.4589 7321	111.4596 5187	108.5683 5262	103.0922 2099	97.9973 0214
136	115.1710 4560	112.1325 9346	109.2043 3816	103.6603 0306	98.5047 7825
137	115.8813 4224	112.8035 8800	109.8382 1079	104.2260 2794	99.0097 2960
138	116.5898 6758	113.4726 2617	110.4699 7754	104.7894 0542	99.5121 6876
139	117.2966 2601	114.1397 1866	111.0996 4538	105.3504 4523	100.0121 0821
140	118.0016 2196	114.8048 7112	111.7272 2131	105.9091 5708	100.5095 6041
141	118.7048 5981	115.4680 8919	112.3527 1227	106.4655 5061	101.0045 3772
142	119.4063 4395	116.1293 7850	112.9761 2519	107.0196 3547	101.4970 5246
143	120.1060 7875	116.7887 4466	113.5974 6696	107.5714 2121	101.9871 1688
144	120.8040 6858	117.4461 9327	114.2167 4448	108.1209 1739	102.4747 4316
145	121.5003 1778	118.1017 2989	114.8339 6460	108.6681 3350	102.9599 4344
146	122.1948 3071	118.7553 6009	115.4491 3415	109.2130 7900	103.4427 2979
147	122.8876 1168	119.4070 8941	116.0622 5995	109.7557 6332	103.9231 1422
148	123.5786 6502	120.0569 2338	116.6733 4879	110.2961 9584	104.4011 0868
149	124.2679 9503	120.7048 6752	117.2824 0743	110.8343 8590	104.8767 2506
150	124.9556 0601	121.3509 2732	117.8894 4262	111.3703 4280	105.3499 7518

$$a_{\overline{n}|i} = \frac{1 - (1 + i)^{-n}}{i} \text{ (Continued)}$$

n	$\frac{1}{4}\%$	$\frac{7}{24}\%$	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$
151	125.6415 0226	121.9951 0825	118.4944 6109	111.9040 7582	105.8208 7083
152	126.3256 8804	122.6374 1579	119.0974 6952	112.4355 9418	106.2894 2371
153	127.0081 6762	123.2778 5538	119.6984 7461	112.9649 0707	106.7556 4548
154	127.6889 4525	123.9164 3245	120.2974 8300	113.4920 2364	107.2195 4774
155	128.3680 2519	124.5531 5242	120.8945 0133	114.0169 5300	107.6811 4203
156	129.0454 1166	125.1880 2069	121.4895 3621	114.5397 0423	108.1404 3983
157	129.7211 0889	125.8210 4265	122.0825 9422	115.0602 8637	108.5974 5257
158	130.3951 2109	126.4522 2367	122.6736 8195	115.5787 0842	109.0521 9161
159	131.0674 5246	127.0815 6909	123.2628 0593	116.0949 7934	109.5046 6827
160	131.7381 0719	127.7090 8426	123.8499 7269	116.6091 0805	109.9548 9380
161	132.4070 8946	128.3347 7450	124.4351 8873	117.1211 0346	110.4028 7940
162	133.0744 0346	128.9586 4512	125.0184 6053	117.6309 7440	110.8486 3622
163	133.7400 5332	129.5807 0141	125.5997 9454	118.1387 2969	111.2921 7535
164	134.4040 4321	130.2009 4864	126.1791 9722	118.6443 7811	111.7335 0781
165	135.0663 7727	130.8193 9208	126.7566 7497	119.1479 2841	112.1726 4458
166	135.7270 5962	131.4360 3697	127.3322 3419	119.6493 8929	112.6095 9660
167	136.3860 9439	132.0508 8855	127.9058 8125	120.1487 6942	113.0443 7473
168	137.0434 8567	132.6639 5202	128.4776 2251	120.6460 7743	113.4769 8978
169	137.6992 3758	133.2752 3259	129.0474 6430	121.1413 2192	113.9074 5252
170	138.3533 5419	133.8847 3545	129.6154 1292	121.6345 1146	114.3357 7865
171	139.0058 3959	134.4924 6576	130.1814 7467	122.1256 5456	114.7619 6383
172	139.6566 9785	135.0984 2867	130.7456 5582	122.6147 5973	115.1860 3366
173	140.3059 3302	135.7026 2934	131.3079 6261	123.1018 3542	115.6079 9369
174	140.9535 4914	136.3050 7287	131.8684 0127	123.5868 9004	116.0278 5442
175	141.5995 5027	136.9057 6439	132.4269 7801	124.0699 3199	116.4456 2629
176	142.2439 4042	137.5047 0899	132.9836 9901	124.5509 6962	116.8613 1969
177	142.8867 2361	138.1019 1175	133.5385 7045	125.0300 1124	117.2749 4496
178	143.5279 0385	138.6973 7773	134.0915 9845	125.5070 6513	117.6865 1240
179	144.1674 8514	139.2911 1199	134.6427 8915	125.9821 3955	118.0960 3224
180	144.8054 7146	139.8831 1956	135.1921 4866	126.4552 4271	118.5035 1467
181	145.4418 6679	140.4734 0546	135.7396 8305	126.9263 8278	118.9089 6982
182	146.0766 7510	141.0619 7470	136.2853 9839	127.3955 6791	119.3124 0778
183	146.7099 0035	141.6488 3227	136.8293 0072	127.8628 0622	119.7138 3859
184	147.3415 4649	142.2339 8315	137.3713 9606	128.3281 0578	120.1132 7223
185	147.9716 1744	142.8174 3231	137.9116 9043	128.7914 7463	120.5107 1863
186	148.6001 1715	143.3991 8469	138.4501 8980	129.2529 2080	120.9061 8769
187	149.2270 4952	143.9792 4522	138.9869 0013	129.7124 5225	121.2996 8925
188	149.8524 1848	144.5576 1883	139.5218 2737	130.1700 7693	121.6912 3308
189	150.4762 2791	145.1343 1043	140.0549 7745	130.6258 0275	122.0808 2894
190	151.0984 8170	145.7093 2490	140.5863 5626	131.0796 3759	122.4684 8651
191	151.7191 8375	146.2826 6712	141.1159 6969	131.5315 8930	122.8542 1543
192	152.3383 3790	146.8543 4195	141.6438 2362	131.9816 6570	123.2380 2530
193	152.9559 4803	147.4243 5425	142.1699 2387	132.4298 7455	123.6199 2567
194	153.5720 1799	147.9927 0885	142.6942 7628	132.8762 2362	123.9999 2604
195	154.1865 5161	148.5594 1057	143.2168 8666	133.3207 2062	124.3780 3586
196	154.7995 5272	149.1244 6422	143.7377 6079	133.7633 7323	124.7542 6454
197	155.4110 2516	149.6878 7458	144.2569 0444	134.2041 8911	125.1286 2143
198	156.0209 7273	150.2496 4645	144.7743 2336	134.6431 7587	125.5011 1585
199	156.6293 9923	150.8097 8458	145.2900 2329	135.0803 4112	125.8717 5707
200	157.2363 0846	151.3682 9372	145.8040 0992	135.5156 9240	126.2405 5430

$$a_{\overline{n}|i} = \frac{1 - (1 + i)^{-n}}{i} \text{ (Continued)}$$

n	$7/12\%$	$2/3\%$	$3/4\%$	$7/8\%$	1%
1	0.9942 0050	0.9933 7748	0.9925 5583	0.9913 2590	0.9900 9901
2	1.9826 3513	1.9801 7631	1.9777 2291	1.9740 5294	1.9703 9506
3	2.9653 3732	2.9604 4004	2.9555 5624	2.9482 5570	2.9409 8521
4	3.9423 4034	3.9342 1196	3.9261 1041	3.9140 0813	3.9019 6555
5	4.9136 7722	4.9015 3506	4.8894 3961	4.8713 8352	4.8534 3124
6	5.8793 8083	5.8624 5205	5.8455 9763	5.8204 5454	5.7954 7647
7	6.8394 8384	6.8170 0535	6.7946 3785	6.7612 9323	6.7281 9453
8	7.7940 1874	7.7652 3710	7.7366 1325	7.6939 7098	7.6516 7775
9	8.7430 1780	8.7071 8917	8.6715 7642	8.6185 5859	8.5660 1758
10	9.6865 1314	9.6429 0315	9.5995 7958	9.5351 2624	9.4713 0453
11	10.6245 3667	10.5724 2035	10.5206 7452	10.4437 4348	10.3676 2825
12	11.5571 2014	11.4957 8180	11.4349 1267	11.3444 7929	11.2550 7747
13	12.4842 9509	12.4130 2828	12.3423 4508	12.2374 0202	12.1337 4007
14	13.4060 9288	13.3242 0028	13.2430 2242	13.1225 7945	13.0037 0304
15	14.3225 4470	14.2293 3802	14.1369 9495	14.0000 7876	13.8650 5252
16	15.2336 8156	15.1284 8148	15.0243 1261	14.8699 6656	14.7178 7378
17	16.1395 3427	16.0216 7035	15.9050 2492	15.7323 0885	15.5622 5127
18	17.0401 3350	16.9089 4405	16.7791 8107	16.5871 7111	16.3982 6858
19	17.9355 0969	17.7903 4177	17.6468 2984	17.4346 1820	17.2260 0850
20	18.8256 9315	18.6659 0242	18.5080 1969	18.2747 1445	18.0455 5297
21	19.7107 1398	19.5356 6466	19.3627 9870	19.1075 2361	18.8569 8313
22	20.5906 0213	20.3996 6688	20.2112 1459	19.9331 0891	19.6603 7934
23	21.4653 8738	21.2579 4723	21.0533 1473	20.7515 3300	20.4558 2113
24	22.3350 9930	22.1105 4361	21.8891 4614	21.5628 5799	21.2433 8726
25	23.1997 6732	22.9574 9365	22.7187 5547	22.3671 4547	22.0231 5570
26	24.0594 2070	23.7988 3475	23.5421 8905	23.1644 5647	22.7952 0366
27	24.9140 8852	24.6346 0406	24.3594 9286	23.9548 5152	23.5596 0759
28	25.7637 9968	25.4648 3847	25.1707 1251	24.7383 9060	24.3164 4316
29	26.6085 8295	26.2895 7464	25.9758 9331	25.5151 3319	25.0657 8530
30	27.4484 6689	27.1088 4898	26.7750 8021	26.2851 3823	25.8077 0822
31	28.2834 7993	27.9226 9766	27.5683 1783	27.0484 6417	26.5422 8537
32	29.1136 5030	28.7311 5662	28.3556 5045	27.8051 6894	27.2695 8947
33	29.9390 0610	29.5342 6154	29.1371 2203	28.5553 0998	27.9896 9255
34	30.7595 7524	30.3320 4789	29.9127 7621	29.2989 4422	28.7026 6589
35	31.5753 8549	31.1245 5088	30.6826 5629	30.0361 2809	29.4085 8009
36	32.3864 6445	31.9118 0551	31.4468 0525	30.7669 1757	30.1075 0504
37	33.1928 3955	32.6938 4653	32.2052 6576	31.4913 6810	30.7995 0994
38	33.9945 3808	33.4707 0848	32.9580 8016	32.2095 3467	31.4846 6330
39	34.7915 8716	34.2424 2564	33.7052 9048	32.9214 7179	32.1630 3298
40	35.5840 1374	35.0090 3209	34.4469 3844	33.6272 3350	32.8346 8611
41	36.3718 4465	35.7705 6168	35.1830 6545	34.3268 7335	33.4996 8922
42	37.1551 0653	36.5270 4803	35.9137 1260	35.0204 4446	34.1581 0814
43	37.9338 2588	37.2785 2453	36.6389 2070	35.7079 9947	34.8100 0806
44	38.7080 2904	38.0250 2437	37.3587 3022	36.3895 9055	35.4554 5352
45	39.4777 4221	38.7665 8050	38.0731 8136	37.0652 6944	36.0945 0844
46	40.2429 9143	39.5032 2566	38.7823 1401	37.7350 8743	36.7272 3608
47	41.0038 0258	40.2349 9238	39.4861 6775	38.3990 9535	37.3536 9909
48	41.7602 0141	40.9619 1296	40.1847 8189	39.0573 4359	37.9739 5949
49	42.5122 1349	41.6840 1949	40.8781 9542	39.7098 8212	38.5880 7871
50	43.2598 6428	42.4013 4387	41.5664 4707	40.3567 6047	39.1961 1753

$$a_{\overline{n}|i} = \frac{1 - (1 + i)^{-n}}{i} \text{ (Continued)}$$

n	$\frac{7}{12}\%$	$\frac{2}{3}\%$	$\frac{3}{4}\%$	$\frac{7}{8}\%$	1%
151	100.2001 0819	95.0013 1128	90.1878 3795	83.6183 6499	77.7426 9594
152	100.6131 9786	95.3655 4100	90.5090 2029	83.8843 7670	77.9630 6529
153	101.0238 9183	95.7273 5861	90.8278 1171	84.1480 8099	78.1812 5276
154	101.4322 0397	96.0867 8008	91.1442 2998	84.4094 9788	78.3972 7996
155	101.8381 4811	96.4438 2127	91.4582 9279	84.6686 4722	78.6111 6828
156	102.2417 3797	96.7984 9795	91.7700 1765	84.9255 4867	78.8229 3889
157	102.6429 8721	97.1508 2578	92.0794 2199	85.1802 2173	79.0326 1276
158	103.0419 0941	97.5008 2031	92.3865 2307	85.4326 8573	79.2402 1065
159	103.4385 1805	97.8484 9700	92.6913 3803	85.6829 5983	79.4457 5312
160	103.8328 2656	98.1938 7119	92.9938 8390	85.9310 6303	79.6492 6052
161	104.2248 4828	98.5369 5813	93.2941 7757	86.1770 1415	79.8507 5299
162	104.6145 9647	98.8777 7298	93.5922 3580	86.4208 3187	80.0502 5048
163	105.0020 8431	99.2163 3078	93.8880 7524	86.6625 3470	80.2477 7275
164	105.3873 2491	99.5526 4647	94.1817 1239	86.9021 4096	80.4433 3936
165	105.7703 3132	99.8867 3490	94.4731 6367	87.1396 6886	80.6369 6966
166	106.1511 1647	100.2186 1083	94.7624 4533	87.3751 3642	80.8286 8284
167	106.5296 9326	100.5482 8890	95.0495 7352	87.6085 6150	81.0184 9786
168	106.9060 7449	100.8757 8368	95.3345 6429	87.8399 6184	81.2064 3352
169	107.2802 7290	101.2011 0961	95.6174 3354	88.0693 5498	81.3925 0844
170	107.6523 0114	101.5242 8107	95.8981 9706	88.2967 5835	81.5767 4103
171	108.0221 7181	101.8453 1232	96.1768 7053	88.5221 8919	81.7591 4953
172	108.3898 9741	102.1642 1754	96.4534 6951	88.7456 6462	81.9397 5201
173	108.7554 9038	102.4810 1080	96.7280 0944	88.9672 0161	82.1185 6635
174	109.1189 6309	102.7957 0609	97.0005 0565	89.1868 1696	82.2956 1025
175	109.4803 2785	103.1083 1731	97.2709 7335	89.4045 2735	82.4709 0123
176	109.8395 9687	103.4188 5826	97.5394 2764	89.6203 4929	82.6444 5667
177	110.1967 8230	103.7273 4264	97.8058 8352	89.8342 9917	82.8162 9373
178	110.5518 9624	104.0337 8408	98.0703 5585	90.0463 9323	82.9864 2944
179	110.9049 5070	104.3381 9610	98.3328 5940	90.2566 4757	83.1548 8063
180	111.2559 5761	104.6405 9216	98.5934 0884	90.4650 7813	83.3216 6399
181	111.6049 2886	104.9409 8559	98.8520 1869	90.6717 0075	83.4867 9603
182	111.9518 7625	105.2393 8966	99.1087 0342	90.8765 3110	83.6502 9310
183	112.2968 1151	105.5358 1754	99.3634 7734	91.0795 8474	83.8121 7138
184	112.6397 4633	105.8302 8232	99.6163 5468	91.2808 7706	83.9724 4691
185	112.9806 9229	106.1227 9701	99.8673 4956	91.4804 2336	84.1311 3556
186	113.3196 6093	106.4133 7451	100.1164 7599	91.6782 3877	84.2882 5303
187	113.6566 6373	106.7020 2766	100.3637 4788	91.8743 3831	84.4438 1488
188	113.9917 1207	106.9887 6920	100.6091 7904	92.0687 3686	84.5978 3651
189	114.3248 1731	107.2736 1179	100.8527 8316	92.2614 4918	84.7503 3318
190	114.6559 9069	107.5565 6800	101.0945 7386	92.4524 8989	84.9013 1998
191	114.9852 4344	107.8376 5033	101.3345 6462	92.6418 7350	85.0508 1186
192	115.3125 8668	108.1168 7119	101.5727 6886	92.8296 1438	85.1988 2363
193	115.6380 3150	108.3942 4291	101.8091 9986	93.0157 2677	85.3453 6993
194	115.9615 8890	108.6697 7772	102.0438 7083	93.2002 2480	85.4904 6528
195	116.2832 6982	108.9434 8780	102.2767 9487	93.3831 2248	85.6341 2404
196	116.6030 8516	109.2153 8523	102.5079 8498	93.5644 3368	85.7763 6043
197	116.9210 4573	109.4854 8202	102.7374 5407	93.7441 7218	85.9171 8855
198	117.2371 6228	109.7537 9009	102.9652 1496	93.9223 5160	86.0566 2232
199	117.5514 4552	110.0203 2128	103.1912 8036	94.0989 8548	86.1946 7557
200	117.8639 0606	110.2850 8736	103.4156 6289	94.2740 8721	86.3313 6195

$$a_{\overline{n}|i} = \frac{1 - (1 + i)^{-n}}{i} \text{ (Continued)}$$

n	1⅛ %	1¼ %	1⅜ %	1½ %	1¾ %
1	0.9888 7515	0.9876 5432	0.9864 3650	0.9852 2167	0.9828 0098
2	1.9667 4923	1.9631 1538	1.9594 9346	1.9558 8342	1.9486 9875
3	2.9337 4460	2.9265 3371	2.9193 5237	2.9122 0042	2.8979 8403
4	3.8899 8230	3.8780 5798	3.8661 9222	3.8543 8465	3.8309 4254
5	4.8355 8200	4.8178 3504	4.8001 8962	4.7826 4497	4.7478 5508
6	5.7706 6205	5.7460 0992	5.7215 1874	5.6971 8717	5.6489 9762
7	6.6953 3948	6.6627 2585	6.6303 5140	6.5982 1396	6.5346 4139
8	7.6097 3002	7.5681 2429	7.5268 5712	7.4859 2508	7.4050 5297
9	8.5139 4810	8.4623 4498	8.4112 0308	8.3605 1732	8.2604 9432
10	9.4081 0690	9.3455 2591	9.2835 5421	9.2221 8455	9.1012 2291
11	10.2923 1832	10.2178 0337	10.1440 7320	10.0711 1779	9.9274 9181
12	11.1666 9302	11.0793 1197	10.9929 2054	10.9075 0521	10.7395 4969
13	12.0313 4044	11.9301 8466	11.8302 5454	11.7315 3222	11.5376 4097
14	12.8863 6880	12.7705 5275	12.6562 3136	12.5433 8150	12.3220 0587
15	13.7318 8509	13.6005 4592	13.4710 0504	13.3432 3301	13.0928 8046
16	14.5679 9514	14.4202 9227	14.2747 2754	14.1312 6405	13.8504 9677
17	15.3948 0360	15.2299 1829	15.0675 4874	14.9076 4931	14.5950 8282
18	16.2124 1395	16.0295 4893	15.8496 1651	15.6725 6089	15.3268 6272
19	17.0209 2850	16.8193 0759	16.6210 7671	16.4261 6837	16.0460 5673
20	17.8204 4845	17.5993 1613	17.3820 7320	17.1686 3879	16.7528 8130
21	18.6110 7387	18.3696 9495	18.1327 4792	17.9001 3673	17.4475 4919
22	19.3929 0371	19.1305 6291	18.8732 4086	18.6208 2437	18.1302 6948
23	20.1660 3580	19.8820 3744	19.6036 9012	19.3308 6145	18.8012 4764
24	20.9305 6693	20.6242 3451	20.3242 3193	20.0304 0537	19.4606 8565
25	21.6865 9276	21.3572 6865	21.0350 0067	20.7196 1120	20.1087 8196
26	22.4342 0792	22.0812 5299	21.7361 2890	21.3986 3172	20.7457 3166
27	23.1735 0598	22.7962 9925	22.4277 4737	22.0676 1746	21.3717 2644
28	23.9045 7946	23.5025 1778	23.1099 8508	22.7267 1671	21.9869 5474
29	24.6275 1986	24.2000 1756	23.7829 6925	23.3760 7558	22.5916 0171
30	25.3424 1766	24.8889 0623	24.4468 2540	24.0158 3801	23.1858 4934
31	26.0493 6233	25.5692 9010	25.1016 7734	24.6461 4582	23.7698 7650
32	26.7484 4236	26.2412 7418	25.7476 4719	25.2671 3874	24.3438 5897
33	27.4397 4522	26.9049 6215	26.3848 5543	25.8789 5442	24.9079 6951
34	28.1233 5745	27.5604 5644	27.0134 2089	26.4817 2849	25.4623 7789
35	28.7993 6460	28.2078 5822	27.6334 6080	27.0755 9458	26.0072 5100
36	29.4678 5127	28.8472 6737	28.2450 9080	27.6606 8431	26.5427 5283
37	30.1289 0114	29.4787 8259	28.8484 2496	28.2371 2740	27.0690 4455
38	30.7825 9692	30.1025 0133	29.4435 7579	28.8050 5163	27.5862 8457
39	31.4290 2044	30.7185 1983	30.0306 5430	29.3645 8288	28.0946 2857
40	32.0682 5260	31.3269 3316	30.6097 6996	29.9158 4520	28.5942 2955
41	32.7003 7340	31.9278 3522	31.1810 3079	30.4589 6079	29.0852 3789
42	33.3254 6195	32.5213 1874	31.7445 4332	30.9940 5004	29.5678 0136
43	33.9435 9649	33.1074 7530	32.3004 1264	31.5212 3157	30.0420 6522
44	34.5548 5438	33.6863 9536	32.8487 4243	32.0406 2223	30.5081 7221
45	35.1593 1212	34.2581 6825	33.3896 3495	32.5523 3718	30.9662 6261
46	35.7570 4536	34.8228 8222	33.9231 9108	33.0564 8983	31.4164 7431
47	36.3481 2891	35.3806 2442	34.4495 1031	33.5531 9195	31.8589 4281
48	36.9326 3674	35.9314 8091	34.9686 9081	34.0425 5365	32.2938 0129
49	37.5106 4202	36.4755 3670	35.4808 2941	34.5246 8339	32.7211 8063
50	38.0822 1708	37.0128 7575	35.9860 2161	34.9996 8807	33.1412 0946

$$a_{\overline{n}|i} = \frac{1 - (1+i)^{-n}}{i} \text{ (Continued)}$$

n	7/12 %	2/3 %	3/4 %	7/8 %	1 %
51	44.0031 7907	43.1139 1775	42.2495 7525	40.9980 2772	39.7981 3617
52	44.7421 8301	43.8217 7260	42.9276 1812	41.6337 3256	40.3941 9423
53	45.4769 0108	44.5249 3967	43.6006 1351	42.2639 2324	40.9843 5072
54	46.2073 5816	45.2234 5000	44.2685 9902	42.8886 4757	41.5686 6408
55	46.9335 7895	45.9173 3444	44.9316 1193	43.5079 5298	42.1471 9216
56	47.6555 8802	46.6066 2362	45.5896 8926	44.1218 8647	42.7199 9224
57	48.3734 0980	47.2913 4796	46.2428 6776	44.7304 9465	43.2871 2102
58	49.0870 6856	47.9715 3771	46.8911 8388	45.3338 2369	43.8486 3468
59	49.7965 8846	48.6472 2289	47.5346 7382	45.9319 1939	44.4045 8879
60	50.5019 9350	49.3184 3334	48.1733 7352	46.5248 2716	44.9550 3841
61	51.2033 0754	49.9851 9868	48.8073 1863	47.1125 9198	45.5000 3803
62	51.9005 5431	50.6475 4836	49.4365 4455	47.6952 5846	46.0396 4161
63	52.5937 5739	51.3055 1161	50.0610 8640	48.2728 7084	46.5739 0258
64	53.2829 4024	51.9591 1749	50.6809 7906	48.8454 7296	47.1028 7385
65	53.9681 2617	52.6083 9486	51.2962 5713	49.4131 0826	47.6266 0777
66	54.6493 3836	53.2533 7238	51.9069 5497	49.9758 1984	48.1451 5621
67	55.3265 9986	53.8940 7852	52.5131 0667	50.5336 5039	48.6585 7050
68	55.9999 3358	54.5305 4158	53.1147 4607	51.0866 4227	49.1669 0149
69	56.6693 6230	55.1627 8965	53.7119 0677	51.6348 3745	49.6701 9949
70	57.3349 0867	55.7908 5064	54.3046 2210	52.1782 7752	50.1685 1435
71	57.9965 9520	56.4147 5230	54.8929 2516	52.7170 0374	50.6618 9539
72	58.6544 4427	57.0345 2215	55.4768 4880	53.2510 5699	51.1503 9148
73	59.3084 7815	57.6501 8756	56.0564 2561	53.7804 7781	51.6340 5097
74	59.9587 1896	58.2617 7573	56.6316 8795	54.3053 0638	52.1129 2175
75	60.6051 8869	58.8693 1363	57.2026 6794	54.8255 8253	52.5870 5124
76	61.2479 0922	59.4728 2811	57.7693 9746	55.3413 4575	53.0564 8638
77	61.8869 0229	60.0723 4581	58.3319 0815	55.8526 3520	53.5212 7364
78	62.5221 8952	60.6678 9319	58.8902 3141	56.3594 8966	53.9814 5905
79	63.1537 9239	61.2594 9654	59.4443 9842	56.8619 4762	54.4370 8817
80	63.7817 3229	61.8471 8200	59.9944 4012	57.3600 4721	54.8882 0611
81	64.4060 3044	62.4309 7549	60.5403 8722	57.8538 2623	55.3348 5753
82	65.0267 0798	63.0109 0281	61.0822 7019	58.3433 2216	55.7770 8666
83	65.6437 8590	63.5869 8954	61.6201 1930	58.8285 7215	56.2149 3729
84	66.2572 8507	64.1592 6114	62.1539 6456	59.3096 1304	56.6484 5276
85	66.8672 2625	64.7277 4285	62.6838 3579	59.7864 8133	57.0776 7600
86	67.4736 3007	65.2924 5979	63.2097 6257	60.2592 1321	57.5026 4951
87	68.0765 1706	65.8534 3687	63.7317 7427	60.7278 4457	57.9234 1535
88	68.6759 0759	66.4106 9888	64.2499 0002	61.1924 1097	58.3400 1520
89	69.2718 2197	66.9642 7041	64.7641 6875	61.6529 4768	58.7524 9030
90	69.8642 8033	67.5141 7591	65.2746 0918	62.1094 8965	59.1608 8148
91	70.4533 0273	68.0604 3964	65.7812 4981	62.5620 7152	59.5652 2919
92	71.0389 0910	68.6030 8574	66.2841 1892	63.0107 2765	59.9655 7346
93	71.6211 1923	69.1421 3815	66.7832 4458	63.4554 9210	60.3619 5392
94	72.1999 5284	69.6776 2068	67.2786 5467	63.8963 9861	60.7544 0982
95	72.7754 2950	70.2095 5696	67.7703 7685	64.3334 8066	61.1429 8002
96	73.3475 6869	70.7379 7049	68.2584 3856	64.7667 7141	61.5277 0299
97	73.9163 8975	71.2628 8460	68.7428 6705	65.1963 0375	61.9086 1682
98	74.4819 1193	71.7843 2245	69.2236 8938	65.6221 1028	62.2857 5923
99	75.0441 5436	72.3023 0707	69.7009 3239	66.0442 2333	62.6591 6755
100	75.6031 3606	72.8168 6132	70.1746 2272	66.4626 7492	63.0288 7877

$$a_{\overline{n}|i} = \frac{1 - (1+i)^{-n}}{i} \text{ (Continued)}$$

n	$\frac{7}{12}\%$	$\frac{2}{3}\%$	$\frac{3}{4}\%$	$\frac{7}{8}\%$	1%
101	76.1588 7596	73.3280 0794	70.6447 8682	66.8774 9683	63.3949 2947
102	76.7113 9283	73.8357 6948	71.1114 5094	67.2887 2052	63.7573 5591
103	77.2607 0538	74.3401 6835	71.5746 4113	67.6963 7722	64.1161 9397
104	77.8068 3219	74.8412 2684	72.0343 8325	68.1004 9786	64.4714 7918
105	78.3497 9174	75.3389 6706	72.4907 0298	68.5011 1312	64.8232 4671
106	78.8896 0240	75.8334 1099	72.9436 2579	68.8982 5341	65.1715 3140
107	79.4262 8241	76.3245 8045	73.3931 7696	69.2919 4885	65.5163 6772
108	79.9598 4996	76.8124 9714	73.8393 8160	69.6822 2935	65.8577 8983
109	80.4903 2307	77.2971 8259	74.2822 6461	70.0691 2451	66.1958 3151
110	81.0177 1971	77.7786 5820	74.7218 5073	70.4526 6370	66.5305 2625
111	81.5420 5770	78.2569 4523	75.1581 6450	70.8328 7604	66.8619 0718
112	82.0633 5480	78.7320 6480	75.5912 3027	71.2097 9037	67.1900 0710
113	82.5816 2863	79.2040 3788	76.0210 7223	71.5834 3531	67.5148 5852
114	83.0968 9674	79.6728 8531	76.4477 1437	71.9538 3922	67.8364 9358
115	83.6091 7654	80.1386 2779	76.8711 8052	72.3210 3020	68.1549 4414
116	84.1184 8537	80.6012 8589	77.2914 9431	72.6850 3614	68.4702 4172
117	84.6248 4047	81.0608 8002	77.7086 7922	73.0458 8465	68.7824 1755
118	85.1282 5896	81.5174 3048	78.1227 5853	73.4036 0312	69.0915 0252
119	85.6287 5787	81.9709 5743	78.5337 5536	73.7582 1871	69.3975 2725
120	86.1263 5414	82.4214 8089	78.9416 9267	74.1097 5832	69.7005 2203
121	86.6210 6460	82.8690 2076	79.3465 9322	74.4582 4864	70.0005 1686
122	87.1129 0598	83.3135 9678	79.7484 7962	74.8037 1613	70.2975 4145
123	87.6018 9493	83.7552 2859	80.1473 7432	75.1461 8699	70.5916 2520
124	88.0880 4798	84.1939 3568	80.5432 9957	75.4856 8723	70.8827 9722
125	88.5713 8159	84.6297 3743	80.9362 7749	75.8222 4261	71.1710 8636
126	89.0519 1210	85.0626 5308	81.3263 3001	76.1558 7867	71.4565 2115
127	89.5296 5577	85.4927 0173	81.7134 7892	76.4866 2074	71.7391 2985
128	90.0046 2877	85.9199 0238	82.0977 4583	76.8144 9391	72.0189 4045
129	90.4768 4716	86.3442 7389	82.4791 5219	77.1395 2309	72.2959 8064
130	90.9463 2692	86.7658 3499	82.8577 1929	77.4617 3292	72.5702 7786
131	91.4130 8393	87.1846 0430	83.2334 6828	77.7811 4788	72.8418 5927
132	91.8771 3399	87.6006 0029	83.6064 2013	78.0977 9220	73.1107 5175
133	92.3384 9278	88.0138 4135	83.9765 9566	78.4116 8991	73.3769 8193
134	92.7971 7592	88.4243 4571	84.3440 1554	78.7228 6484	73.6405 7617
135	93.2531 9893	88.8321 3150	84.7087 0029	79.0313 4061	73.9015 6056
136	93.7065 7722	89.2372 1673	85.0706 7026	79.3371 4063	74.1599 6095
137	94.1573 2616	89.6396 1926	85.4299 4567	79.6402 8811	74.4158 0293
138	94.6054 6097	90.0393 5688	85.7865 4657	79.9408 0606	74.6691 1181
139	95.0509 9682	90.4364 4724	86.1404 9288	80.2387 1728	74.9199 1268
140	95.4939 4878	90.8309 0785	86.4918 0434	80.5340 4440	75.1682 3038
141	95.9343 3185	91.2227 5614	86.8405 0059	80.8268 0981	75.4140 8948
142	96.3721 6091	91.6120 0941	87.1866 0108	81.1170 3575	75.6575 1434
143	96.8074 5078	91.9986 8485	87.5301 2514	81.4047 4423	75.8985 2905
144	97.2402 1619	92.3827 9952	87.8710 9195	81.6899 5711	76.1371 5747
145	97.6704 7177	92.7643 7038	88.2095 2055	81.9726 9602	76.3734 2324
146	98.0982 3208	93.1434 1429	88.5454 2982	82.2529 8242	76.6073 4974
147	98.5235 1160	93.5199 4797	88.8788 3854	82.5308 3759	76.8389 6014
148	98.9463 2470	93.8939 8805	89.2097 6530	82.8062 8262	77.0682 7737
149	99.3666 8570	94.2655 5104	89.5382 2858	83.0793 3841	77.2953 2413
150	99.7846 0882	94.6346 5335	89.8642 4673	83.3500 2569	77.5201 2290

$$a_{\overline{n}|i} = \frac{1 - (1 + i)^{-n}}{i} \text{ (Continued)}$$

n	1⅛ %	1¼ %	1⅜ %	1½ %	1¾ %
51	38.6474 3345	37.5435 8099	36.4843 6164	35.4676 7298	33.5540 1421
52	39.2063 6188	38.0677 3431	36.9759 4243	35.9287 4185	33.9597 1913
53	39.7590 7232	38.5854 1660	37.4608 5566	36.3829 9690	34.3584 4632
54	40.3056 3394	39.0967 0776	37.9391 9178	36.8305 3882	34.7503 1579
55	40.8461 1514	39.6016 8667	38.4110 3998	37.2714 6681	35.1354 4550
56	41.3805 8358	40.1004 3128	38.8764 8826	37.7058 7863	35.5139 5135
57	41.9091 0613	40.5930 1855	39.3356 2344	38.1338 7058	35.8859 4727
58	42.4317 4896	41.0795 2449	39.7885 3114	38.5555 3751	36.2515 4523
59	42.9485 7746	41.5600 2419	40.2352 9582	38.9709 7292	36.6108 5526
60	43.4596 5633	42.0345 9179	40.6760 0081	39.3802 6889	36.9639 8552
61	43.9650 4952	42.5033 0054	41.1107 2829	39.7835 1614	37.3110 4228
62	44.4648 2029	42.9662 2275	41.5395 5935	40.1808 0408	37.6521 3000
63	44.9590 3119	43.4234 2988	41.9625 7396	40.5722 2077	37.9873 5135
64	45.4477 4407	43.8749 9247	42.3798 5101	40.9578 5298	38.3168 0723
65	45.9310 2009	44.3209 8022	42.7914 6832	41.3377 8618	38.6405 9678
66	46.4089 1975	44.7614 6195	43.1975 0266	41.7121 0461	38.9588 1748
67	46.8815 0284	45.1965 0563	43.5980 2975	42.0808 9125	39.2715 6509
68	47.3488 2852	45.6261 7840	43.9931 2429	42.4442 2783	39.5789 3375
69	47.8109 5527	46.0505 4656	44.3828 5997	42.8021 9490	39.8810 1597
70	48.2679 4094	46.4696 7562	44.7673 0946	43.1548 7183	40.1779 0267
71	48.7198 4270	46.8836 3024	45.1465 4448	43.5023 3678	40.4696 8321
72	49.1667 1714	47.2924 7431	45.5206 3573	43.8446 6677	40.7564 4542
73	49.6086 2016	47.6962 7093	45.8896 5300	44.1819 3771	41.0382 7560
74	50.0456 0708	48.0950 8240	46.2536 6511	44.5142 2434	41.3152 5857
75	50.4777 3259	48.4889 7027	46.6127 3994	44.8416 0034	41.5874 7771
76	50.9050 5077	48.8779 9533	46.9669 4445	45.1641 3826	41.8550 1495
77	51.3276 1510	49.2622 1761	47.3163 4471	45.4819 0962	42.1179 5081
78	51.7454 7847	49.6416 9640	47.6610 0588	45.7949 8485	42.3763 6443
79	52.1586 9317	50.0164 9027	48.0009 9224	46.1034 3335	42.6303 3359
80	52.5673 1092	50.3866 5706	48.3363 6719	46.4073 2349	42.8799 3474
81	52.9713 8286	50.7522 5389	48.6671 9328	46.7067 2265	43.1252 4298
82	53.3709 5957	51.1133 3717	48.9935 3221	47.0016 9720	43.3663 3217
83	53.7660 9104	51.4699 6264	49.3154 4484	47.2923 1251	43.6032 7486
84	54.1568 2674	51.8221 8532	49.6329 9122	47.5786 3301	43.8361 4237
85	54.5432 1557	52.1700 5958	49.9462 3055	47.8607 2218	44.0650 0479
86	54.9253 0588	52.5136 3909	50.2552 2125	48.1386 4254	44.2899 3099
87	55.3031 4549	52.8529 7688	50.5600 2096	48.4124 5571	44.5109 8869
88	55.6767 8169	53.1881 2531	50.8606 8653	48.6822 2237	44.7282 4441
89	56.0462 6126	53.5191 3611	51.1572 7401	48.9480 0234	44.9417 6355
90	56.4116 3041	53.8460 6035	51.4498 3873	49.2098 5452	45.1516 1037
91	56.7729 3490	54.1689 4850	51.7384 3524	49.4678 3696	45.3578 4803
92	57.1302 1992	54.4878 5037	52.0231 1738	49.7220 0686	45.5605 3861
93	57.4835 3021	54.8028 1518	52.3039 3823	49.9724 2055	45.7597 4310
94	57.8329 0997	55.1138 9154	52.5809 5016	50.2191 3355	45.9555 2147
95	58.1784 0294	55.4211 2744	52.8542 0484	50.4622 0054	46.1479 3265
96	58.5200 5235	55.7245 7031	53.1237 5324	50.7016 7541	46.3370 3455
97	58.8579 0096	56.0242 6698	53.3896 4561	50.9376 1124	46.5228 8408
98	59.1919 9106	56.3202 6368	53.6519 3155	51.1700 6034	46.7055 3718
99	59.5223 6446	56.6126 0610	53.9106 5998	51.3990 7422	46.8850 4882
100	59.8490 6251	56.9013 3936	54.1658 7914	51.6247 0367	47.0614 7304

$$a_{\overline{n}|i} = \frac{1 - (1+i)^{-n}}{i} \text{ (Continued)}$$

n	2%	2¼%	2½%	2¾%	3%
1	0.9803 9216	0.9779 9511	0.9756 0976	0.9732 3601	0.9708 7379
2	1.9415 6094	1.9344 6955	1.9274 2415	1.9204 2434	1.9134 6970
3	2.8838 8327	2.8698 9687	2.8560 2356	2.8422 6213	2.8286 1135
4	3.8077 2870	3.7847 4021	3.7619 7421	3.7394 2787	3.7170 9840
5	4.7134 5951	4.6794 5253	4.6458 2850	4.6125 8186	4.5797 0719
6	5.6014 3089	5.5544 7680	5.5081 2536	5.4623 6678	5.4171 9144
7	6.4719 9107	6.4102 4626	6.3493 9060	6.2894 0806	6.2302 8296
8	7.3254 8144	7.2471 8461	7.1701 3717	7.0943 1441	7.0196 9219
9	8.1622 3671	8.0657 0622	7.9708 6553	7.8776 7826	7.7861 0892
10	8.9825 8501	8.8662 1635	8.7520 6393	8.6400 7616	8.5302 0284
11	9.7868 4805	9.6491 1134	9.5142 0871	9.3820 6926	9.2526 2411
12	10.5753 4122	10.4147 7882	10.2577 6460	10.1042 0366	9.9540 0399
13	11.3483 7375	11.1635 9787	10.9831 8497	10.8070 1086	10.6349 5533
14	12.1062 4877	11.8959 3924	11.6909 1217	11.4910 0814	11.2960 7314
15	12.8492 6350	12.6121 6551	12.3813 7773	12.1566 9892	11.9379 3509
16	13.5777 0931	13.3126 3131	13.0550 0266	12.8045 7315	12.5611 0203
17	14.2918 7188	13.9976 8343	13.7121 9772	13.4351 0769	13.1661 1847
18	14.9920 3125	14.6676 6106	14.3533 6363	14.0487 6661	13.7535 1308
19	15.6784 6201	15.3228 9590	14.9788 9134	14.6460 0157	14.3237 9911
20	16.3514 3334	15.9637 1237	15.5891 6229	15.2272 5213	14.8774 7486
21	17.0112 0916	16.5904 2775	16.1845 4857	15.7929 4612	15.4150 2414
22	17.6580 4820	17.2033 5232	16.7654 1324	16.3434 9987	15.9369 1664
23	18.2922 0412	17.8027 8955	17.3321 1048	16.8793 1861	16.4436 0839
24	18.9139 2560	18.3890 3624	17.8849 8583	17.4007 9670	16.9355 4212
25	19.5234 5647	18.9623 8263	18.4243 7642	17.9083 1795	17.4131 4769
26	20.1210 3576	19.5231 1260	18.9506 1114	18.4022 5592	17.8768 4242
27	20.7068 9780	20.0715 0376	19.4640 1087	18.8829 7413	18.3270 3147
28	21.2812 7236	20.6078 2764	19.9648 8866	19.3508 2640	18.7641 0823
29	21.8443 8466	21.1323 4977	20.4535 4991	19.8061 5708	19.1884 5459
30	22.3964 5555	21.6453 2985	20.9302 9259	20.2493 0130	19.6004 4135
31	22.9377 0152	22.1470 2186	21.3954 0741	20.6805 8520	20.0004 2849
32	23.4683 3482	22.6376 7419	21.8491 7796	21.1003 2623	20.3887 6553
33	23.9885 6355	23.1175 2977	22.2918 8094	21.5088 3332	20.7657 9178
34	24.4985 9172	23.5868 2618	22.7237 8628	21.9064 0712	21.1318 3668
35	24.9986 1933	24.0457 9577	23.1451 5734	22.2933 4026	21.4872 2007
36	25.4888 4248	24.4946 6579	23.5562 5107	22.6699 1753	21.8322 5250
37	25.9694 5341	24.9336 5848	23.9573 1812	23.0364 1609	22.1672 3544
38	26.4406 4060	25.3629 9118	24.3486 0304	23.3931 0568	22.4924 6159
39	26.9025 8883	25.7828 7646	24.7303 4443	23.7402 4884	22.8082 1513
40	27.3554 7924	26.1935 2221	25.1027 7505	24.0781 0106	23.1147 7197
41	27.7994 8945	26.5951 3174	25.4661 2200	24.4069 1101	23.4123 9997
42	28.2347 9358	26.9879 0390	25.8206 0683	24.7269 2069	23.7013 5920
43	28.6615 6233	27.3720 3316	26.1664 4569	25.0383 6563	23.9819 0213
44	29.0799 6307	27.7477 0969	26.5038 4945	25.3414 7507	24.2542 7392
45	29.4901 5987	28.1151 1950	26.8330 2386	25.6364 7209	24.5187 1254
46	29.8923 1360	28.4744 4450	27.1541 6962	25.9235 7381	24.7754 4907
47	30.2865 8196	28.8258 6259	27.4674 8255	26.2029 9154	25.0247 0783
48	30.6731 1957	29.1695 4777	27.7731 5371	26.4749 3094	25.2667 0664
49	31.0520 7801	29.5056 7019	28.0713 6947	26.7395 9215	25.5016 5693
50	31.4236 0589	29.8343 9627	28.3623 1168	26.9971 6998	25.7297 6401

$$a_{\overline{n}|i} = \frac{1 - (1 + i)^{-n}}{i} \quad (Continued)$$

n	2%	2¼%	2½%	2¾%	3%
51	31.7878 4892	30.1558 8877	28.6461 5774	27.2478 5400	25.9512 2719
52	32.1449 4992	30.4703 0687	28.9230 8072	27.4918 2871	26.1662 3999
53	32.4950 4894	30.7778 0623	29.1932 4948	27.7292 7368	26.3749 9028
54	32.8382 8327	31.0785 3910	29.4568 2876	27.9603 6368	26.5776 6047
55	33.1747 8752	31.3726 5438	29.7139 7928	28.1852 6879	26.7744 2764
56	33.5046 9365	31.6602 9768	29.9648 5784	28.4041 5454	26.9654 6373
57	33.8281 3103	31.9416 1142	30.2096 1740	28.6171 8203	27.1509 3566
58	34.1452 2650	32.2167 3489	30.4484 0722	28.8245 0806	27.3310 0549
59	34.4561 0441	32.4858 0429	30.6813 7290	29.0262 8522	27.5058 3058
60	34.7608 8668	32.7489 5285	30.9086 5649	29.2226 6201	27.6755 6367
61	35.0596 9282	33.0063 1086	31.1303 9657	29.4137 8298	27.8403 5307
62	35.3526 4002	33.2580 0573	31.3467 2836	29.5997 8879	28.0003 4279
63	35.6398 4316	33.5041 6208	31.5577 8377	29.7808 1634	28.1556 7261
64	35.9214 1486	33.7449 0179	31.7636 9148	29.9569 9887	28.3064 7826
65	36.1974 6555	33.9803 4405	31.9645 7705	30.1284 6605	28.4528 9152
66	36.4681 0348	34.2106 0543	32.1605 6298	30.2953 4409	28.5950 4031
67	36.7334 3478	34.4357 9993	32.3517 6876	30.4577 5581	28.7330 4884
68	36.9935 6351	34.6560 3905	32.5383 1099	30.6158 2074	28.8670 3771
69	37.2485 9168	34.8714 3183	32.7203 0340	30.7696 5522	28.9971 2399
70	37.4986 1929	35.0820 8492	32.8978 5698	30.9193 7247	29.1234 2135
71	37.7437 4441	35.2881 0261	33.0710 7998	31.0650 8270	29.2460 4015
72	37.9840 6314	35.4895 8691	33.2400 7803	31.2068 9314	29.3650 8752
73	38.2196 6975	35.6866 3756	33.4049 5417	31.3449 0816	29.4806 6750
74	38.4506 5662	35.8793 5214	33.5658 0895	31.4792 2936	29.5928 8107
75	38.6771 1433	36.0678 2605	33.7227 4044	31.6099 5558	29.7018 2628
76	38.8991 3170	36.2521 5262	33.8758 4433	31.7371 8304	29.8075 9833
77	39.1167 9578	36.4324 2310	34.0252 1398	31.8610 0540	29.9102 8964
78	39.3301 9194	36.6087 2675	34.1709 4047	31.9815 1377	30.0099 8994
79	39.5394 0386	36.7811 5085	34.3131 1265	32.0987 9685	30.1067 8635
80	39.7445 1359	36.9497 8079	34.4518 1722	32.2129 4098	30.2007 6345
81	39.9456 0156	37.1147 0004	34.5871 3875	32.3240 3015	30.2920 0335
82	40.1427 4663	37.2759 9026	34.7191 5976	32.4321 4613	30.3805 8577
83	40.3360 2611	37.4337 3130	34.8479 6074	32.5373 6850	30.4665 8813
84	40.5255 1579	37.5880 0127	34.9736 2023	32.6397 7469	30.5500 8556
85	40.7112 8999	37.7388 7655	35.0962 1486	32.7394 4009	30.6311 5103
86	40.8934 2156	37.8864 3183	35.2158 1938	32.8364 3804	30.7098 5537
87	41.0719 8192	38.0307 4018	35.3325 0671	32.9308 3994	30.7862 6735
88	41.2470 4110	38.1718 7304	35.4463 4801	33.0227 1527	30.8604 5374
89	41.4186 6774	38.3099 0028	35.5574 1269	33.1121 3165	30.9324 7936
90	41.5869 2916	38.4448 9025	35.6657 6848	33.1991 5489	31.0024 0714
91	41.7518 9133	38.5769 0978	35.7714 8144	33.2838 4905	31.0702 9820
92	41.9136 1895	38.7060 2423	35.8746 1604	33.3662 7644	31.1362 1184
93	42.0721 7545	38.8322 9754	35.9752 3516	33.4464 9776	31.2002 0567
94	42.2276 2299	38.9557 9221	36.0734 0016	33.5245 7202	31.2623 3560
95	42.3800 2254	39.0765 6940	36.1691 7089	33.6005 5671	31.3226 5592
96	42.5294 3386	39.1946 8890	36.2626 0574	33.6745 0775	31.3812 1934
97	42.6759 1555	39.3102 0920	36.3537 6170	33.7464 7956	31.4380 7703
98	42.8195 2505	39.4231 8748	36.4426 9434	33.8165 2512	31.4932 7867
99	42.9603 1867	39.5336 7968	36.5294 5790	33.8846 9598	31.5468 7250
100	43.0983 5164	39.6417 4052	36.6141 0526	33.9510 4232	31.5989 0534

$$a_{\overline{n}|i} = \frac{1 - (1 + i)^{-n}}{i} \text{ (Continued)}$$

n	3½ %	4 %	4½ %	5 %	5½ %
1	0.9661 8357	0.9615 3846	0.9569 3780	0.9523 8095	0.9478 6730
2	1.8996 9428	1.8860 9467	1.8726 6775	1.8594 1043	1.8463 1971
3	2.8016 3698	2.7750 9103	2.7489 6435	2.7232 4803	2.6979 3338
4	3.6730 7921	3.6298 9522	3.5875 2570	3.5459 5050	3.5051 5012
5	4.5150 5238	4.4518 2233	4.3899 7674	4.3294 7667	4.2702 8448
6	5.3285 5302	5.2421 3686	5.1578 7248	5.0756 9207	4.9955 3031
7	6.1145 4398	6.0020 5467	5.8927 0094	5.7863 7340	5.6829 6712
8	6.8739 5554	6.7327 4487	6.5958 8607	6.4632 1276	6.3345 6599
9	7.6076 8651	7.4353 3161	7.2687 9050	7.1078 2168	6.9521 9525
10	8.3166 0532	8.1108 9578	7.9127 1818	7.7217 3493	7.5376 2583
11	9.0015 5104	8.7604 7671	8.5289 1692	8.3064 1422	8.0925 3633
12	9.6633 3433	9.3850 7376	9.1185 8078	8.8632 5164	8.6185 1785
13	10.3027 3849	9.9856 4785	9.6828 5242	9.3935 7299	9.1170 7853
14	10.9205 2028	10.5631 2293	10.2228 2528	9.8986 4094	9.5896 4790
15	11.5174 1090	11.1183 8743	10.7395 4573	10.3796 5804	10.0375 8094
16	12.0941 1681	11.6522 9561	11.2340 1505	10.8377 6956	10.4621 6203
17	12.6513 2059	12.1656 6885	11.7071 9143	11.2740 6625	10.8646 0856
18	13.1896 8173	12.6592 9697	12.1599 9180	11.6895 8690	11.2460 7447
19	13.7098 3742	13.1339 3940	12.5932 9359	12.0853 2086	11.6076 5352
20	14.2124 0330	13.5903 2634	13.0079 3645	12.4622 1034	11.9503 8248
21	14.6979 7420	14.0291 5995	13.4047 2388	12.8211 5271	12.2752 4406
22	15.1671 2484	14.4511 1533	13.7844 2476	13.1630 0258	12.5831 6973
23	15.6204 1047	14.8568 4167	14.1477 7489	13.4885 7388	12.8750 4239
24	16.0583 6760	15.2469 6314	14.4954 7837	13.7986 4179	13.1516 9895
25	16.4815 1459	15.6220 7994	14.8282 0896	14.0939 4457	13.4139 3266
26	16.8903 5226	15.9827 6918	15.1466 1145	14.3751 8530	13.6624 9541
27	17.2853 6451	16.3295 8575	15.4513 0282	14.6430 3362	13.8980 9991
28	17.6670 1885	16.6630 6322	15.7428 7351	14.8981 2726	14.1214 2172
29	18.0357 6700	16.9837 1463	16.0218 8853	15.1410 7358	14.3331 0116
30	18.3920 4541	17.2920 3330	16.2888 8854	15.3724 5103	14.5337 4517
31	18.7362 7576	17.5884 9356	16.5443 9095	15.5928 1050	14.7239 2907
32	19.0688 6547	17.8735 5150	16.7888 9086	15.8026 7667	14.9041 9817
33	19.3902 0818	18.1476 4567	17.0228 6207	16.0025 4921	15.0750 6936
34	19.7006 8423	18.4111 9776	17.2467 5796	16.1929 0401	15.2370 3257
35	20.0006 6110	18.6646 1323	17.4610 1240	16.3741 9429	15.3905 5220
36	20.2904 9381	18.9082 8195	17.6660 4058	16.5468 5171	15.5360 6843
37	20.5705 2542	19.1425 7880	17.8622 3979	16.7112 8734	15.6739 9851
38	20.8410 8736	19.3678 6423	18.0499 9023	16.8678 9271	15.8047 3793
39	21.1024 9987	19.5844 8484	18.2296 5572	17.0170 4067	15.9286 6154
40	21.3550 7234	19.7927 7388	18.4015 8442	17.1590 8635	16.0461 2469
41	21.5991 0371	19.9930 5181	18.5661 0949	17.2943 6796	16.1574 6416
42	21.8348 8281	20.1856 2674	18.7235 4975	17.4232 0758	16.2629 9920
43	22.0626 8870	20.3707 9494	18.8742 1029	17.5459 1198	16.3630 3242
44	22.2827 9102	20.5488 4129	19.0183 8305	17.6627 7331	16.4578 5063
45	22.4954 5026	20.7200 3970	19.1563 4742	17.7740 6982	16.5477 2572
46	22.7009 1813	20.8846 5356	19.2883 7074	17.8800 6650	16.6329 1537
47	22.8994 3780	21.0429 3612	19.4147 0884	17.9810 1571	16.7136 6386
48	23.0912 4425	21.1951 3088	19.5356 0654	18.0771 5782	16.7902 0271
49	23.2765 6450	21.3414 7200	19.6512 9813	18.1687 2173	16.8627 5139
50	23.4556 1787	21.4821 8462	19.7620 0778	18.2559 2546	16.9315 1790

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i} \quad (Continued)$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	$5\frac{1}{2}\%$
51	23.6286 1630	21.6174 8521	19.8679 5003	18.3389 7663	16.9966 9943
52	23.7957 6454	21.7475 8193	19.9693 3017	18.4180 7298	17.0584 8287
53	23.9572 6043	21.8726 7493	20.0663 4466	18.4934 0284	17.1170 4538
54	24.1132 9510	21.9929 5667	20.1591 8149	18.5651 4556	17.1725 5486
55	24.2640 5323	22.1086 1218	20.2480 2057	18.6334 7196	17.2251 7048
56	24.4097 1327	22.2198 1940	20.3330 3404	18.6985 4473	17.2750 4311
57	24.5504 4760	22.3267 4943	20.4143 8664	18.7605 1879	17.3223 1575
58	24.6864 2281	22.4295 6676	20.4922 3602	18.8195 4170	17.3671 2393
59	24.8177 9981	22.5284 2957	20.5667 3303	18.8757 5400	17.4095 9614
60	24.9447 3412	22.6234 8997	20.6380 2204	18.9292 8953	17.4498 5416
61	25.0673 7596	22.7148 9421	20.7062 4118	18.9802 7574	17.4880 1343
62	25.1858 7049	22.8027 8289	20.7715 2266	19.0288 3404	17.5241 8334
63	25.3003 5796	22.8872 9124	20.8339 9298	19.0750 8003	17.5584 6762
64	25.4109 7388	22.9685 4927	20.8937 7319	19.1191 2384	17.5909 6457
65	25.5178 4916	23.0466 8199	20.9509 7913	19.1610 7033	17.6217 6737
66	25.6211 1030	23.1218 0961	21.0057 2165	19.2010 1936	17.6509 6433
67	25.7208 7951	23.1940 4770	21.0581 0684	19.2390 6606	17.6786 3917
68	25.8172 7489	23.2635 0740	21.1082 3621	19.2753 0101	17.7048 7125
69	25.9104 1052	23.3302 9558	21.1562 0690	19.3098 1048	17.7297 3579
70	26.0003 9664	23.3945 1498	21.2021 1187	19.3426 7665	17.7533 0406
71	26.0873 3975	23.4562 6440	21.2460 4007	19.3739 7776	17.7756 4366
72	26.1713 4275	23.5156 3885	21.2880 7662	19.4037 8834	17.7968 1864
73	26.2525 0508	23.5727 2966	21.3283 0298	19.4321 7937	17.8168 8970
74	26.3309 2278	23.6276 2468	21.3667 9711	19.4592 1845	17.8359 1441
75	26.4066 8868	23.6804 0834	21.4036 3360	19.4849 6995	17.8539 4731
76	26.4798 9244	23.7311 6187	21.4388 8383	19.5094 9519	17.8710 4010
77	26.5506 2072	23.7799 6333	21.4726 1611	19.5328 5257	17.8872 4180
78	26.6189 5721	23.8268 8782	21.5048 9579	19.5550 9768	17.9025 9887
79	26.6849 8281	23.8720 0752	21.5357 8545	19.5762 8351	17.9171 5532
80	26.7487 7567	23.9153 9185	21.5653 4493	19.5964 6048	17.9309 5291
81	26.8104 1127	23.9571 0754	21.5936 3151	19.6156 7665	17.9440 3120
82	26.8699 6258	23.9972 1879	21.6207 0001	19.6339 7776	17.9564 2767
83	26.9275 0008	24.0357 8730	21.6466 0288	19.6514 0739	17.9681 7789
84	26.9830 9186	24.0728 7240	21.6713 9032	19.6680 0704	17.9793 1554
85	27.0368 0373	24.1085 3116	21.6951 1035	19.6838 1623	17.9898 7255
86	27.0886 9926	24.1428 1842	21.7178 0895	19.6988 7260	17.9998 7919
87	27.1388 3986	24.1757 8694	21.7395 3009	19.7132 1200	18.0093 6416
88	27.1872 8489	24.2074 8745	21.7603 1588	19.7268 6857	18.0183 5466
89	27.2340 9168	24.2379 6870	21.7802 0658	19.7398 7483	18.0268 7645
90	27.2793 1564	24.2672 7759	21.7992 4075	19.7522 6174	18.0349 5398
91	27.3230 1028	24.2954 5923	21.8174 5526	19.7640 5880	18.0426 1041
92	27.3652 2732	24.3225 5695	21.8348 8542	19.7752 9410	18.0498 6769
93	27.4060 1673	24.3486 1245	21.8515 6499	19.7859 9438	18.0567 4662
94	27.4454 2680	24.3736 6582	21.8675 2631	19.7961 8512	18.0632 6694
95	27.4835 0415	24.3977 5559	21.8828 0030	19.8058 9059	18.0694 4734
96	27.5202 9387	24.4209 1884	21.8974 1655	19.8151 3390	18.0753 0553
97	27.5558 3948	24.4431 9119	21.9114 0340	19.8239 3705	18.0808 5833
98	27.5901 8308	24.4646 0692	21.9247 8794	19.8323 2100	18.0861 2164
99	27.6233 6529	24.4851 9896	21.9375 9612	19.8403 0571	18.0911 1055
100	27.6554 2540	24.5049 9900	21.9498 5274	19.8479 1020	18.0958 3939

$$a_{\overline{n}|i} = \frac{1 - (1 + i)^{-n}}{i} \text{ (Continued)}$$

n	6%	6½%	7%	7½%	8%
1	0.9433 9623	0.9389 6714	0.9345 7944	0.9302 3256	0.9259 2593
2	1.8333 9267	1.8206 2642	1.8080 1817	1.7955 6517	1.7832 6475
3	2.6730 1195	2.6484 7551	2.6243 1604	2.6005 2574	2.5770 9699
4	3.4651 0561	3.4257 9860	3.3872 1126	3.3493 2627	3.3121 2684
5	4.2123 6379	4.1556 7944	4.1001 9744	4.0458 8490	3.9927 1004
6	4.9173 2433	4.8410 1356	4.7665 3966	4.6938 4642	4.6228 7966
7	5.5823 8144	5.4845 1977	5.3892 8940	5.2966 0132	5.2063 7006
8	6.2097 9381	6.0887 5096	5.9712 9851	5.8573 0355	5.7466 3894
9	6.8016 9227	6.6561 0419	6.5152 3225	6.3788 8703	6.2468 8791
10	7.3600 8705	7.1888 3022	7.0235 8154	6.8640 8096	6.7100 8140
11	7.8868 7458	7.6890 4246	7.4986 7434	7.3154 2415	7.1389 6426
12	8.3838 4394	8.1587 2532	7.9426 8630	7.7352 7827	7.5360 7802
13	8.8526 8296	8.5997 4208	8.3576 5074	8.1258 4026	7.9037 7594
14	9.2949 8393	9.0138 4233	8.7454 6799	8.4891 5373	8.2442 3698
15	9.7122 4899	9.4026 6885	9.1079 1401	8.8271 1974	8.5594 7869
16	10.1058 9527	9.7677 6418	9.4466 4860	9.1415 0674	8.8513 6916
17	10.4772 5969	10.1105 7670	9.7632 2299	9.4339 5976	9.1216 3811
18	10.8276 0348	10.4324 6638	10.0590 8691	9.7060 0908	9.3718 8714
19	11.1581 1649	10.7347 1022	10.3355 9524	9.9590 7821	9.6035 9920
20	11.4699 2122	11.0185 0725	10.5940 1425	10.1944 9136	9.8181 4741
21	11.7640 7662	11.2849 8333	10.8355 2733	10.4134 8033	10.0168 0316
22	12.0415 8172	11.5351 9562	11.0612 4050	10.6171 9101	10.2007 4366
23	12.3033 7898	11.7701 3673	11.2721 8738	10.8066 8931	10.3710 5895
24	12.5503 5753	11.9907 3871	11.4693 3400	10.9829 6680	10.5287 5828
25	12.7833 5616	12.1978 7673	11.6535 8318	11.1469 4586	10.6747 7619
26	13.0031 6619	12.3923 7251	11.8257 7867	11.2994 8452	10.8099 7795
27	13.2105 3414	12.5749 9766	11.9867 0904	11.4413 8095	10.9351 6477
28	13.4061 6428	12.7464 7668	12.1371 1125	11.5733 7763	11.0510 7849
29	13.5907 2102	12.9074 8984	12.2776 7407	11.6961 6524	11.1584 0601
30	13.7648 3115	13.0586 7591	12.4090 4118	11.8103 8627	11.2577 8334
31	13.9290 8599	13.2006 3465	12.5318 1419	11.9166 3839	11.3497 9939
32	14.0840 4339	13.3339 2925	12.6465 5532	12.0154 7757	11.4349 9944
33	14.2302 2961	13.4590 8850	12.7537 9002	12.1074 2099	11.5138 8837
34	14.3681 4114	13.5766 0892	12.8540 0936	12.1929 4976	11.5869 3367
35	14.4982 4636	13.6869 5673	12.9476 7230	12.2725 1141	11.6545 6822
36	14.6209 8713	13.7905 6970	13.0352 0776	12.3465 2224	11.7171 9279
37	14.7367 8031	13.8878 5887	13.1170 1660	12.4153 6952	11.7751 7851
38	14.8460 1916	13.9792 1021	13.1934 7345	12.4794 1351	11.8288 6899
39	14.9490 7468	14.0649 8611	13.2649 2846	12.5389 8931	11.8785 8240
40	15.0462 9687	14.1455 2687	13.3317 0884	12.5944 0866	11.9246 1333
41	15.1380 1592	14.2211 5199	13.3941 2041	12.6459 6155	11.9672 3457
42	15.2245 4332	14.2921 6149	13.4524 4898	12.6939 1772	12.0066 9867
43	15.3061 7294	14.3588 3708	13.5069 6167	12.7385 2811	12.0432 3951
44	15.3831 8202	14.4214 4327	13.5579 0810	12.7800 2615	12.0770 7362
45	15.4558 3209	14.4802 2842	13.6055 2159	12.8186 2898	12.1084 0150
46	15.5243 6990	14.5354 2575	13.6500 2018	12.8545 3858	12.1374 0880
47	15.5890 2821	14.5872 5422	13.6916 0764	12.8879 4287	12.1642 6741
48	15.6500 2661	14.6359 1946	13.7304 7443	12.9190 1662	12.1891 3649
49	15.7075 7227	14.6816 1451	13.7667 9853	12.9479 2244	12.2121 6341
50	15.7618 6064	14.7245 2067	13.8007 4629	12.9748 1157	12.2334 8464

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i} \text{ (Continued)}$$

n	$8\frac{1}{2}\%$	9%	$9\frac{1}{2}\%$	10%	$10\frac{1}{2}\%$
1	0.9216 5899	0.9174 3119	0.9132 4201	0.9090 9091	0.9049 7738
2	1.7711 1427	1.7591 1119	1.7472 5298	1.7355 3719	1.7239 6143
3	2.5540 2237	2.5312 9467	2.5089 0683	2.4868 5199	2.4651 2346
4	3.2755 9666	3.2397 1988	3.2044 8112	3.1698 6545	3.1358 5834
5	3.9406 4208	3.8896 5126	3.8397 0879	3.7907 8677	3.7428 5822
6	4.5535 8717	4.4859 1859	4.4198 2538	4.3552 6070	4.2921 7939
7	5.1185 1352	5.0329 5284	4.9496 1222	4.8684 1882	4.7893 0261
8	5.6391 8297	5.5348 1911	5.4334 3581	5.3349 2620	5.2391 8789
9	6.1190 6264	5.9952 4689	5.8752 8385	5.7590 2382	5.6463 2388
10	6.5613 4806	6.4176 5770	6.2787 9803	6.1445 6711	6.0147 7274
11	6.9689 8439	6.8051 9055	6.6473 0414	6.4950 6101	6.3482 1062
12	7.3446 8607	7.1607 2528	6.9838 3940	6.8136 9182	6.6499 6437
13	7.6909 5490	7.4869 0392	7.2911 7753	7.1033 5620	6.9230 4468
14	8.0100 9668	7.7861 5039	7.5718 5163	7.3666 8746	7.1701 7618
15	8.3042 3658	8.0606 8843	7.8281 7500	7.6060 7951	7.3938 2459
16	8.5753 3325	8.3125 5819	8.0622 6028	7.8237 0864	7.5962 2135
17	8.8251 9194	8.5436 3137	8.2760 3678	8.0215 5331	7.7793 8584
18	9.0554 7644	8.7556 2511	8.4712 6647	8.2014 1210	7.9451 4556
19	9.2677 2022	8.9501 1478	8.6495 5842	8.3649 2009	8.0951 5435
20	9.4633 3661	9.1285 4567	8.8123 8212	8.5135 6372	8.2309 0891
21	9.6436 2821	9.2922 4373	8.9610 7956	8.6486 9429	8.3537 6372
22	9.8097 9559	9.4424 2544	9.0968 7631	8.7715 4026	8.4649 4455
23	9.9629 4524	9.5802 0683	9.2208 9161	8.8832 1842	8.5655 6067
24	10.1040 9700	9.7066 1177	9.3341 4759	8.9847 4402	8.6566 1599
25	10.2341 9078	9.8225 7960	9.4375 7770	9.0770 4002	8.7390 1900
26	10.3540 9288	9.9289 7211	9.5320 3443	9.1609 4547	8.8135 9186
27	10.4646 0174	10.0265 7992	9.6182 9629	9.2372 2316	8.8810 7860
28	10.5664 5321	10.1161 2837	9.6970 7423	9.3065 6651	8.9421 5258
29	10.6603 2554	10.1982 8291	9.7690 1756	9.3696 0591	8.9974 2315
30	10.7468 4382	10.2736 5404	9.8347 1924	9.4269 1447	9.0474 4176
31	10.8265 8416	10.3428 0187	9.8947 2076	9.4790 1315	9.0927 0748
32	10.9000 7757	10.4062 4025	9.9495 1668	9.5263 7559	9.1336 7193
33	10.9678 1343	10.4644 4060	9.9995 5861	9.5694 3236	9.1707 4383
34	11.0302 4279	10.5178 3541	10.0452 5901	9.6085 7487	9.2042 9305
35	11.0877 8137	10.5668 2148	10.0869 9453	9.6441 5897	9.2346 5435
36	11.1408 1233	10.6117 6282	10.1251 0916	9.6765 0816	9.2621 3063
37	11.1896 8878	10.6529 9342	10.1599 1704	9.7059 1651	9.2869 9605
38	11.2347 3620	10.6908 1965	10.1917 0506	9.7326 5137	9.3094 9869
39	11.2762 5457	10.7255 2261	10.2207 3521	9.7569 5579	9.3298 6306
40	11.3145 2034	10.7573 6020	10.2472 4677	9.7790 5072	9.3482 9237
41	11.3497 8833	10.7865 6899	10.2714 5824	9.7991 3702	9.3649 7047
42	11.3822 9339	10.8133 6604	10.2935 6917	9.8173 9729	9.3800 6377
43	11.4122 5197	10.8379 5050	10.3137 6180	9.8339 9753	9.3937 2287
44	11.4398 6357	10.8605 0504	10.3322 0255	9.8490 8867	9.4060 8404
45	11.4653 1205	10.8811 9729	10.3490 4343	9.8628 0788	9.4172 7063
46	11.4887 6686	10.9001 8100	10.3644 2322	9.8752 7989	9.4273 9423
47	11.5103 8420	10.9175 9725	10.3784 6870	9.8866 1808	9.4365 5587
48	11.5303 0802	10.9335 7546	10.3912 9561	9.8969 2553	9.4448 4694
49	11.5486 7099	10.9482 3436	10.4030 0969	9.9062 9594	9.4523 5017
50	11.5655 9538	10.9616 8290	10.4137 0748	9.9148 1449	9.4591 4043

* Source : Taken from, MATHEMATICS OF FINANCE, Third Ed. : Hummel and Seebeck, 1971, pp. 308 - 323.

Appendix F

Annuity Table

Values of $\dfrac{1}{s_{\overline{n}|i}}$

n	¼ % (0.0025)	7⁄24 % (0.00291667)	⅓ % (0.00333333)	5⁄12 % (0.00416667)	½ % (0.005)
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	0.4993 7578	0.4992 7190	0.4991 6805	0.4989 6050	0.4987 5312
3	0.3325 0139	0.3323 6300	0.3322 2469	0.3319 4829	0.3316 7221
4	0.2490 6445	0.2489 0890	0.2487 5347	0.2484 4291	0.2481 3279
5	0.1990 0250	0.1988 3673	0.1986 7110	0.1983 4026	0.1980 0997
6	0.1656 2803	0.1654 5552	0.1652 8317	0.1649 3898	0.1645 9546
7	0.1417 8928	0.1416 1200	0.1414 3491	0.1410 8133	0.1407 2854
8	0.1239 1035	0.1237 2953	0.1235 4895	0.1231 8845	0.1228 2886
9	0.1100 0462	0.1098 2111	0.1096 3785	0.1092 7209	0.1089 0736
10	0.0988 8015	0.0986 9451	0.0985 0915	0.0981 3929	0.0977 7057
11	0.0897 7840	0.0895 9106	0.0894 0402	0.0890 3090	0.0886 5903
12	0.0821 9370	0.0820 0496	0.0818 1657	0.0814 4082	0.0810 6643
13	0.0757 7595	0.0755 8607	0.0753 9656	0.0750 1866	0.0746 4224
14	0.0702 7510	0.0700 8426	0.0698 9383	0.0695 1416	0.0691 3609
15	0.0655 0777	0.0653 1613	0.0651 2491	0.0647 4378	0.0643 6436
16	0.0613 3642	0.0611 4409	0.0609 5223	0.0605 6988	0.0601 8937
17	0.0576 5587	0.0574 6297	0.0572 7056	0.0568 8720	0.0565 0579
18	0.0543 8433	0.0541 9094	0.0539 9807	0.0536 1387	0.0532 3173
19	0.0514 5722	0.0512 6341	0.0510 7015	0.0506 8525	0.0503 0253
20	0.0488 2288	0.0486 2870	0.0484 3511	0.0480 4963	0.0476 6645
21	0.0464 3947	0.0462 4499	0.0460 5111	0.0456 6517	0.0452 8163
22	0.0442 7278	0.0440 7804	0.0438 8393	0.0434 9760	0.0431 1380
23	0.0422 9455	0.0420 9958	0.0419 0528	0.0415 1865	0.0411 3465
24	0.0404 8121	0.0402 8606	0.0400 9159	0.0397 0472	0.0393 2061
25	0.0388 1298	0.0386 1767	0.0384 2307	0.0380 3603	0.0376 5186
26	0.0372 7312	0.0370 7767	0.0368 8297	0.0364 9581	0.0361 1163
27	0.0358 4736	0.0356 5180	0.0354 5702	0.0350 6978	0.0346 8565
28	0.0345 2347	0.0343 2783	0.0341 3299	0.0337 4572	0.0333 6167
29	0.0332 9093	0.0330 9521	0.0329 0033	0.0325 1307	0.0321 2914
30	0.0321 4059	0.0319 4482	0.0317 4992	0.0313 6270	0.0309 7892
31	0.0310 6449	0.0308 6869	0.0306 7378	0.0302 8663	0.0299 0304
32	0.0300 5569	0.0298 5987	0.0296 6496	0.0292 7791	0.0288 9453
33	0.0291 0806	0.0289 1222	0.0287 1734	0.0283 3041	0.0279 4727
34	0.0282 1620	0.0280 2037	0.0278 2551	0.0274 3873	0.0270 5586
35	0.0273 7533	0.0271 7951	0.0269 8470	0.0265 9809	0.0262 1550
36	0.0265 8121	0.0263 8541	0.0261 9065	0.0258 0423	0.0254 2194
37	0.0258 3004	0.0256 3428	0.0254 3957	0.0250 5336	0.0246 7139
38	0.0251 1843	0.0249 2271	0.0247 2808	0.0243 4208	0.0239 6045
39	0.0244 4335	0.0242 4767	0.0240 5311	0.0236 6736	0.0232 8607
40	0.0238 0204	0.0236 0642	0.0234 1194	0.0230 2644	0.0226 4552
41	0.0231 9204	0.0229 9648	0.0228 0209	0.0224 1685	0.0220 3631
42	0.0226 1112	0.0224 1562	0.0222 2133	0.0218 3637	0.0214 5622
43	0.0220 5724	0.0218 6181	0.0216 6762	0.0212 8295	0.0209 0320
44	0.0215 2855	0.0213 3321	0.0211 3912	0.0207 5474	0.0203 7541
45	0.0210 2339	0.0208 2813	0.0206 3415	0.0202 5008	0.0198 7117
46	0.0205 4022	0.0203 4504	0.0201 5118	0.0197 6743	0.0193 8894
47	0.0200 7762	0.0198 8254	0.0196 8880	0.0193 0537	0.0189 2733
48	0.0196 3433	0.0194 3933	0.0192 4572	0.0188 6263	0.0184 8503
49	0.0192 0915	0.0190 1425	0.0188 2077	0.0184 3801	0.0180 6087
50	0.0188 0099	0.0186 0620	0.0184 1285	0.0180 3044	0.0176 5376

$$\frac{1}{s_{\overline{n}|i}} = \frac{i}{(1+i)^n - 1} \qquad \left(\frac{1}{a_{\overline{n}|i}} = \frac{1}{s_{\overline{n}|i}} + i\right) \text{ (Continued)}$$

n	¼% (0.0025)	7⁄24% (0.00291667)	⅓% (0.00333333)	5⁄12% (0.00416667)	½% (0.005)
51	0.0184 0886	0.0182 1418	0.0180 2096	0.0176 3891	0.0172 6269
52	0.0180 3184	0.0178 3726	0.0176 4418	0.0172 6249	0.0168 8675
53	0.0176 6906	0.0174 7460	0.0172 8165	0.0169 0033	0.0165 2507
54	0.0173 1974	0.0171 2539	0.0169 3259	0.0165 5164	0.0161 7686
55	0.0169 8314	0.0167 8890	0.0165 9625	0.0162 1567	0.0158 4139
56	0.0166 5858	0.0164 6446	0.0162 7196	0.0158 9176	0.0155 1797
57	0.0163 4542	0.0161 5143	0.0159 5907	0.0155 7927	0.0152 0598
58	0.0160 4308	0.0158 4922	0.0156 5701	0.0152 7760	0.0149 0481
59	0.0157 5101	0.0155 5727	0.0153 6522	0.0149 8620	0.0146 1392
60	0.0154 6869	0.0152 7508	0.0150 8319	0.0147 0457	0.0143 3280
61	0.0151 9564	0.0150 0216	0.0148 1043	0.0144 3221	0.0140 6096
62	0.0149 3142	0.0147 3807	0.0145 4650	0.0141 6869	0.0137 9796
63	0.0146 7561	0.0144 8239	0.0142 9098	0.0139 1358	0.0135 4337
64	0.0144 2780	0.0142 3472	0.0140 4348	0.0136 6649	0.0132 9681
65	0.0141 8764	0.0139 9469	0.0138 0361	0.0134 2704	0.0130 5789
66	0.0139 5476	0.0137 6196	0.0135 7105	0.0131 9489	0.0128 2627
67	0.0137 2886	0.0135 3619	0.0133 4545	0.0129 6972	0.0126 0163
68	0.0135 0961	0.0133 1709	0.0131 2652	0.0127 5121	0.0123 8366
69	0.0132 9674	0.0131 0436	0.0129 1395	0.0125 3908	0.0121 7206
70	0.0130 8996	0.0128 9772	0.0127 0749	0.0123 3304	0.0119 6657
71	0.0128 8902	0.0126 9693	0.0125 0687	0.0121 3285	0.0117 6693
72	0.0126 9368	0.0125 0173	0.0123 1185	0.0119 3827	0.0115 7289
73	0.0125 0370	0.0123 1190	0.0121 2220	0.0117 4905	0.0113 8422
74	0.0123 1887	0.0121 2722	0.0119 3769	0.0115 6498	0.0112 0070
75	0.0121 3898	0.0119 4748	0.0117 5813	0.0113 8586	0.0110 2214
76	0.0119 6385	0.0117 7250	0.0115 8332	0.0112 1150	0.0108 4832
77	0.0117 9327	0.0116 0207	0.0114 1308	0.0110 4170	0.0106 7908
78	0.0116 2708	0.0114 3603	0.0112 4722	0.0108 7629	0.0105 1423
79	0.0114 6511	0.0112 7422	0.0110 8559	0.0107 1510	0.0103 5360
80	0.0113 0721	0.0111 1647	0.0109 2802	0.0105 5798	0.0101 9704
81	0.0111 5321	0.0109 6263	0.0107 7436	0.0104 0477	0.0100 4439
82	0.0110 0298	0.0108 1256	0.0106 2447	0.0102 5534	0.0098 9552
83	0.0108 5639	0.0106 6612	0.0104 7822	0.0101 0954	0.0097 5028
84	0.0107 1330	0.0105 2318	0.0103 3547	0.0099 6724	0.0096 0855
85	0.0105 7359	0.0103 8363	0.0101 9610	0.0098 2833	0.0094 7021
86	0.0104 3714	0.0102 4734	0.0100 6000	0.0096 9268	0.0093 3513
87	0.0103 0384	0.0101 1419	0.0099 2704	0.0095 6018	0.0092 0320
88	0.0101 7357	0.0099 8409	0.0097 9713	0.0094 3073	0.0090 7431
89	0.0100 4625	0.0098 5693	0.0096 7015	0.0093 0422	0.0089 4837
90	0.0099 2177	0.0097 3261	0.0095 4602	0.0091 8055	0.0088 2527
91	0.0098 0004	0.0096 1104	0.0094 2464	0.0090 5962	0.0087 0493
92	0.0096 8096	0.0094 9212	0.0093 0592	0.0089 4136	0.0085 8724
93	0.0095 6446	0.0093 7578	0.0091 8976	0.0088 2568	0.0084 7213
94	0.0094 5044	0.0092 6193	0.0090 7610	0.0087 1248	0.0083 5950
95	0.0093 3884	0.0091 5049	0.0089 6485	0.0086 0170	0.0082 4930
96	0.0092 2957	0.0090 4139	0.0088 5594	0.0084 9325	0.0081 4143
97	0.0091 2257	0.0089 3455	0.0087 4929	0.0083 8707	0.0080 3583
98	0.0090 1776	0.0088 2990	0.0086 4484	0.0082 8309	0.0079 3242
99	0.0089 1508	0.0087 2738	0.0085 4252	0.0081 8124	0.0078 3115
100	0.0088 1446	0.0086 2693	0.0084 4226	0.0080 8145	0.0077 3194

$$\frac{1}{s_{\overline{n}|i}} = \frac{i}{(1+i)^n - 1} \qquad \left(\frac{1}{a_{\overline{n}|i}} = \frac{1}{s_{\overline{n}|i}} + i\right) \text{ (Continued)}$$

n	¼% (0.0025)	7/24% (0.00291667)	⅓% (0.00333333)	5/12% (0.00416667)	½% (0.005)
101	0.0087 1584	0.0085 2848	0.0083 4400	0.0079 8366	0.0076 3473
102	0.0086 1917	0.0084 3198	0.0082 4769	0.0078 8782	0.0075 3947
103	0.0085 2439	0.0083 3736	0.0081 5327	0.0077 9387	0.0074 4610
104	0.0084 3144	0.0082 4457	0.0080 6068	0.0077 0175	0.0073 5457
105	0.0083 4027	0.0081 5357	0.0079 6987	0.0076 1142	0.0072 6481
106	0.0082 5082	0.0080 6430	0.0078 8079	0.0075 2281	0.0071 7679
107	0.0081 6307	0.0079 7670	0.0077 9340	0.0074 3589	0.0070 9045
108	0.0080 7694	0.0078 9075	0.0077 0764	0.0073 5061	0.0070 0575
109	0.0079 9241	0.0078 0638	0.0076 2346	0.0072 6691	0.0069 2264
110	0.0079 0942	0.0077 2356	0.0075 4084	0.0071 8476	0.0068 4107
111	0.0078 2793	0.0076 4225	0.0074 5972	0.0071 0412	0.0067 6102
112	0.0077 4791	0.0075 6239	0.0073 8007	0.0070 2495	0.0066 8242
113	0.0076 6932	0.0074 8397	0.0073 0184	0.0069 4720	0.0066 0526
114	0.0075 9211	0.0074 0693	0.0072 2500	0.0068 7083	0.0065 2948
115	0.0075 1626	0.0073 3125	0.0071 4952	0.0067 9582	0.0064 5506
116	0.0074 4172	0.0072 5688	0.0070 7535	0.0067 2213	0.0063 8195
117	0.0073 6846	0.0071 8380	0.0070 0246	0.0066 4973	0.0063 1013
118	0.0072 9646	0.0071 1196	0.0069 3082	0.0065 7857	0.0062 3956
119	0.0072 2567	0.0070 4135	0.0068 6041	0.0065 0863	0.0061 7021
120	0.0071 5607	0.0069 7192	0.0067 9118	0.0064 3988	0.0061 0205
121	0.0070 8764	0.0069 0365	0.0067 2311	0.0063 7230	0.0060 3505
122	0.0070 2033	0.0068 3652	0.0066 5617	0.0063 0584	0.0059 6918
123	0.0069 5412	0.0067 7048	0.0065 9034	0.0062 4049	0.0059 0441
124	0.0068 8899	0.0067 0552	0.0065 2558	0.0061 7621	0.0058 4072
125	0.0068 2491	0.0066 4162	0.0064 6188	0.0061 1298	0.0057 7808
126	0.0067 6186	0.0065 7874	0.0063 9919	0.0060 5078	0.0057 1647
127	0.0066 9981	0.0065 1686	0.0063 3751	0.0059 8959	0.0056 5586
128	0.0066 3873	0.0064 5595	0.0062 7681	0.0059 2937	0.0055 9623
129	0.0065 7861	0.0063 9601	0.0062 1707	0.0058 7010	0.0055 3755
130	0.0065 1942	0.0063 3699	0.0061 5825	0.0058 1177	0.0054 7981
131	0.0064 6115	0.0062 7889	0.0061 0035	0.0057 5435	0.0054 2298
132	0.0064 0376	0.0062 2168	0.0060 4334	0.0056 9782	0.0053 6703
133	0.0063 4725	0.0061 6534	0.0059 8720	0.0056 4216	0.0053 1197
134	0.0062 9159	0.0061 0985	0.0059 3191	0.0055 8736	0.0052 5775
135	0.0062 3675	0.0060 5519	0.0058 7745	0.0055 3339	0.0052 0436
136	0.0061 8274	0.0060 0135	0.0058 2381	0.0054 8023	0.0051 5179
137	0.0061 2952	0.0059 4830	0.0057 7097	0.0054 2787	0.0051 0002
138	0.0060 7707	0.0058 9603	0.0057 1890	0.0053 7628	0.0050 4902
139	0.0060 2539	0.0058 4453	0.0056 6760	0.0053 2546	0.0049 9879
140	0.0059 7446	0.0057 9377	0.0056 1704	0.0052 7539	0.0049 4930
141	0.0059 2425	0.0057 4373	0.0055 6721	0.0052 2604	0.0049 0055
142	0.0058 7476	0.0056 9442	0.0055 1809	0.0051 7741	0.0048 5250
143	0.0058 2597	0.0056 4580	0.0054 6968	0.0051 2948	0.0048 0516
144	0.0057 7787	0.0055 9787	0.0054 2195	0.0050 8224	0.0047 5850
145	0.0057 3043	0.0055 5061	0.0053 7489	0.0050 3566	0.0047 1252
146	0.0056 8365	0.0055 0401	0.0053 2849	0.0049 8975	0.0046 6718
147	0.0056 3752	0.0054 5805	0.0052 8273	0.0049 4447	0.0046 2250
148	0.0055 9201	0.0054 1272	0.0052 3760	0.0048 9983	0.0045 7844
149	0.0055 4712	0.0053 6800	0.0051 9309	0.0048 5580	0.0045 3500
150	0.0055 0284	0.0053 2390	0.0051 4919	0.0048 1238	0.0044 9217

$$\frac{1}{s_{\overline{n}|i}} = \frac{i}{(1+i)^n - 1} \qquad \left(\frac{1}{a_{\overline{n}|i}} = \frac{1}{s_{\overline{n}|i}} + i\right) \text{(Continued)}$$

n	¼ % (0.0025)	7/24 % (0.00291667)	⅓ % (0.00333333)	5/12 % (0.00416667)	½ % (0.005)
151	0.0054 5915	0.0052 8038	0.0051 0588	0.0047 6956	0.0044 4993
152	0.0054 1605	0.0052 3745	0.0050 6315	0.0047 2731	0.0044 0827
153	0.0053 7351	0.0051 9509	0.0050 2099	0.0046 8564	0.0043 6719
154	0.0053 3153	0.0051 5329	0.0049 7939	0.0046 4453	0.0043 2666
155	0.0052 9010	0.0051 1203	0.0049 3834	0.0046 0396	0.0042 8668
156	0.0052 4921	0.0050 7132	0.0048 9783	0.0045 6393	0.0042 4723
157	0.0052 0885	0.0050 3113	0.0048 5784	0.0045 2443	0.0042 0832
158	0.0051 6900	0.0049 9146	0.0048 1837	0.0044 8545	0.0041 6992
159	0.0051 2966	0.0049 5230	0.0047 7941	0.0044 4697	0.0041 3203
160	0.0050 9082	0.0049 1363	0.0047 4095	0.0044 0889	0.0040 9464
161	0.0050 5247	0.0048 7545	0.0047 0298	0.0043 7150	0.0040 5773
162	0.0050 1459	0.0048 3776	0.0046 6549	0.0043 3450	0.0040 2131
163	0.0049 7719	0.0048 0053	0.0046 2846	0.0042 9796	0.0039 8536
164	0.0049 4025	0.0047 6377	0.0045 9190	0.0042 6188	0.0039 4987
165	0.0049 0377	0.0047 2746	0.0045 5580	0.0042 2626	0.0039 1483
166	0.0048 6773	0.0046 9160	0.0045 2014	0.0041 9109	0.0038 8024
167	0.0048 3213	0.0046 5617	0.0044 8492	0.0041 5635	0.0038 4608
168	0.0047 9695	0.0046 2118	0.0044 5012	0.0041 2204	0.0038 1236
169	0.0047 6220	0.0045 8660	0.0044 1575	0.0040 8815	0.0037 7906
170	0.0047 2787	0.0045 5244	0.0043 8180	0.0040 5468	0.0037 4617
171	0.0046 9394	0.0045 1869	0.0043 4825	0.0040 2162	0.0037 1369
172	0.0046 6042	0.0044 8534	0.0043 1510	0.0039 8896	0.0036 8161
173	0.0046 2728	0.0044 5239	0.0042 8235	0.0039 5669	0.0036 4992
174	0.0045 9454	0.0044 1982	0.0042 4998	0.0039 2481	0.0036 1862
175	0.0045 6217	0.0043 8763	0.0042 1800	0.0038 9330	0.0035 8770
176	0.0045 3018	0.0043 5581	0.0041 8639	0.0038 6217	0.0035 5715
177	0.0044 9855	0.0043 2436	0.0041 5514	0.0038 3141	0.0035 2697
178	0.0044 6729	0.0042 9327	0.0041 2426	0.0038 0101	0.0034 9715
179	0.0044 3638	0.0042 6254	0.0040 9373	0.0037 7097	0.0034 6768
180	0.0044 0582	0.0042 3216	0.0040 6355	0.0037 4127	0.0034 3857
181	0.0043 7560	0.0042 0212	0.0040 3371	0.0037 1192	0.0034 0979
182	0.0043 4572	0.0041 7242	0.0040 0421	0.0036 8290	0.0033 8136
183	0.0043 1617	0.0041 4305	0.0039 7504	0.0036 5422	0.0033 5325
184	0.0042 8695	0.0041 1400	0.0039 4620	0.0036 2586	0.0033 2547
185	0.0042 5805	0.0040 8528	0.0039 1768	0.0035 9782	0.0032 9802
186	0.0042 2947	0.0040 5687	0.0038 8948	0.0035 7010	0.0032 7088
187	0.0042 0120	0.0040 2878	0.0038 6159	0.0035 4269	0.0032 4404
188	0.0041 7323	0.0040 0099	0.0038 3400	0.0035 1559	0.0032 1752
189	0.0041 4557	0.0039 7350	0.0038 0672	0.0034 8879	0.0031 9129
190	0.0041 1820	0.0039 4631	0.0037 7973	0.0034 6228	0.0031 6537
191	0.0040 9112	0.0039 1941	0.0037 5304	0.0034 3607	0.0031 3973
192	0.0040 6434	0.0038 9280	0.0037 2663	0.0034 1014	0.0031 1438
193	0.0040 3783	0.0038 6647	0.0037 0050	0.0033 8450	0.0030 8931
194	0.0040 1160	0.0038 4042	0.0036 7466	0.0033 5913	0.0030 6452
195	0.0039 8565	0.0038 1465	0.0036 4908	0.0033 3404	0.0030 4000
196	0.0039 5997	0.0037 8914	0.0036 2378	0.0033 0922	0.0030 1576
197	0.0039 3455	0.0037 6390	0.0035 9874	0.0032 8467	0.0029 9178
198	0.0039 0939	0.0037 3892	0.0035 7397	0.0032 6037	0.0029 6806
199	0.0038 8450	0.0037 1420	0.0035 4945	0.0032 3634	0.0029 4459
200	0.0038 5985	0.0036 8974	0.0035 2519	0.0032 1255	0.0029 2138

$$\frac{1}{s_{\overline{n}|i}} = \frac{i}{(1+i)^n - 1} \qquad \left(\frac{1}{a_{\overline{n}|i}} = \frac{1}{s_{\overline{n}|i}} + i\right) \text{ (Continued)}$$

n	7⁄12 % (0.00583333)	2⁄3 % (0.00666667)	3⁄4 % (0.0075)	7⁄8 % (0.00875)	1 % (0.01)
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	0.4985 4591	0.4983 3887	0.4981 3200	0.4978 2203	0.4975 1244
3	0.3313 9643	0.3311 2095	0.3308 4579	0.3304 3361	0.3300 2211
4	0.2478 2310	0.2475 1384	0.2472 0501	0.2467 4257	0.2462 8109
5	0.1976 8024	0.1973 5105	0.1970 2242	0.1965 3049	0.1960 3980
6	0.1642 5260	0.1639 1042	0.1635 6891	0.1630 5789	0.1625 4837
7	0.1403 7653	0.1400 2531	0.1396 7488	0.1391 5070	0.1386 2828
8	0.1224 7018	0.1221 1240	0.1217 5552	0.1212 2190	0.1206 9029
9	0.1085 4365	0.1081 8096	0.1078 1929	0.1072 7868	0.1067 4036
10	0.0974 0299	0.0970 3654	0.0966 7123	0.0961 2538	0.0955 8208
11	0.0882 8842	0.0879 1905	0.0875 5094	0.0870 0111	0.0864 5408
12	0.0806 9341	0.0803 2176	0.0799 5148	0.0793 9860	0.0788 4879
13	0.0742 6730	0.0738 9385	0.0735 2188	0.0729 6669	0.0724 1482
14	0.0687 5962	0.0638 8474	0.0680 1146	0.0674 5453	0.0669 0117
15	0.0639 8666	0.0636 1067	0.0632 3639	0.0626 7817	0.0621 2378
16	0.0598 1068	0.0594 3382	0.0590 5879	0.0584 9965	0.0579 4460
17	0.0561 2632	0.0557 4880	0.0553 7321	0.0548 1346	0.0542 5806
18	0.0528 5165	0.0524 7363	0.0520 9766	0.0515 3756	0.0509 8205
19	0.0499 2198	0.0495 4361	0.0491 6740	0.0486 0715	0.0480 5175
20	0.0472 8556	0.0469 0696	0.0465 3063	0.0459 7042	0.0454 1531
21	0.0449 0050	0.0445 2176	0.0441 4543	0.0435 8541	0.0430 3075
22	0.0427 3251	0.0423 5374	0.0419 7748	0.0414 1779	0.0408 6372
23	0.0407 5329	0.0403 7456	0.0399 9846	0.0394 3921	0.0388 8584
24	0.0389 3925	0.0385 6062	0.0381 8474	0.0376 2604	0.0370 7347
25	0.0372 7055	0.0368 9210	0.0365 1650	0.0359 5843	0.0354 0675
26	0.0357 3043	0.0353 5220	0.0349 7693	0.0344 1959	0.0338 6888
27	0.0343 0460	0.0339 2664	0.0335 5176	0.0329 9520	0.0324 4553
28	0.0329 8082	0.0326 0317	0.0322 2871	0.0316 7300	0.0311 2444
29	0.0317 4853	0.0313 7123	0.0309 9723	0.0304 4243	0.0298 9502
30	0.0305 9857	0.0302 2166	0.0298 4816	0.0292 9431	0.0287 4811
31	0.0295 2299	0.0291 4649	0.0287 7352	0.0282 2068	0.0276 7573
32	0.0285 1482	0.0281 3875	0.0277 6634	0.0272 1454	0.0266 7089
33	0.0275 6791	0.0271 9231	0.0268 2048	0.0262 6976	0.0257 2744
34	0.0266 7687	0.0263 0176	0.0259 3053	0.0253 8092	0.0248 3997
35	0.0258 3691	0.0254 6231	0.0250 9170	0.0245 4324	0.0240 0368
36	0.0250 4376	0.0246 6970	0.0242 9973	0.0237 5244	0.0232 1431
37	0.0242 9365	0.0239 2013	0.0235 5082	0.0230 0473	0.0224 6805
38	0.0235 8316	0.0232 1020	0.0228 4157	0.0222 9671	0.0217 6150
39	0.0229 0925	0.0225 3687	0.0221 6893	0.0216 2531	0.0210 9160
40	0.0222 6917	0.0218 9739	0.0215 3016	0.0209 8780	0.0204 5560
41	0.0216 6046	0.0212 8928	0.0209 2276	0.0203 8169	0.0198 5102
42	0.0210 8087	0.0207 1031	0.0203 4452	0.0198 0475	0.0192 7563
43	0.0205 2836	0.0201 5843	0.0197 9338	0.0192 5493	0.0187 2737
44	0.0200 0110	0.0196 3180	0.0192 6751	0.0187 3039	0.0182 0441
45	0.0194 9740	0.0191 2875	0.0187 6521	0.0182 2943	0.0177 0505
46	0.0190 1571	0.0186 4772	0.0182 8495	0.0177 5053	0.0172 2775
47	0.0185 5465	0.0181 8732	0.0178 2532	0.0172 9228	0.0167 7111
48	0.0181 1291	0.0177 4626	0.0173 8504	0.0168 5338	0.0163 3384
49	0.0176 8932	0.0173 2334	0.0169 6292	0.0164 3265	0.0159 1474
50	0.0172 8278	0.0169 1749	0.0165 5787	0.0160 2900	0.0155 1273

$$\frac{1}{s_{\overline{n}|i}} = \frac{i}{(1+i)^n - 1} \qquad \left(\frac{1}{a_{\overline{n}|i}} = \frac{1}{s_{\overline{n}|i}} + i\right) \text{ (Continued)}$$

n	$\frac{7}{12}$ % (0.0058 3333)	$\frac{2}{3}$ % (0.0066 6667)	$\frac{3}{4}$ % (0.0075)	$\frac{7}{8}$ % (0.00875)	1 % (0.01)
51	0.0168 9230	0.0165 2770	0.0161 6888	0.0156 4142	0.0151 2680
52	0.0165 1694	0.0161 5304	0.0157 9503	0.0152 6899	0.0147 5603
53	0.0161 5585	0.0157 9266	0.0154 3546	0.0149 1084	0.0143 9956
54	0.0158 0824	0.0154 4576	0.0150 8938	0.0145 6619	0.0140 5658
55	0.0154 7337	0.0151 1160	0.0147 5605	0.0142 3430	0.0137 2637
56	0.0151 5056	0.0147 8951	0.0144 3478	0.0139 1449	0.0134 0824
57	0.0148 3918	0.0144 7885	0.0141 2496	0.0136 0611	0.0131 0156
58	0.0145 3863	0.0141 7903	0.0138 2597	0.0133 0858	0.0128 0573
59	0.0142 4836	0.0138 8949	0.0135 3727	0.0130 2135	0.0125 2020
60	0.0139 6787	0.0136 0973	0.0132 5836	0.0127 4390	0.0122 4445
61	0.0136 9666	0.0133 3926	0.0129 8873	0.0124 7575	0.0119 7800
62	0.0134 3428	0.0130 7763	0.0127 2795	0.0122 1644	0.0117 2041
63	0.0131 8033	0.0128 2442	0.0124 7560	0.0119 6557	0.0114 7125
64	0.0129 3440	0.0125 7923	0.0122 3127	0.0117 2273	0.0112 3013
65	0.0126 9612	0.0123 4171	0.0119 9460	0.0114 8754	0.0109 9667
66	0.0124 6515	0.0121 1149	0.0117 6524	0.0112 5968	0.0107 7052
67	0.0122 4116	0.0118 8825	0.0115 4286	0.0110 3879	0.0105 5136
68	0.0120 2383	0.0116 7168	0.0113 2716	0.0108 2459	0.0103 3889
69	0.0118 1289	0.0114 6150	0.0111 1785	0.0106 1677	0.0101 3280
70	0.0116 0805	0.0112 5742	0.0109 1464	0.0104 1506	0.0099 3282
71	0.0114 0906	0.0110 5919	0.0107 1728	0.0102 1921	0.0097 3870
72	0.0112 1567	0.0108 6657	0.0105 2554	0.0100 2897	0.0095 5019
73	0.0110 2766	0.0106 7933	0.0103 3917	0.0098 4411	0.0093 6706
74	0.0108 4481	0.0104 9725	0.0101 5796	0.0096 6441	0.0091 8910
75	0.0106 6690	0.0103 2011	0.0099 8170	0.0094 8966	0.0090 1609
76	0.0104 9375	0.0101 4773	0.0098 1020	0.0093 1967	0.0088 4784
77	0.0103 2517	0.0099 7993	0.0096 4328	0.0091 5426	0.0086 8416
78	0.0101 6099	0.0098 1652	0.0094 8074	0.0089 9324	0.0085 2488
79	0.0100 0103	0.0096 5733	0.0093 2244	0.0088 3645	0.0083 6983
80	0.0098 4514	0.0095 0222	0.0091 6821	0.0086 8374	0.0082 1885
81	0.0096 9316	0.0093 5102	0.0090 1790	0.0085 3494	0.0080 7179
82	0.0095 4496	0.0092 0360	0.0088 7136	0.0083 8992	0.0079 2851
83	0.0094 0040	0.0090 5982	0.0087 2847	0.0082 4854	0.0077 8887
84	0.0092 5935	0.0089 1955	0.0085 8908	0.0081 1067	0.0076 5273
85	0.0091 2168	0.0087 8266	0.0084 5308	0.0079 7619	0.0075 1998
86	0.0089 8727	0.0086 4904	0.0083 2034	0.0078 4497	0.0073 9050
87	0.0088 5602	0.0085 1857	0.0081 9076	0.0077 1691	0.0072 6418
88	0.0087 2781	0.0083 9115	0.0080 6423	0.0075 9190	0.0071 4089
89	0.0086 0255	0.0082 6667	0.0079 4064	0.0074 6982	0.0070 2056
90	0.0084 8013	0.0081 4504	0.0078 1989	0.0073 5060	0.0069 0306
91	0.0083 6047	0.0080 2616	0.0077 0190	0.0072 3413	0.0067 8832
92	0.0082 4346	0.0079 0994	0.0075 8657	0.0071 2031	0.0066 7624
93	0.0081 2903	0.0077 9629	0.0074 7382	0.0070 0908	0.0065 6673
94	0.0080 1709	0.0076 8514	0.0073 6356	0.0069 0033	0.0064 5971
95	0.0079 0757	0.0075 7641	0.0072 5571	0.0067 9401	0.0063 5511
96	0.0078 0038	0.0074 7001	0.0071 5020	0.0066 9002	0.0062 5284
97	0.0076 9547	0.0073 6588	0.0070 4696	0.0065 8829	0.0061 5284
98	0.0075 9275	0.0072 6394	0.0069 4592	0.0064 8877	0.0060 5503
99	0.0074 9216	0.0071 6415	0.0068 4701	0.0063 9137	0.0059 5936
100	0.0073 9363	0.0070 6642	0.0067 5017	0.0062 9604	0.0058 6574

* Source : Taken from, MATHEMATICS OF FINANCE, Third Ed. : Hummel and Seebeck, 1971, pp. 324 - 329.

Appendix G

Table of Present Worth of 1

And Present Worth of 1 per period for High Rates.

n	4%	6%	8%	10%	12%	14%	16%	18%	20%	22%	24%
1	.962	.943	.926	.909	.893	.877	.862	.847	.833	.820	.806
2	.925	.890	.857	.826	.797	.769	.743	.718	.694	.672	.650
3	.889	.840	.794	.751	.712	.675	.641	.609	.579	.551	.524
4	.855	.792	.735	.683	.636	.592	.552	.516	.482	.451	.423
5	.822	.747	.681	.621	.567	.519	.476	.437	.402	.370	.341
6	.790	.705	.630	.564	.507	.456	.410	.370	.335	.303	.275
7	.760	.665	.583	.513	.452	.400	.354	.314	.279	.249	.222
8	.731	.627	.540	.467	.404	.351	.305	.266	.233	.204	.179
9	.703	.592	.500	.424	.361	.308	.263	.225	.194	.167	.144
10	.676	.558	.463	.386	.322	.270	.227	.191	.162	.137	.116
11	.650	.527	.429	.350	.287	.237	.195	.162	.135	.112	.094
12	.625	.497	.397	.319	.257	.208	.168	.137	.112	.092	.076
13	.601	.469	.368	.290	.229	.182	.145	.116	.093	.075	.061
14	.577	.442	.340	.263	.205	.160	.125	.099	.078	.062	.049
15	.555	.417	.315	.239	.183	.140	.108	.084	.065	.051	.040
16	.534	.394	.292	.218	.163	.123	.093	.071	.054	.042	.032
17	.513	.371	.270	.198	.146	.108	.080	.060	.045	.034	.026
18	.494	.350	.250	.180	.130	.095	.069	.051	.038	.028	.021
19	.475	.331	.232	.164	.116	.083	.060	.043	.031	.023	.017
20	.456	.312	.215	.149	.104	.073	.051	.037	.026	.019	.014
21	.439	.294	.199	.135	.093	.064	.044	.031	.022	.015	.011
22	.422	.278	.184	.123	.083	.056	.038	.026	.018	.013	.009
23	.406	.262	.170	.112	.074	.049	.033	.022	.015	.010	.007
24	.390	.247	.158	.102	.066	.043	.028	.019	.013	.008	.006
25	.375	.233	.146	.092	.059	.038	.024	.016	.010	.007	.005
26	.361	.220	.135	.084	.053	.033	.021	.014	.009	.006	.004
27	.347	.207	.125	.076	.047	.029	.018	.011	.007	.005	.003
28	.333	.196	.116	.069	.042	.026	.016	.010	.006	.004	.002
29	.321	.185	.107	.063	.037	.022	.014	.008	.005	.003	.002
30	.308	.174	.099	.057	.033	.020	.012	.007	.004	.003	.002
31	.296	.164	.092	.052	.030	.017	.010	.006	.004	.002	.001
32	.285	.155	.085	.047	.027	.015	.009	.005	.003	.002	.001
33	.274	.146	.079	.043	.024	.013	.007	.004	.002	.001	.001
34	.264	.138	.073	.039	.021	.012	.006	.004	.002	.001	.001
35	.253	.130	.068	.036	.019	.010	.006	.003	.002	.001	.001
36	.244	.123	.063	.032	.017	.009	.005	.003	.001	.001	—
37	.234	.116	.058	.029	.015	.008	.004	.002	.001	.001	—
38	.225	.109	.054	.027	.013	.007	.004	.002	.001	.001	—
39	.217	.103	.050	.024	.012	.006	.003	.002	.001	—	—
40	.208	.097	.046	.022	.011	.005	.003	.001	.001	—	—
41	.200	.092	.043	.020	.010	.005	.002	.001	.001	—	—
42	.193	.087	.039	.018	.009	.004	.002	.001	—	—	—
43	.185	.082	.037	.017	.008	.004	.002	.001	—	—	—
44	.178	.077	.034	.015	.007	.003	.001	.001	—	—	—
45	.171	.073	.031	.014	.006	.003	.001	.001	—	—	—
46	.165	.069	.029	.012	.005	.002	.001	—	—	—	—
47	.158	.065	.027	.011	.005	.002	.001	—	—	—	—
48	.152	.061	.025	.010	.004	.002	.001	—	—	—	—
49	.146	.058	.023	.009	.004	.002	.001	—	—	—	—
50	.141	.054	.021	.009	.003	.001	.001	—	—	—	—

	25%	30%	35%	40%	45%	50%	60%	70%	80%	90%	100%
1	.800	.769	.741	.714	.690	.667	.625	.588	.556	.526	.500
2	.640	.592	.549	.510	.476	.444	.391	.346	.309	.277	.250
3	.512	.455	.406	.364	.328	.296	.244	.204	.171	.146	.125
4	.410	.350	.301	.260	.226	.198	.153	.120	.095	.077	.063
5	.328	.269	.223	.186	.156	.132	.095	.070	.053	.040	.031
6	.262	.207	.165	.133	.108	.088	.060	.041	.029	.021	.016
7	.210	.159	.122	.095	.074	.059	.037	.024	.016	.011	.008
8	.168	.123	.091	.068	.051	.039	.023	.014	.009	.006	.004
9	.134	.094	.067	.048	.035	.026	.015	.008	.005	.003	.002
10	.107	.073	.050	.035	.024	.017	.009	.005	.003	.002	.001
11	.086	.056	.037	.025	.017	.012	.006	.003	.002	.001	—
12	.069	.043	.027	.018	.012	.008	.004	.002	.001	—	—
13	.055	.033	.020	.013	.008	.005	.002	.001	—	—	—
14	.044	.025	.015	.009	.006	.003	.001	.001	—	—	—
15	.035	.020	.011	.006	.004	.002	.001	—	—	—	—
16	.028	.015	.008	.005	.003	.002	.001	—	—	—	—
17	.023	.012	.006	.003	.002	.001	—	—	—	—	—
18	.018	.009	.005	.002	.001	.001	—	—	—	—	—
19	.014	.007	.003	.002	.001	—	—	—	—	—	—
20	.012	.005	.002	.001	.001	—	—	—	—	—	—

Present Worth of 1 Per Period

n	4%	6%	8%	10%	12%	14%	16%	18%	20%	22%	24%
1	.962	.943	.926	.909	.893	.877	.862	.847	.833	.820	.806
2	1.886	1.833	1.783	1.736	1.690	1.647	1.605	1.566	1.528	1.492	1.457
3	2.775	2.673	2.577	2.487	2.402	2.322	2.246	2.174	2.106	2.042	1.981
4	3.630	3.465	3.312	3.170	3.037	2.914	2.798	2.690	2.589	2.494	2.404
5	4.452	4.212	3.993	3.791	3.605	3.433	3.274	3.127	2.991	2.864	2.745
6	5.242	4.917	4.623	4.355	4.111	3.889	3.685	3.498	3.326	3.167	3.020
7	6.002	5.582	5.206	4.868	4.564	4.288	4.039	3.812	3.605	3.416	3.242
8	6.733	6.210	5.747	5.335	4.968	4.639	4.344	4.078	3.837	3.619	3.421
9	7.435	6.802	6.247	5.759	5.328	4.946	4.607	4.303	4.031	3.786	3.566
10	8.111	7.360	6.710	6.145	5.650	5.216	4.833	4.494	4.192	3.923	3.682
11	8.760	7.887	7.139	6.495	5.938	5.453	5.029	4.656	4.327	4.035	3.776
12	9.385	8.384	7.536	6.814	6.194	5.660	5.197	4.793	4.439	4.127	3.851
13	9.986	8.853	7.904	7.103	6.424	5.842	5.342	4.910	4.533	4.203	3.912
14	10.563	9.295	8.244	7.367	6.628	6.002	5.468	5.008	4.611	4.265	3.962
15	11.118	9.712	8.559	7.606	6.811	6.142	5.575	5.092	4.675	4.315	4.001
16	11.652	10.106	8.851	7.824	6.974	6.265	5.668	5.162	4.730	4.357	4.033
17	12.166	10.477	9.122	8.022	7.120	6.373	5.749	5.222	4.775	4.391	4.059
18	12.659	10.828	9.372	8.201	7.250	6.467	5.818	5.273	4.812	4.419	4.080
19	13.134	11.158	9.604	8.365	7.366	6.550	5.877	5.316	4.843	4.442	4.097
20	13.590	11.470	9.818	8.514	7.469	6.623	5.929	5.353	4.870	4.460	4.110
21	14.029	11.764	10.017	8.649	7.562	6.687	5.973	5.384	4.891	4.476	4.121
22	14.451	12.042	10.201	8.772	7.645	6.743	6.011	5.410	4.909	4.488	4.130
23	14.857	12.303	10.371	8.883	7.718	6.792	6.044	5.432	4.925	4.499	4.137
24	15.247	12.550	10.529	8.985	7.784	6.835	6.073	5.451	4.937	4.507	4.143
25	15.622	12.783	10.675	9.077	7.843	6.873	6.097	5.467	4.948	4.514	4.147
26	15.983	13.003	10.810	9.161	7.896	6.906	6.118	5.480	4.956	4.520	4.151
27	16.330	13.211	10.935	9.237	7.943	6.935	6.136	5.492	4.964	4.524	4.154
28	16.663	13.406	11.051	9.307	7.984	6.961	6.152	5.502	4.970	4.528	4.157
29	16.984	13.591	11.158	9.370	8.022	6.983	6.166	5.510	4.975	4.531	4.159
30	17.292	13.765	11.258	9.427	8.055	7.003	6.177	5.517	4.979	4.534	4.160
31	17.588	13.929	11.350	9.479	8.085	7.020	6.187	5.523	4.982	4.536	4.161
32	17.874	14.084	11.435	9.526	8.112	7.035	6.196	5.528	4.985	4.538	4.162
33	18.148	14.230	11.514	9.569	8.135	7.048	6.203	5.532	4.988	4.539	4.163
34	18.411	14.368	11.587	9.609	8.157	7.060	6.210	5.536	4.990	4.540	4.164
35	18.665	14.498	11.655	9.644	8.176	7.070	6.215	5.539	4.992	4.541	4.164
36	18.908	14.621	11.717	9.677	8.192	7.079	6.220	5.541	4.993	4.542	4.165
37	19.143	14.737	11.775	9.706	8.208	7.087	6.224	5.543	4.994	4.543	4.165
38	19.368	14.846	11.829	9.733	8.221	7.094	6.228	5.545	4.995	4.543	4.165
39	19.584	14.949	11.879	9.757	8.233	7.100	6.231	5.547	4.996	4.544	4.166
40	19.793	15.046	11.925	9.779	8.244	7.105	6.233	5.548	4.997	4.544	4.166
41	19.993	15.138	11.967	9.799	8.253	7.110	6.236	5.549	4.997	4.544	4.166
42	20.186	15.225	12.007	9.817	8.262	7.114	6.238	5.550	4.998	4.544	4.166
43	20.371	15.306	12.043	9.834	8.270	7.117	6.239	5.551	4.998	4.545	4.166
44	20.549	15.383	12.077	9.849	8.276	7.120	6.241	5.552	4.998	4.545	4.166
45	20.720	15.456	12.108	9.863	8.283	7.123	6.242	5.552	4.999	4.545	4.166
46	20.885	15.524	12.137	9.875	8.288	7.126	6.243	5.553	4.999	4.545	4.166
47	21.043	15.589	12.164	9.887	8.293	7.128	6.244	5.553	4.999	4.545	4.166
48	21.195	15.650	12.189	9.897	8.297	7.130	6.245	5.554	4.999	4.545	4.167
49	21.341	15.708	12.212	9.906	8.301	7.131	6.246	5.554	4.999	4.545	4.167
50	21.482	15.762	12.233	9.915	8.304	7.133	6.246	5.554	4.999	4.545	4.167

	25%	30%	35%	40%	45%	50%	60%	70%	80%	90%	100%
1	.800	.769	.741	.714	.690	.667	.625	.588	.556	.526	.500
2	1.440	1.361	1.289	1.224	1.165	1.111	1.016	.934	.864	.803	.750
3	1.952	1.816	1.696	1.589	1.493	1.407	1.260	1.138	1.036	.949	.875
4	2.362	2.166	1.997	1.849	1.720	1.605	1.412	1.258	1.131	1.026	.938
5	2.689	2.436	2.220	2.035	1.876	1.737	1.508	1.328	1.184	1.066	.969
6	2.951	2.643	2.385	2.168	1.983	1.824	1.567	1.369	1.213	1.087	.984
7	3.161	2.802	2.508	2.263	2.057	1.883	1.605	1.394	1.230	1.099	.992
8	3.329	2.925	2.598	2.331	2.109	1.922	1.628	1.408	1.239	1.105	.996
9	3.463	3.019	2.665	2.379	2.144	1.948	1.642	1.417	1.244	1.108	.998
10	3.571	3.092	2.715	2.414	2.168	1.965	1.652	1.421	1.246	1.109	.999
11	3.656	3.147	2.752	2.438	2.185	1.977	1.657	1.424	1.248	1.110	1.000
12	3.725	3.190	2.779	2.456	2.196	1.985	1.661	1.426	1.249	1.111	1.000
13	3.780	3.223	2.799	2.469	2.204	1.990	1.663	1.427	1.249	1.111	1.000
14	3.824	3.249	2.814	2.478	2.210	1.993	1.664	1.428	1.250	1.111	1.000
15	3.859	3.268	2.825	2.484	2.214	1.995	1.665	1.428	1.250	1.111	1.000
16	3.887	3.283	2.834	2.489	2.216	1.997	1.666	1.428	1.250	1.111	1.000
17	3.910	3.295	2.840	2.492	2.218	1.998	1.666	1.428	1.250	1.111	1.000
18	3.928	3.304	2.844	2.494	2.219	1.999	1.666	1.428	1.250	1.111	1.000
19	3.942	3.311	2.848	2.496	2.220	1.999	1.666	1.429	1.250	1.111	1.000
20	3.954	3.316	2.850	2.497	2.221	1.999	1.667	1.429	1.250	1.111	1.000

* **Source : Taken from, MATHEMATICS OF FINANCE,**

Fourth Ed. : Cissell and Cissell, 1973, pp. 77 - 78.

Appendix H

Table of Common Logarithm

N	0	1	2	3	4	5	6	7	8	9
0.00	−∞	−6‡ .90776	−6 .21461	−5 .80914	−5 .52146	−5 .29832	−5 .11000	−4 .96185	−4 .82831	−4 .71053
.01	−4.60517	.50986	.42285	.34281	.26870	.19971	.13517	.07454	.01738	*.96332
.02	−3.91202	.86323	.81671	.77226	.72970	.68888	.64966	.61192	.57555	.54046
.03	.50656	.47377	.44202	.41125	.38139	.35241	.32424	.29684	.27017	.24419
.04	.21888	.19418	.17009	.14656	.12357	.10109	.07911	.05761	.03655	.01593
.05	−2.99573	.97593	.95651	.93746	.91877	.90042	.88240	.86470	.84731	.83022
.06	.81341	.79688	.78062	.76462	.74887	.73337	.71810	.70306	.68825	.67365
.07	.65926	.64508	.63109	.61730	.60369	.59027	.57702	.56395	.55105	.53831
.08	.52573	.51331	.50104	.48891	.47694	.46510	.45341	.44185	.43042	.41912
.09	.40795	.39690	.38597	.37516	.36446	.35388	.34341	.33304	.32279	.31264
0.10	−2.30259	.29263	.28278	.27303	.26336	.25379	.24432	.23493	.22562	.21641
.11	.20727	.19823	.18926	.18037	.17156	.16282	.15417	.14558	.13707	.12863
.12	.12026	.11196	.10373	.09557	.08747	.07944	.07147	.06357	.05573	.04794
.13	.04022	.03256	.02495	.01741	.00992	.00248	*.99510	*.98777	*.98050	*.97328
.14	−1.96611	.95900	.95193	.94491	.93794	.93102	.92415	.91732	.91054	.90381
.15	.89712	.89048	.88387	.87732	.87080	.86433	.85790	.85151	.84516	.83885
.16	.83258	.82635	.82016	.81401	.80789	.80181	.79577	.78976	.78379	.77786
.17	.77196	.76609	.76026	.75446	.74870	.74297	.73727	.73161	.72597	.72037
.18	.71480	.70926	.70375	.69827	.69282	.68740	.68201	.67665	.67131	.66601
.19	.66073	.65548	.65026	.64507	.63990	.63476	.62964	.62455	.61949	.61445
0.20	−1.60944	.60445	.59949	.59455	.58964	.58475	.57988	.57504	.57022	.56542
.21	.56065	.55590	.55117	.54646	.54178	.53712	.53248	.52786	.52326	.51868
.22	.51413	.50959	.50508	.50058	.49611	.49165	.48722	.48281	.47841	.47403
.23	.46968	.46534	.46102	.45672	.45243	.44817	.44392	.43970	.43548	.43129
.24	.42712	.42296	.41882	.41469	.41059	.40650	.40242	.39837	.39433	.39030
.25	.38629	.38230	.37833	.37437	.37042	.36649	.36258	.35868	.35480	.35093
.26	.34707	.34323	.33941	.33560	.33181	.32803	.32426	.32051	.31677	.31304
.27	.30933	.30564	.30195	.29828	.29463	.29098	.28735	.28374	.28013	.27654
.28	.27297	.26940	.26585	.26231	.25878	.25527	.25176	.24827	.24479	.24133
.29	.23787	.23443	.23100	.22758	.22418	.22078	.21740	.21402	.21066	.20731
0.30	−1.20397	.20065	.19733	.19402	.19073	.18744	.18417	.18091	.17766	.17441
.31	.17118	.16796	.16475	.16155	.15836	.15518	.15201	.14885	.14570	.14256
.32	.13943	.13631	.13320	.13010	.12701	.12393	.12086	.11780	.11474	.11170
.33	.10866	.10564	.10262	.09961	.09661	.09362	.09064	.08767	.08471	.08176
.34	.07881	.07587	.07294	.07002	.06711	.06421	.06132	.05843	.05555	.05268
.35	−1.04982	.04697	.04412	.04129	.03846	.03564	.03282	.03002	.02722	.02443
.36	.02165	.01888	.01611	.01335	.01060	.00786	.00512	.00239	*.99967	*.99696
.37	−0.99425	.99155	.98886	.98618	.98350	.98083	.97817	.97551	.97286	.97022
.38	.96758	.96496	.96233	.95972	.95711	.95451	.95192	.94933	.94675	.94418
.39	.94161	.93905	.93649	.93395	.93140	.92887	.92634	.92382	.92130	.91879
0.40	−0.91629	.91379	.91130	.90882	.90634	.90387	.90140	.89894	.89649	.89404
.41	.89160	.88916	.88673	.88431	.88189	.87948	.87707	.87467	.87227	.86988
.42	.86750	.86512	.86275	.86038	.85802	.85567	.85332	.85097	.84863	.84630
.43	.84397	.84165	.83933	.83702	.83471	.83241	.83011	.82782	.82554	.82326
.44	.82098	.81871	.81645	.81419	.81193	.80968	.80744	.80520	.80296	.80073
.45	.79851	.79629	.79407	.79186	.78966	.78746	.78526	.78307	.78089	.77871
.46	.77653	.77436	.77219	.77003	.76787	.76572	.76357	.76143	.75929	.75715
.47	.75502	.75290	.75078	.74866	.74655	.74444	.74234	.74024	.73814	.73605
.48	.73397	.73189	.72981	.72774	.72567	.72361	.72155	.71949	.71744	.71539
.49	.71335	.71131	.70928	.70725	.70522	.70320	.70118	.69917	.69716	.69515

0.500–0.999

N	0	1	2	3	4	5	6	7	8	9
0.50	−0.69315	.69115	.68916	.68717	.68518	.68320	.68122	.67924	.67727	.67531
.51	.67334	.67139	.66934	.66748	.66553	.66359	.66165	.65971	.65778	.65585
.52	.65393	.65201	.65009	.64817	.64626	.64436	.64245	.64055	.63866	.63677
.53	.63488	.63299	.63111	.62923	.62736	.62549	.62362	.62176	.61990	.61804
.54	.61619	.61434	.61249	.61065	.60881	.60697	.60514	.60331	.60148	.59966
.55	.59784	.59602	.59421	.59240	.59059	.58879	.58699	.58519	.58340	.58161
.56	.57982	.57803	.57625	.57448	.57270	.57093	.56916	.56740	.56563	.56387
.57	.56212	.56037	.55862	.55687	.55513	.55339	.55165	.54991	.54818	.54645
.58	.54473	.54300	.54128	.53957	.53785	.53614	.53444	.53273	.53103	.52933
.59	.52763	.52594	.52425	.52256	.52088	.51919	.51751	.51584	.51416	.51249
0.60	−0.51083	.50916	.50750	.50584	.50418	.50253	.50088	.49923	.49758	.49594
.61	.49430	.49266	.49102	.48939	.48776	.48613	.48451	.48289	.48127	.47965
.62	.47804	.47642	.47482	.47321	.47160	.47000	.46840	.46681	.46522	.46362
.63	.46204	.46045	.45887	.45728	.45571	.45413	.45256	.45099	.44942	.44785
.64	.44629	.44473	.44317	.44161	.44006	.43850	.43696	.43541	.43386	.43232
.65	.43078	.42925	.42771	.42618	.42465	.42312	.42159	.42007	.41855	.41703
.66	.41552	.41400	.41249	.41098	.40947	.40797	.40647	.40497	.40347	.40197
.67	.40048	.39899	.39750	.39601	.39453	.39304	.39156	.39008	.38861	.38713
.68	.38566	.38419	.38273	.38126	.37980	.37834	.37688	.37542	.37397	.37251
.69	.37106	.36962	.36817	.36673	.36528	.36384	.36241	.36097	.35954	.35810
0.70	−0.35667	.35525	.35382	.35240	.35098	.34956	.34814	.34672	.34531	.34390
.71	.34249	.34108	.33968	.33827	.33687	.33547	.33408	.33268	.33129	.32989
.72	.32850	.32712	.32573	.32435	.32296	.32158	.32021	.31883	.31745	.31608
.73	.31471	.31334	.31197	.31061	.30925	.30788	.30653	.30517	.30381	.30246
.74	.30111	.29975	.29841	.29706	.29571	.29437	.29303	.29169	.29035	.28902
.75	.28768	.28635	.28502	.28369	.28236	.28104	.27971	.27839	.27707	.27575
.76	.27444	.27312	.27181	.27050	.26919	.26788	.26657	.26527	.26397	.26266
.77	.26136	.26007	.25877	.25748	.25618	.25489	.25360	.25231	.25103	.24974
.78	.24846	.24718	.24590	.24462	.24335	.24207	.24080	.23953	.23826	.23699
.79	.23572	.23446	.23319	.23193	.23067	.22941	.22816	.22690	.22565	.22439
0.80	−0.22314	.22189	.22065	.21940	.21816	.21691	.21567	.21433	.21319	.21196
.81	.21072	.20949	.20825	.20702	.20579	.20457	.20334	.20212	.20089	.19967
.82	.19845	.19723	.19601	.19480	.19358	.19237	.19116	.18995	.18874	.18754
.83	.18633	.18513	.18392	.18272	.18152	.18032	.17913	.17793	.17674	.17554
.84	.17435	.17316	.17198	.17079	.16960	.16842	.16724	.16605	.16487	.16370
.85	−0.16252	.16134	.16017	.15900	.15782	.15665	.15548	.15432	.15315	.15199
.86	.15032	.14966	.14850	.14734	.14618	.14503	.14387	.14272	.14156	.14041
.87	.13926	.13811	.13697	.13582	.13467	.13353	.13239	.13125	.13011	.12897
.88	.12783	.12670	.12556	.12443	.12330	.12217	.12104	.11991	.11878	.11766
.89	.11653	.11541	.11429	.11317	.11205	.11093	.10981	.10870	.10759	.10647
0.90	−0.10536	.10425	.10314	.10203	.10093	.09982	.09872	.09761	.09651	.09541
.91	.09431	.09321	.09212	.09102	.08992	.08883	.08744	.08665	.08556	.08447
.92	.08338	.08230	.08121	.08013	.07904	.07796	.07688	.07580	.07472	.07365
.93	.07257	.07150	.07042	.06935	.06828	.06721	.06614	.06507	.06401	.06294
.94	.06188	.06081	.05975	.05869	.05763	.05657	.05551	.05446	.05340	.05235
.95	.05129	.05024	.04919	.04814	.04709	.04604	.04500	.04395	.04291	.04186
.96	.04082	.03978	.03874	.03770	.03666	.03563	.03459	.03356	.03252	.03149
.97	.03046	.02943	.02840	.02737	.02634	.02532	.02429	.02327	.02225	.02122
.98	.02020	.01918	.01816	.01715	.01613	.01511	.01410	.01309	.01207	.01106
.99	.01005	.00904	.00803	.00702	.00602	.00501	.00401	.00300	.00200	.00100

0–499

N	0	1	2	3	4	5	6	7	8	9
0	$-\infty$	0.00000	0.69315	1.09861	.38629	60944	.79176	.94591	*.07944	*.19722
1	2.30259	.39790	.48491	.56495	.63906	.70805	.77259	.83321	.89037	.94444
2	.99573	*.04452	*.09104	*.13549	*.17805	*.21888	*.25810	*.29584	*.33220	*.36730
3	3.40120	.43399	.46574	.49651	.52636	.55535	.58352	.61092	.63759	.66356
4	.68888	.71357	.73767	.76120	.78419	.80666	.82864	.85015	.87120	.89182
5	.91202	.93183	.95124	.97029	.98898	*.00733	*.02535	*.04305	*.06044	*.07754
6	4.09434	.11087	.12713	.14313	.15888	.17439	.18965	.20469	.21951	.23411
7	.24850	.26268	.27667	.29046	.30407	.31749	.33073	.34381	.35671	.36945
8	.38203	.39445	.40672	.41884	.43082	.44265	.45435	.46591	.47734	.48864
9	.49981	.51086	.52179	.53260	.54329	.55388	.56435	.57471	.58497	.59512
10	4.60517	.61512	.62497	.63473	.64439	.65396	.66344	.67283	.68213	.69135
11	.70048	.70953	.71850	.72739	.73620	.74493	.75359	.76217	.77068	.77912
12	.78749	.79579	.80402	.81218	.82028	.82831	.83628	.84419	.85203	.85981
13	.86753	.87520	.88280	.89035	.89784	.90527	.91265	.91998	.92725	.93447
14	.94164	.94876	.95583	.96284	.96981	.97673	.98361	.99043	.99721	*.00395
15	5.01064	.01728	.02388	.03044	.03695	.04343	.04986	.05625	.06260	.06890
16	.07517	.08140	.08760	.09375	.09987	.10595	.11199	.11799	.12396	.12990
17	.13580	.14166	.14749	.15329	.15906	.16479	.17048	.17615	.18178	.18739
18	.19296	.19850	.20401	.20949	.21494	.22036	.22575	.23111	.23644	.24175
19	.24702	.25227	.25750	.26269	.26786	.27300	.27811	.28320	.28827	.29330
20	5.29832	.30330	.30827	.31321	.31812	.32301	.32788	.33272	.33754	.34233
21	.34711	.35186	.35659	.36129	.36598	.37064	.37528	.37990	.38450	.38907
22	.39363	.39816	.40268	.40717	.41165	.41610	.42053	.42495	.42935	.43372
23	.43808	.44242	.44674	.45104	.45532	.45959	.46383	.46806	.47227	.47646
24	.48064	.48480	.48894	.49306	.49717	.50126	.50533	.50939	.51343	.51745
25	.52146	.52545	.52943	.53339	.53733	.54126	.54518	.54908	.55296	.55683
26	.56068	.56452	.56834	.57215	.57595	.57973	.58350	.58725	.59099	.59471
27	.59842	.60212	.60580	.60947	.61313	.61677	.62040	.62402	.62762	.63121
28	.63479	.63835	.64191	.64545	.64897	.65249	.65599	.65948	.66296	.66643
29	.66988	.67332	.67675	.68017	.68358	.68698	.69036	.69373	.69709	.70044
30	5.70378	.70711	.71043	.71373	.71703	.72031	.72359	.72685	.73010	.73334
31	.73657	.73979	.74300	.74620	.74939	.75257	.75574	.75890	.76205	.76519
32	.76832	.77144	.77455	.77765	.78074	.78383	.78690	.78996	.79301	.79606
33	.79909	.80212	.80513	.80814	.81114	.81413	.81711	.82008	.82305	.82600
34	.82895	.83188	.83481	.83773	.84064	.84354	.84644	.84932	.85220	.85507
35	.85793	.86079	.86363	.86647	.86930	.87212	.87493	.87774	.88053	.88332
36	.88610	.88888	.89164	.89440	.89715	.89990	.90263	.90536	.90808	.91080
37	.91350	.91620	.91889	.92158	.92426	.92693	.92959	.93225	.93489	.93754
38	.94017	.94280	.94542	.94803	.95064	.95324	.95584	.95842	.96101	.96358
39	.96615	.96871	.97126	.97381	.97635	.97889	.98141	.98394	.98645	.98896
40	5.99146	.99396	.99645	.99894	*.00141	*.00389	*.00635	*.00881	*.01127	*.01372
41	6.01616	.01859	.02102	.02345	.02587	.02828	.03069	.03309	.03548	.03787
42	.04025	.04263	.04501	.04737	.04973	.05209	.05444	.05678	.05192	.06146
43	.06379	.06611	.06843	.07074	.07304	.07535	.07764	.07993	.08222	.08450
44	.08677	.08904	.09131	.09357	.09582	.09807	.10032	.10256	.10479	.10702
45	.10925	.11147	.11368	.11589	.11810	.12030	.12249	.12468	.12687	.12905
46	.13123	.13340	.13556	.13773	.13988	.14204	.14419	.14633	.14847	.15060
47	.15273	.15486	.15698	.15910	.16121	.16331	.16542	.16752	.16961	.17170
48	.17379	.17587	.17794	.18002	.18208	.18415	.18621	.18826	.19032	.19236
49	.19441	.19644	.19848	.20051	.20254	.20456	.20658	.20859	.21060	.21261

500–999

N	0	1	2	3	4	5	6	7	8	9
50	6.21461	.21661	.21860	.22059	.22258	.22456	.22654	.22851	.23048	.23245
51	.23441	.23637	.23832	.24028	.24222	.24417	.24611	.24804	.24998	.25190
52	.25383	.25575	.25767	.25958	.26149	.26340	.26530	.26720	.26910	.27099
53	.27288	.27476	.27664	.27852	.28040	.28227	.28413	.28600	.28786	.28972
54	.29157	.29342	.29527	.29711	.29895	.30079	.30262	.30445	.30628	.30810
55	.30992	.31173	.31355	.31536	.31716	.31897	.32077	.32257	.32436	.32615
56	.32794	.32972	.33150	.33328	.33505	.33683	.33859	.34036	.34212	.34388
57	.34564	.34739	.34914	.35089	.35263	.35437	.35611	.35784	.35957	.36130
58	.36303	.36475	.36647	.36819	.36990	.37161	.37332	.37502	.37673	.37843
59	.38012	.38182	.38351	.38519	.38688	.38856	.39024	.39192	.39359	.39526
60	6.30693	.39859	.40026	.40192	.40357	.40523	.40688	.40853	.41017	.41182
61	.41346	.41510	.41673	.41836	.41999	.42162	.42325	.42487	.42649	.42811
62	.42972	.43133	.43294	.43455	.43615	.43775	.43935	.44095	.44254	.44413
63	.44572	.44731	.44889	.45047	.45205	.45362	.45520	.45677	.45834	.45990
64	.46147	.46303	.46459	.46614	.46770	.46925	.47080	.47235	.47389	.47543
65	.47697	.47851	.48004	.48158	.48311	.48464	.48616	.48768	.48920	.49072
66	.49224	.49375	.49527	.49677	.49828	.49979	.50129	.50279	.50429	.50578
67	.50728	.50877	.51026	.51175	.51323	.51471	.51619	.51767	.51915	.52062
68	.52209	.52356	.52503	.52649	.52796	.52942	.53088	.53233	.53379	.53524
69	.53669	.53814	.53959	.54103	.54247	.54391	.54535	.54679	.54822	.54965
70	6.55108	.55251	.55393	.55536	.55678	.55820	.55962	.56103	.56244	.56386
71	.56526	.56667	.56808	.56948	.57088	.57228	.57368	.57508	.57647	.57786
72	.57925	.58064	.58203	.58341	.58479	.58617	.58755	.58893	.59030	.59167
73	.59304	.59441	.59578	.59715	.59851	.59987	.60123	.60259	.60394	.60530
74	.60665	.60800	.60935	.61070	.61204	.61338	.61473	.61607	.61740	.61874
75	.62007	.62141	.62274	.62407	.62539	.62672	.62804	.62936	.63068	.63200
76	.63332	.63463	.63595	.63726	.63857	.63988	.64118	.64249	.64379	.64509
77	.64639	.64769	.64898	.65028	.65157	.65286	.65415	.65544	.65673	.65801
78	.65929	.66058	.66185	.66313	.66441	.66568	.66696	.66823	.66950	.67077
79	.67203	.67330	.67456	.67582	.67708	.67834	.67960	.68085	.68211	.68336
80	6.68461	.68586	.68711	.68835	.68960	.69084	.69208	.69332	.69456	.69580
81	.69703	.69827	.69950	.70073	.70196	.70319	.70441	.70564	.70686	.70808
82	.70930	.71052	.71174	.71296	.71417	.71538	.71659	.71780	.71901	.72022
83	.72143	.72263	.72383	.72503	.72623	.72743	.72863	.72982	.73102	.73221
84	.73340	.73459	.73578	.73697	.73815	.73934	.74052	.74170	.74288	.74406
85	.74524	.74641	.74759	.74876	.74993	.75110	.75227	.75344	.75460	.75577
86	.75693	.75809	.75926	.76041	.76157	.76273	.76388	.76504	.76619	.76734
87	.76849	.76964	.77079	.77194	.77308	.77422	.77537	.77651	.77765	.77878
88	.77992	.78106	.78219	.78333	.78446	.78559	.78672	.78784	.78897	.79010
89	.79122	.79234	.79347	.79459	.79571	.79682	.79794	.79906	.80017	.80128
90	6.80239	.80351	.80461	.80572	.80683	.80793	.80904	.81014	.81124	.81235
91	.81344	.81454	.81564	.81674	.81783	.81892	.82002	.82111	.82220	.82329
92	.82437	.82546	.82655	.82763	.82871	.82979	.83087	.83195	.83303	.83411
93	.83518	.83626	.83733	.83841	.83948	.84055	.84162	.84268	.84375	.84482
94	.84588	.84694	.84801	.84907	.85013	.85118	.85224	.85330	.85435	.85541
95	.85646	.85751	.85857	.85961	.86066	.86171	.86276	.86380	.86485	.86589
96	.86693	.86797	.86901	.87005	.87109	.87213	.87316	.87420	.87523	.87626
97	.87730	.87833	.87936	.88038	.88141	.88244	.88346	.88449	.88551	.88653
98	.88755	.88857	.88959	.89061	.89163	.89264	.89366	.89467	.89568	.89669
99	.89770	.89871	.89972	.90073	.90174	.90274	.90375	.90475	.90575	.90675

* Source : Taken from, ALGEBRA AND CALCULUS FOR BUSINESS · Dyckman and Thomas, 1974, pp. 422 423 .

Appendix I

Table of Natural Logarithm

N	0	1	2	3	4	5	6	7	8	9	PROPORTIONAL PARTS								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	†4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	†3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4834	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9	PROPORTIONAL PARTS 1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	4	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

* **Source : Taken from, CALCULUS FOR BUSINESS AND ECONOMICS : Robert L. Childress, 1972, pp. 240 - 243 .**

Appendix J

Table of E^X and E^{-X}

x	e^x	e^{-x}	x	e^x	e^{-x}
.00	1.00000	1.00000	.60	1.82212	.54881
.01	1.01005	.99005	.61	1.84043	.54335
.02	1.02020	.98020	.62	1.85893	.53794
.03	1.03045	.97045	.63	1.87761	.53259
.04	1.04081	.96079	.64	1.89648	.52729
.05	1.05127	.95123	.65	1.91554	.52205
.06	1.06184	.94176	.66	1.93479	.51685
.07	1.07251	.93239	.67	1.95424	.51171
.08	1.08329	.92312	.68	1.97388	.50662
.09	1.09417	.91393	.69	1.99372	.50158
.10	1.10517	.90484	.70	2.01375	.49659
.11	1.11628	.89583	.71	2.03399	.49164
.12	1.12750	.88692	.72	2.05443	.48675
.13	1.13883	.87810	.73	2.07508	.48191
.14	1.15027	.86936	.74	2.09594	.47711
.15	1.16183	.86071	.75	2.11700	.47237
.16	1.17351	.85214	.76	2.13828	.46767
.17	1.18530	.84366	.77	2.15977	.46301
.18	1.19722	.83527	.78	2.18147	.45841
.19	1.20925	.82696	.79	2.20340	.45384
.20	1.22140	.81873	.80	2.22554	.44933
.21	1.23368	.81058	.81	2.24791	.44486
.22	1.24608	.80252	.82	2.27050	.44043
.23	1.25860	.79453	.83	2.29332	.43605
.24	1.27125	.78663	.84	2.31637	.43171
.25	1.28403	.77880	.85	2.33965	.42741
.26	1.29693	.77105	.86	2.36316	.42316
.27	1.30996	.76338	.87	2.38691	.41895
.28	1.32313	.75578	.88	2.41090	.41478
.29	1.33643	.74826	.89	2.43513	.41066
.30	1.34986	.74082	.90	2.45960	.40657
.31	1.36343	.73345	.91	2.48432	.40252
.32	1.37713	.72615	.92	2.50929	.39852
.33	1.39097	.71892	.93	2.53451	.39455
.34	1.40495	.71177	.94	2.55998	.39063
.35	1.41907	.70469	.95	2.58571	.38674
.36	1.43333	.69768	.96	2.61170	.38289
.37	1.44773	.69073	.97	2.63794	.37908
.38	1.46228	.68386	.98	2.66446	.37531
.39	1.47698	.67706	.99	2.69123	.37158
.40	1.49182	.67032	1.00	2.71828	.36788
.41	1.50682	.66365	1.01	2.74560	.36422
.42	1.52196	.65705	1.02	2.77319	.36059
.43	1.53726	.65051	1.03	2.80107	.35701
.44	1.55271	.64404	1.04	2.82922	.35345
.45	1.56831	.63763	1.05	2.85765	.34994
.46	1.58407	.63128	1.06	2.88637	.34646
.47	1.59999	.62500	1.07	2.91538	.34301
.48	1.61607	.61878	1.08	2.94468	.33960
.49	1.63232	.61263	1.09	2.97427	.33622

x	e^x	e^{-x}	x	e^x	e^{-x}
.50	1.64872	.60653	**1.10**	3.00417	.33287
.51	1.66529	.60050	1.11	3.03436	.32956
.52	1.68203	.59452	1.12	3.06485	.32628
.53	1.69893	.58860	1.13	3.09566	.32303
.54	1.71601	.58275	1.14	3.12677	.31982
.55	1.73325	.57695	1.15	3.15819	.31664
.56	1.75067	.57121	1.16	3.18993	.31349
.57	1.76827	.56553	1.17	3.22199	.31037
.58	1.78604	.55990	1.18	3.25437	.30728
.59	1.80399	.55433	1.19	3.28708	.30422
1.20	3.32012	.30119	**1.80**	6.04965	.16530
1.21	3.35348	.29820	1.81	6.11045	.16365
1.22	3.38719	.29523	1.82	6.17186	.16203
1.23	3.42123	.29229	1.83	6.23389	.16041
1.24	3.45561	.28938	1.84	6.29654	.15882
1.25	3.49034	.28650	1.85	6.35982	.15724
1.26	3.52542	.28365	1.86	6.42374	.15567
1.27	3.56085	.28083	1.87	6.48830	.15412
1.28	3.59664	.27804	1.88	6.55350	.15259
1.29	3.63279	.27527	1.89	6.61937	.15107
1.30	3.66930	.27253	**1.90**	6.68589	.14957
1.31	3.70617	.26982	1.91	6.75309	.14808
1.32	3.74342	.26714	1.92	6.82096	.14661
1.33	3.78104	.26448	1.93	6.88951	.14515
1.34	3.81904	.26185	1.94	6.95875	.14370
1.35	3.85743	.25924	1.95	7.02869	.14227
1.36	3.89619	.25666	1.96	7.09933	.14086
1.37	3.93535	.25411	1.97	7.17068	.13946
1.38	3.97490	.25158	1.98	7.24274	.13807
1.39	4.01485	.24908	1.99	7.31553	.13670
1.40	4.05520	.24660	**2.00**	7.38906	.13534
1.41	4.09596	.24414	2.01	7.46332	.13399
1.42	4.13712	.24171	2.02	7.53832	.13266
1.43	4.17870	.23931	2.03	7.61409	.13134
1.44	4.22070	.23693	2.04	7.69061	.13003
1.45	4.26311	.23457	2.05	7.76790	.12873
1.46	4.30596	.23224	2.06	7.84597	.12745
1.47	4.34924	.22993	2.07	7.92482	.12619
1.48	4.39295	.22764	2.08	8.00447	.12493
1.49	4.43710	.22537	2.09	8.08491	.12369
1.50	4.48169	.22313	**2.10**	8.16617	.12246
1.51	4.52673	.22091	2.11	8.24824	.12124
1.52	4.57223	.21871	2.12	8.33114	.12003
1.53	4.61818	.21654	2.13	8.41487	.11884
1.54	4.66459	.21438	2.14	8.49944	.11765
1.55	4.71147	.21225	2.15	8.58486	.11648
1.56	4.75882	.21014	2.16	8.67114	.11533
1.57	4.80665	.20805	2.17	8.75828	.11418
1.58	4.85496	.20598	2.18	8.84631	.11304
1.59	4.90375	.20393	2.19	8.93521	.11192

x	e^x	e^{-x}	x	e^x	e^{-x}
1.60	4.95303	.20190	**2.20**	9.02501	.11080
1.61	5.00281	.19989	2.21	9.11572	.10970
1.62	5.05309	.19790	2.22	9.20733	.10861
1.63	5.10387	.19593	2.23	9.29987	.10753
1.64	5.15517	.19398	2.24	9.39333	.10646
1.65	5.20698	.19205	2.25	9.48774	.10540
1.66	5.25931	.19014	2.26	9.58309	.10435
1.67	5.31217	.18825	2.27	9.67940	.10331
1.68	5.36556	.18637	2.28	9.77668	.10228
1.69	5.41948	.18452	2.29	9.87494	.10127
1.70	5.47395	.18268	**2.30**	9.97418	.10026
1.71	5.52896	.18087	2.31	10.07442	.09926
1.72	5.58453	.17907	2.32	10.17567	.09827
1.73	5.64065	.17728	2.33	10.27794	.09730
1.74	5.69734	.17552	2.34	10.38124	.09633
1.75	5.75460	.17377	2.35	10.48557	.09537
1.76	5.81244	.17204	2.36	10.59095	.09442
1.77	5.87085	.17033	2.37	10.69739	.09348
1.78	5.92986	.16864	2.38	10.80490	.09255
1.79	5.98945	.16696	2.39	10.91349	.09163
2.40	11.02318	.09072	**3.00**	20.08554	.04979
2.41	11.13396	.08982	3.01	20.28740	.04929
2.42	11.24586	.08892	3.02	20.49129	.04880
2.43	11.35888	.08804	3.03	20.69723	.04832
2.44	11.47304	.08716	3.04	20.90524	.04783
2.45	11.58835	.08629	3.05	21.11534	.04736
2.46	11.70481	.08543	3.06	21.32756	.04689
2.47	11.82245	.08458	3.07	21.54190	.04642
2.48	11.94126	.08374	3.08	21.75840	.04596
2.49	12.06128	.08291	3.09	21.97708	.04550
2.50	12.18249	.08208	**3.10**	22.19795	.04505
2.51	12.30493	.08127	3.11	22.42104	.04460
2.52	12.42860	.08046	3.12	22.64638	.04416
2.53	12.55351	.07966	3.13	22.87398	.04372
2.54	12.67967	.07887	3.14	23.10387	.04328
2.55	12.80710	.07808	3.15	23.33606	.04285
2.56	12.93582	.07730	3.16	23.57060	.04243
2.57	13.06582	.07654	3.17	23.80748	.04200
2.58	13.19714	.07577	3.18	24.04675	.04159
2.59	13.32977	.07502	3.19	24.28843	.04117
2.60	13.46374	.07427	**3.20**	24.53253	.04076
2.61	13.59905	.07353	3.21	24.77909	.04036
2.62	13.73572	.07280	3.22	25.02812	.03996
2.63	13.87377	.07208	3.23	25.27966	.03956
2.64	14.01320	.07136	3.24	25.53372	.03916
2.65	14.15404	.07065	3.25	25.79034	.03877
2.66	14.29629	.06995	3.26	26.04954	.03839
2.67	14.43997	.06925	3.27	26.31134	.03801
2.68	14.58509	.06856	3.28	26.57577	.03763
2.69	14.73168	.06788	3.29	26.84286	.03725

x	e^x	e^{-x}	x	e^x	e^{-x}
2.70	14.87973	.06721	**3.30**	27.11264	.03688
2.71	15.02928	.06654	3.31	27.38512	.03652
2.72	15.18032	.06587	3.32	27.66035	.03615
2.73	15.33289	.06522	3.33	27.93834	.03579
2.74	15.48698	.06457	3.34	28.21913	.03544
2.75	15.64263	.06393	3.35	28.50273	.03508
2.76	15.79984	.06329	3.36	28.78919	.03474
2.77	15.95863	.06266	3.37	29.07853	.03439
2.78	16.11902	.06204	3.38	29.37077	.03405
2.79	16.28102	.06142	3.39	29.66595	.03371
2.80	16.44465	.06081	**3.40**	29.96410	.03337
2.81	16.60992	.06020	3.41	30.26524	.03304
2.82	16.77685	.05961	3.42	30.56941	.03271
2.83	16.94546	.05901	3.43	30.87664	.03239
2.84	17.11577	.05843	3.44	31.18696	.03206
2.85	17.28778	.05784	3.45	31.50039	.03175
2.86	17.46153	.05727	3.46	31.81698	.03143
2.87	17.63702	.05670	3.47	32.13674	.03112
2.88	17.81427	.05613	3.48	32.45972	.03081
2.89	17.99331	.05558	3.49	32.78595	.03050
2.90	18.17414	.05502	**3.50**	33.11545	.03020
2.91	18.35680	.05448	3.51	33.44827	.02990
2.92	18.54129	.05393	3.52	33.78443	.02960
2.93	18.72763	.05340	3.53	34.12397	.02930
2.94	18.91585	.05287	3.54	34.46692	.02901
2.95	19.10595	.05234	3.55	34.81332	.02872
2.96	19.29797	.05182	3.56	35.16320	.02844
2.97	19.49192	.05130	3.57	35.51659	.02816
2.98	19.68782	.05079	3.58	35.87354	.02788
2.99	19.88568	.05029	3.59	36.23408	.02760
3.60	36.59823	.02732	**4.20**	66.68633	.01500
3.61	36.96605	.02705	4.21	67.35654	.01485
3.62	37.33757	.02678	4.22	68.03348	.01470
3.63	37.71282	.02652	4.23	68.71723	.01455
3.64	38.09184	.02625	4.24	69.40785	.01441
3.65	38.47467	.02599	4.25	70.10541	.01426
3.66	38.86134	.02573	4.26	70.80998	.01412
3.67	39.25191	.02548	4.27	71.52163	.01398
3.68	39.64639	.02522	4.28	72.24044	.01384
3.69	40.04485	.02497	4.29	72.96647	.01370
3.70	40.44730	.02472	**4.30**	73.69979	.01357
3.71	40.85381	.02448	4.31	74.44049	.01343
3.72	41.26439	.02423	4.32	75.18863	.01330
3.73	41.67911	.02399	4.33	75.94429	.01317
3.74	42.09799	.02375	4.34	76.70754	.01304
3.75	42.52108	.02352	4.35	77.47846	.01291
3.76	42.94843	.02328	4.36	78.25713	.01278
3.77	43.38006	.02305	4.37	79.04363	.01265
3.78	43.81604	.02282	4.38	79.83803	.01253
3.79	44.25640	.02260	4.39	80.64042	.01240

x	e^x	e^{-x}	x	e^x	e^{-x}
3.80	44.70118	.02237	**4.40**	81.45087	.01228
3.81	45.15044	.02215	4.41	82.26946	.01216
3.82	45.60421	.02193	4.42	83.09628	.01203
3.83	46.06254	.02171	4.43	83.93141	.01191
3.84	46.52547	.02149	4.44	84.77494	.01180
3.85	46.99306	.02128	4.45	85.62694	.01168
3.86	47.46535	.02107	4.46	86.48751	.01156
3.87	47.94238	.02086	4.47	87.35672	.01145
3.88	48.42421	.02065	4.48	88.23467	.01133
3.89	48.91089	.02045	4.49	89.12144	.01122
3.90	49.40245	.02024	**4.50**	90.01713	.01111
3.91	49.89895	.02004	4.51	90.92182	.01100
3.92	50.40044	.01984	4.52	91.83560	.01089
3.93	50.90698	.01964	4.53	92.75856	.01078
3.94	51.41860	.01945	4.54	93.69080	.01067
3.95	51.93537	.01925	4.55	94.63240	.01057
3.96	52.45732	.01906	4.56	95.58347	.01046
3.97	52.98453	.01887	4.57	96.54411	.01036
3.98	53.51703	.01869	4.58	97.51439	.01025
3.99	54.05489	.01850	4.59	98.49443	.01015
4.00	54.59815	.01832	**4.60**	99.48431	.01005
4.01	55.14687	.01813	4.61	100.48415	.00995
4.02	55.70110	.01795	4.62	101.49403	.00985
4.03	56.26091	.01777	4.63	102.51406	.00975
4.04	56.82634	.01760	4.64	103.54435	.00966
4.05	57.39745	.01742	4.65	104.58498	.00956
4.06	57.97431	.01725	4.66	105.63608	.00947
4.07	58.55696	.01708	4.67	106.69774	.00937
4.08	59.14547	.01691	4.68	107.77007	.00928
4.09	59.73989	.01674	4.69	108.85318	.00919
4.10	60.34029	.01657	**4.70**	109.94717	.00910
4.11	60.94671	.01641	4.71	111.05216	.00900
4.12	61.55924	.01624	4.72	112.16825	.00892
4.13	62.17792	.01608	4.73	113.29556	.00883
4.14	62.80282	.01592	4.74	114.43420	.00874
4.15	63.43400	.01576	4.75	115.58428	.00865
4.16	64.07152	.01561	4.76	116.74592	.00857
4.17	64.71545	.01545	4.77	117.91924	.00848
4.18	65.36585	.01530	4.78	119.10435	.00840
4.19	66.02279	.01515	4.79	120.30136	.00831
4.80	121.51041	.00823	**8.00**	2980.95779	.00034
4.81	122.73161	.00815	8.10	3294.46777	.00030
4.82	123.96509	.00807	8.20	3640.95004	.00027
4.83	125.21096	.00799	8.30	4023.87219	.00025
4.84	126.46935	.00791	8.40	4447.06665	.00022
4.85	127.74039	.00783	8.50	4914.76886	.00020
4.86	129.02420	.00775	8.60	5431.65906	.00018
4.87	130.32091	.00767	8.70	6002.91180	.00017
4.88	131.63066	.00760	8.80	6634.24371	.00015
4.89	132.95357	.00752	8.90	7331.97339	.00014

x	e^x	e^{-x}	x	e^x	e^{-x}
4.90	134.28978	.00745	**9.00**	8103.08295	.00012
4.91	135.63941	.00737	9.10	8955.29187	.00011
4.92	137.00261	.00730	9.20	9897.12830	.00010
4.93	138.37951	.00723	9.30	10938.01868	.00009
4.94	139.77024	.00715	9.40	12088.38049	.00008
4.95	141.17496	.00708	9.50	13359.72522	.00007
4.96	142.59379	.00701	9.60	14764.78015	.00007
4.97	144.02688	.00694	9.70	16317.60608	.00006
4.98	145.47438	.00687	9.80	18033.74414	.00006
4.99	146.93642	.00681	9.90	19930.36987	.00005
5.00	148.41316	.00674	**10.00**	22026.46313	.00005
5.10	164.02190	.00610	10.10	24343.00708	.00004
5.20	181.27224	.00552	10.20	26903.18408	.00004
5.30	200.33680	.00499	10.30	29732.61743	.00003
5.40	221.40641	.00452	10.40	32859.62500	.00003
5.50	244.69192	.00409	10.50	36315.49854	.00003
5.60	270.42640	.00370	10.60	40134.83350	.00002
5.70	298.86740	.00335	10.70	44355.85205	.00002
5.80	330.29955	.00303	10.80	49020.79883	.00002
5.90	365.03746	.00274	10.90	54176.36230	.00002
6.00	403.42877	.00248	**11.00**	59874.13477	.00002
6.10	445.85775	.00224	11.10	66171.15430	.00002
6.20	492.74903	.00203	11.20	73130.43652	.00001
6.30	544.57188	.00184	11.30	80821.63379	.00001
6.40	601.84502	.00166	11.40	89321.72168	.00001
6.50	665.14159	.00150	11.50	98715.75879	.00001
6.60	735.09516	.00136	11.60	109097.78906	.00001
6.70	812.40582	.00123	11.70	120571.70605	.00001
6.80	897.84725	.00111	11.80	133252.34570	.00001
6.90	992.27469	.00101	11.90	147266.62109	.00001
7.00	1096.63309	.00091	—	—	—
7.10	1211.96703	.00083	—	—	—
7.20	1339.43076	.00075	—	—	—
7.30	1480.29985	.00068	—	—	—
7.40	1635.98439	.00061	—	—	—
7.50	1808.04231	.00055	—	—	—
7.60	1998.19582	.00050	—	—	—
7.70	2208.34796	.00045	—	—	—
7.80	2440.60187	.00041	—	—	—
7.90	2697.28226	.00037	—	—	—

* Source : Taken from, CALCULUS AND ITS APPLICATIONS : Goldstein, Lay and Schneider, 1977, pp. 459 - 463.

Appendix K

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ANSWERS

To Some Of The Problems

Exercise 1-1

1. \$8.75,\$758.75.
4. Interest \$50.00; principal \$14.43.
6. \$3227.81. 8. 36%. 10. 3 months.
12. 10 months. 14. \$3,608,625.
16. 10.53%. 18. 96%. 20. \$9.00.
22. 4.26% at \$94; 6.56% at \$61.

Exercise 1-2

1. \$6.25,\$6.16. 3. \$6.75,\$6.66.
5. \$5.42,\$5.34. 7. \$7027.22.
9. \$514.51. 11. \$3081.66.
13. \$12.50. 15. 14.7%.

Exercise 1-3

1. \$1456.31. 3. \$582.52.

5. (a) \$96.15; (b) \$95.24; (c) \$94.34.

7. By paying \$3100 in a year he saves \$19.23 now.

9. By paying \$200 cash he saves \$3.92 now.

Exercise 2-1

1. \$1218.99; \$218.99. 3. \$373.48.

5. 21,900. 7. (a) \$8211.41; (b) \$8236.72; (c) \$8249.65; (d) \$8258.37.

9. \$8069.33. 11. \$21,442.54.

13. \$1160.66. 15. \$2588.05.

17. \$3160.06. 19. \$6.84.

21. \$3998.02. 23. \$7607.65.

25. \$2447.80. 27. \$2172.93.

29. \$6991.74. 30. \$987,201.12.

Exercise 2-2

1. \$5564; \$1564. 3. \$1456.88.

5. \$111.63. 7. \$740,820. 9. 22 years.

Exercise 2-3

1. (a) 10.25%; (b) 10.38%.

3. 6% converted quarterly. 5. \$30.

8. \$10,446.14. 10. (a) 18%; (b) 19.56%.

12. \$3198.54. 14. 8% compounded annually.

15. 4.08%. 17. 10.52%. 19. 4.88%.

Exercise 3-1

1. (a) \$57,319.40; (b) \$60,030.54.

3. \$4986.62. 5. \$8122.26.

7. \$3512.17; \$3837.84. 9. \$4802.44.

11. \$983.74. 13. \$380.66.

15. \$5324.19. 17. \$3648.47.

Exercise 3-2

1. (a) \$42,651.02; (b) \$40,554.48.

3. (a) \$10,779.10; (b) \$10,502.48; (c) \$10,236.24. 5. \$240.36.

7. \$2002.43. 9. \$10,356.68.

11. \$14,850.35. 13. Saves \$482.59 now by paying cash.

Exercise 3-3

1. \$12,854.52. 3. \$1208.03.

5. \$35,305.06.

Exercise 3-4

1. \$4581.12. 3. \$5584.36.

5. \$3477.07. 7. \$905.73.

9. $175.16. 11. $63.90.

13. Eleven $500 withdrawals and a 12th withdrawal of $85.45.

15. $89.62. 17. $16,000.

19. $896.27. 21. $136.23.

23. $455.04. 25. $7036.23.

27. $1339.53. 29. $648.33.

Exercise 3-5

1. ฿43,086.60. 3. ฿32,992.60.

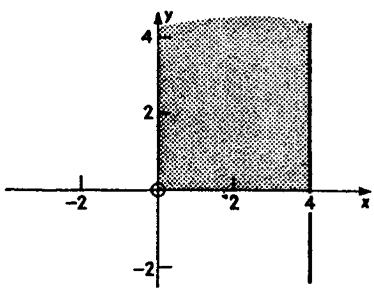
5. ฿119,340.40. 7. ฿53,628.20.

9. ฿128,618. 11. ฿21,595.60.

13. 30. 15. ฿20,153.80.

Exercise 4-1a

1.



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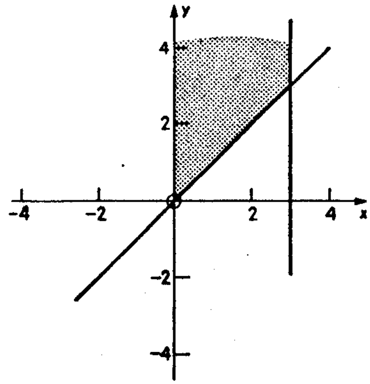

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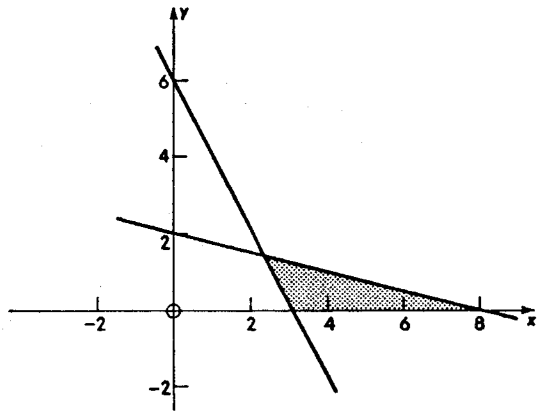

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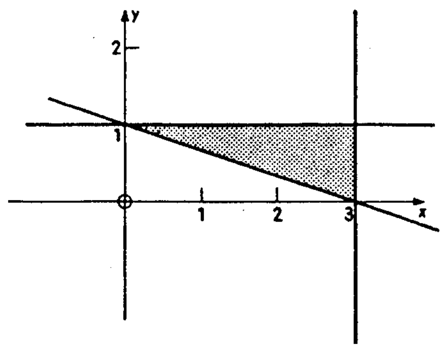
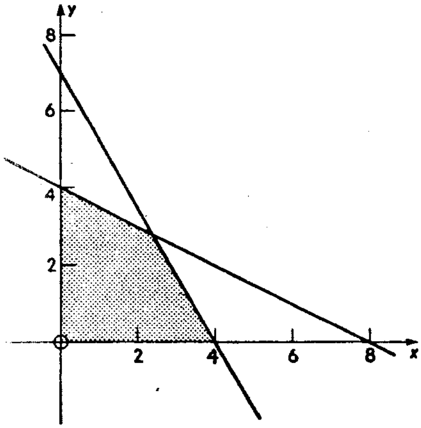




$2 \leqslant x \leqslant \frac{30}{7},\ 0 \leqslant y \leqslant \frac{x - 2}{2}$

$\frac{30}{7} \leqslant x \leqslant 6,\ 0 \leqslant y \leqslant \frac{12 - 2x}{3}$.

$x \leqslant \frac{52}{19},\ \frac{8 - x}{5} \leqslant y \leqslant 12 - 4x.$

$\frac{52}{19} \leqslant x \leqslant 3,\ 12 - 4x \leqslant y \leqslant \frac{8 - x}{5}.$

$3 \leqslant x \leqslant 8,\ 0 \leqslant y \leqslant \frac{8 - x}{5}.$

$0 \leqslant x \leqslant \frac{52}{19},\ y \geqslant 12 - 4x;$

$\frac{52}{19} \leqslant x \leqslant 8,\ y \geqslant \frac{8 - x}{5}$

$x \geqslant 8,\ y \geqslant 0.$

$0 \leqslant x \leqslant \frac{52}{19},\ 0 \leqslant y \leqslant \frac{8 - x}{5};$

$\frac{52}{19} \leqslant x \leqslant 3,\ 0 \leqslant y \leqslant 12 - 4x.$

$x \leqslant 10,\ 8 - 0.5x \leqslant y \leqslant \frac{45 - 3x}{5}.$

$0 \leqslant x \leqslant 320,\ 0 \leqslant y \leqslant \frac{2,400 - 3x}{8}$

$320 \leqslant x \leqslant 500,\ 0 \leqslant y \leqslant 500 - x.$

ting x be the number of units of A,

y the number of units of B,

$x \leqslant \frac{9}{2}\ ;\ 0 \leqslant y \leqslant 2x$

$y \leqslant 18,\ 0 \leqslant y \leqslant \frac{(36 - 2x)}{3}.$

39. $0 \leqslant x \leqslant 13$, $\frac{x}{4} \leqslant y \leqslant 4x$;

$13 \leqslant x \leqslant 88$, $\frac{x}{4} \leqslant y \leqslant \frac{(286 - 2x)}{5}$.

40. $0 \leqslant x \leqslant 20$, $0 \leqslant y \leqslant \frac{40 - x}{2}$;

$20 \leqslant x \leqslant 27$, $0 \leqslant y \leqslant \frac{190 - 6x}{7}$;

$27 \leqslant x \leqslant 30$, $0 \leqslant y \leqslant \frac{120 - 4x}{3}$.

Exercise 4-1b

1. $x \geqslant -3$. 2. No solution.

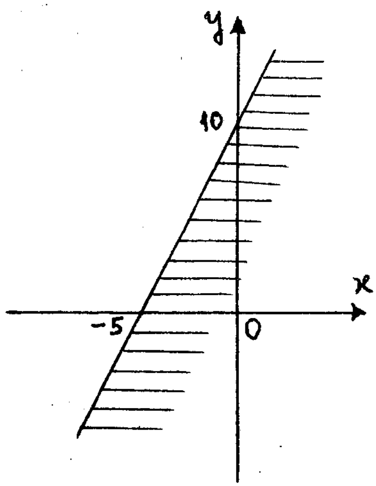
3. $x \leqslant 4$. 4. $x \geqslant -6$.

5. $-36 \leqslant x \leqslant -4$. 6. No solution.

7. (a) $x \leqslant 40 - 4y$; (b) $x \leqslant 20$;

(c) $y \leqslant 10 - \frac{x}{4}$; (d) $y \leqslant 20$.

8.



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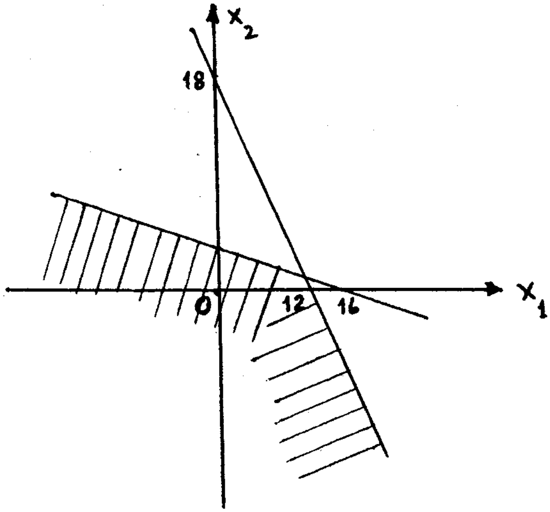

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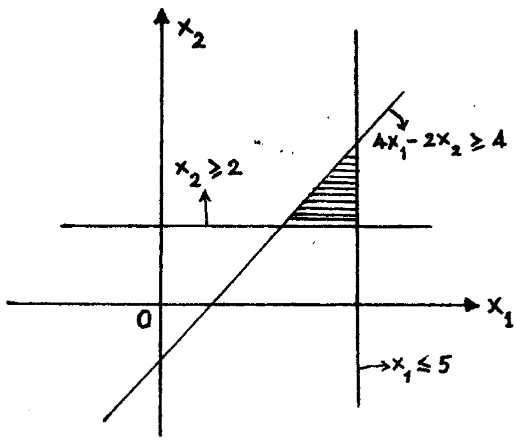


Exercise 4-2

1. 13. **3.** 6. **5.** 5. **7.** $-3 < x < 3$. **9.** $\sqrt{5} - 2$.

11. (a) $|x - 7| < 3$, (b) $|x - 2| < 3$, (c) $|x - 7| \leq 5$, (d) $|x - 7| = 4$,
(e) $|x + 4| < 2$, (f) $|x| < 3$, (g) $|x| > 6$, (h) $|x - 6| > 4$,
(i) $|x - 105| < 3$, (j) $|x - 850| < 100$.

13. $|p_1 - p_2| \leq 2$. **15.** ± 7. **17.** ± 6. **19.** $-3, 13$. **21.** $\frac{2}{5}$. **23.** $\frac{1}{2}, 3$

25. $-4 < x < 4$. **27.** $x < -8, x > 8$. **29.** $-9 < x < -5$.

31. $x < 0, x > 1$. **33.** $2 \leq x \leq 3$.

35. $x \leq 0, x \geq \frac{16}{3}$.

Exercise 4-3a

1. $P = 640$ when $x = 40, y = 20$. **3.** $Z = -10$ when $x = 2, y = 3$.

5. No optimum solution (empty feasible region). **7.** $Z = 3$ when $x = 0, y = 1$.

9. $C = 2.4$ when $x = \frac{3}{5}, y = \frac{6}{5}$. **11.** No optimum solution (unbounded).

13. 15 widgets, 25 wadgits; \$210. **15.** 4 units of Food A, 4 units of Food B; \$8.

17. 10 tons of ore I, 10 tons of ore II; \$1100.

Exercise 4-3b

1. $Z = 30$, $x_1 = 15$, $x_2 = 0$.
2. $Z = 24$, $x_1 = 3$, $x_2 = 4$.
3. $Z = 24$ at $(4,0)$ and $(2,3)$.
4. $Z = 40$ at $(1,4)$ and $(\frac{4}{3}, \frac{8}{3})$.
5. No feasible solution.

(b) Minimize : $z = 540x_1 + 744x_2$

Subject to: $200x_1 + 400x_2 \leqslant 20{,}000$

$100x_1 + 120x_2 \leqslant 10{,}000$

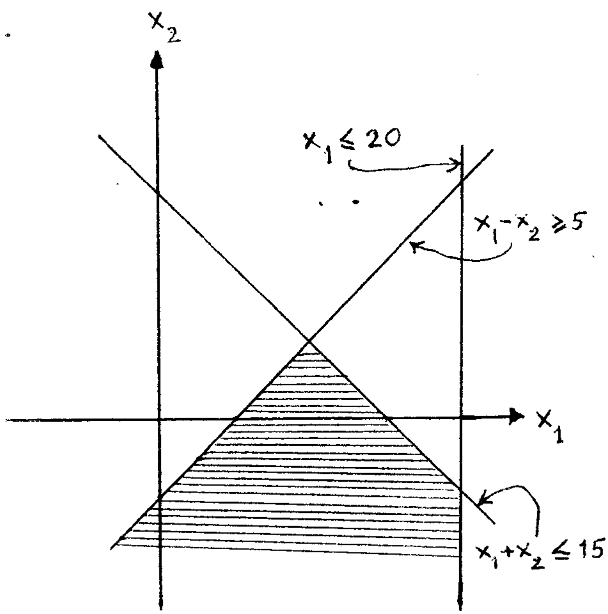
$5x_1 + 8x_2 \leqslant 400$

$x_1 \geqslant 10$

$x_1 + x_2 \geqslant 35$

$x_1,\ x_2 \geqslant 0$

$z = 20,\ x_1 = \frac{20}{7},\ x_2 = \frac{20}{7}$



9. Maximize: $Z = 400A + 200B + 1000C + 500D + 40E$

Subject to: $A + B + C + D + E \leqslant 70$

$C - E = 0$

$A \geqslant 20$

$B \geqslant 10$

$A, B, C, D, E \geqslant 0.$

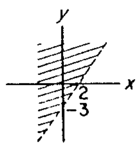
10. at least 161 days.

11. at least 120,001. 13. 12,400.

15. 60,000. 17. \$25,714.29.

19. \$1,000.

Exercise 4-3c

1.


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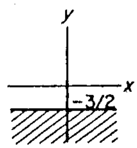

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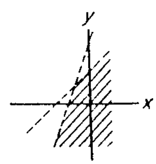

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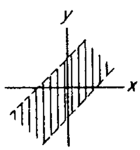

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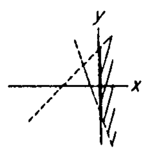


11. $Z = 3$ when $x = 3, y = 0$. 13. $Z = -2$ when $x = 0, y = 2$.

15. No optimum solution (empty feasible region).

17. $Z = 36$ when $x = (1 - t)(2) + 4t = 2 + 2t$, $y = (1 - t)(3) + 0t = 3 - 3t$, and $0 < t < 1$.

19. $Z = 32$ when $x_1 = 8, x_2 = 0$. 21. $Z = 2$ when $x_1 = 0, x_2 = 0, x_3 = 2$.

23. $Z = 24$ when $x_1 = 0, x_2 = 12$. 25. $Z = \frac{7}{2}$ when $x_1 = \frac{5}{4}, x_2 = 0, x_3 = \frac{9}{4}$.

27. No optimum solution (unbounded). 29. $Z = 70$ when $x_1 = 35, x_2 = 0, x_3 = 0$.

31. 0 units of X, 6 units of Y, 14 units of Z; \$398.

33. 500,000 gal from A to D, 100,000 gal from A to C, 400,000 gal from B to C; \$19,000.

Exercise 4-4

1. 181,250. 3. ฿80,000 at 6 percent, ฿320,000 at $7\frac{1}{2}$ percent.
5. ฿85. 7. 4 percent. 9. 40 units.
11. 46,000 units. 12. either ฿8,800 or ฿9,200. 13. ฿2,000. 14. p = 77.
15. 80 ft. by 140 ft.
16. Either 125 units of A and 100 units of B. or 150 units of A and 125 units of B.
17. 60 rai.

Exercise 5-1

1. $V = 50{,}000 - 6250t$.
3. (a) yes; (b) $V = 5{,}500{,}000 - 550{,}000t$; (c) 10 years.
5. (a) $c = 800 - 4.5x$, where x = number of patrol cars and c = number of serious crimes per week; (b) $0 \leqslant x \leqslant 120$; (c) reduces number of serious crimes per week by 4.5.
7. (a) $q = 235{,}000 - 4000p$; (b) \$43.75; (c) for every dollar increase in price, demand increases by 4000 units.
9. (a) $q = 6000p$; (b) \$8.33; (c) for every dollar increase in market price, supply will increase by 6000 units.

11. (a) $g = 2.42 + 0.06t$; (b) $t = 9.67$ or sometime between 1982 and 1983; (c) 2.84; (d) Cumulative grade point average increases 0.06 per year.

Exercise 5-2

2. $p^* = \$20$; $q^* = 70{,}000$ units.
4. (a) $R = 10x_1 + 15x_2 + 8.5x_3$;

   (b) $C = 7.5x_1 + 10.5x_2 + 6x_3 + 50{,}000$;

   (c) $P = R - C = 2.5x_1 + 4.5x_2 + 2.5x_3 - 50{,}000$;

   (d) \$120,000.
6. (a) $R = 25.6 - 0.25t$;

   (b) births per thousand of population are decreasing by 0.25 per year;

   (c) 21.6; (d) $0 \leqslant t \leqslant 102.4$.
8. (a) $q = 60{,}000p - 150{,}000$; (b) \$4.167;

   (c) quantity supplied will increase by 60,000 units for every increase in market price by \$1;

   (d) 2.5 (price must be greater than \$2.50 if any quantities are to be supplied).
10. (a) 36,000; (b) -\$15,000.
13. $p^* = 400$; $q^* = 20{,}000$.   15. $y = 2x^2$.
17. (a) $P = 15x - 75{,}000$; (b) 15,000.
18. (a) \$75,000; (b) \$7500; (c) 10 years.

19. (a) 80; (b) 56.

20. $p^* = 100$, $q^* = 35{,}000$.

21. $$\begin{aligned} 6x_1 + 2x_2 + 2x_3 &= 80 \\ 7x_1 + 4x_2 + x_3 &= 60 \\ 5x_1 + 5x_2 + 3x_3 &= 100. \end{aligned}$$

Exercise 5-3

1. (a) $f(x) = 0.72x: x \geqslant 0$;

   (b) $f(y) = 1.39y: y \geqslant 0$.

2. (a) $x = -\frac{5}{2} + \frac{1}{2}y$ for $0 \leqslant x \leqslant 5$ and $5 \leqslant y \leqslant 15$.

   (b) The inverse is not defined.

   (c) If I is an even integer, the inverse is not defined.

   If I is an odd integer, the inverse is defined $x = y^{\frac{1}{I}}$.

   (d) $x = \sqrt{y}$.

3. (a) $y = \frac{1}{36}.x.(1+a)^3$.

   (b) $x = \frac{36.y}{(1+a)^3}$

4. Profit, y = sales − costs

$$= p.x - (4000 + 5x)$$

But $x = 10{,}000 - p$. Hence

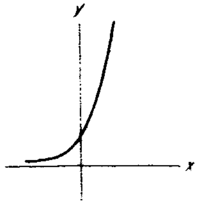
$$y = p.10{,}000 - p^2 - 4000 - 50{,}000 + 5p$$

$$y = -p^2 + 10{,}005p - 54{,}000$$

for $0 < p < 10{,}000$.

Exercise 5-4

1.



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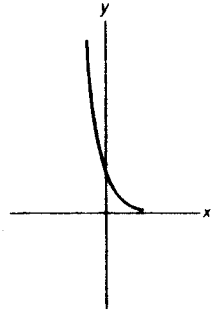
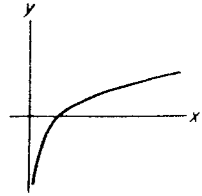


5.



7. 4.4817. 9. 0.67032. 11. 1.60944.

13. 2.00013. 15. $\log_{25} 5 = \frac{1}{2}$.

17. $\log 10{,}000 = 4$. 19. $2^6 = 64$.

21. $2^{14} = x$. 23. $\ln 7.3891 = 2$.

25. $e^{1.0986} = 3$. 27. 9. 29. 125.

31. $\frac{1}{10}$. 33. e^2. 35. 2. 37. 6.

39. 2. 41. 4. 43. $\frac{1}{2}$. 45. $\frac{1}{81}$.

47. 2. 49. $\frac{5}{3}$. 51. $\log_2 5$.

53. $\frac{\ln 2}{3}$. 55. $\log_3 8$. 57. $\frac{5 + \ln 3}{2}$.

59. 140,000. 61. \$1491.80.

62. \$26,960. 63. 41.50.

65. .399; .242; .242.

66. (a) 91; (b) 432; (c) 8.

Exercise 6-1

1. 2. 3. $-\frac{8}{13}$. 5. not defined.

7. 0. 9. $6x - y - 4 = 0$.

11. $x + 4y - 18 = 0$. 13. $3x - 7y + 25 = 0$.

15. $8x - 5y - 29 = 0$. 17. $y - 5 = 0$.

19. $x - 2 = 0$. 21. $4x - y + 7 = 0$.

23. 2; $(0,-1)$. 25. $\frac{3}{8}$; $(0,-1)$.

27. $-\frac{1}{2}$; $(0,\frac{3}{2})$. 29. No slope; no y-intercept.

31. 3; $(0,0)$. 33. 0; $(0,1)$.

35. $\frac{1}{40}$; $(0,-\frac{1}{2})$.

37. $x + 2y - 4 = 0$; $y = -\frac{1}{2}x + 2$.

39. $4x + 9y - 5 = 0$; $y = -\frac{4}{9}x + \frac{5}{9}$.

41. $9x - 28y - 3 = 0$; $y = \frac{9}{28}x - \frac{3}{28}$.

43. $3x - 2y + 24 = 0$; $y = \frac{3}{2}x + 12$.

45. $x + y - 1 = 0$; $y = -x + 1$.

47. 1; $(0,1)$. 49. -3; $(0,5)$.

51. $f(x) = 5x - 14$. 53. $f(x) = -3x + 9$.

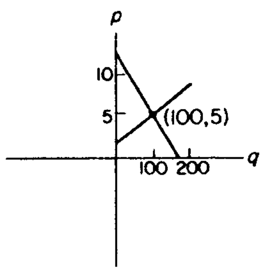
55. $(5,-4)$. 57. $p = -\frac{2}{5}q + 28$; 16.

59. $x + 10y = 100$.

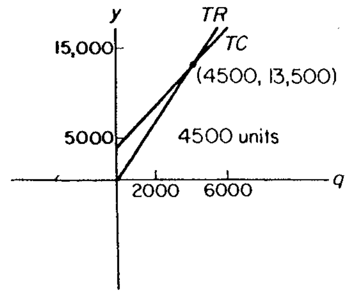
61. (b) $-\frac{2}{3}$; (c) $-\frac{2}{3}$; (d) they are parallel.

Exercise 6-2

1. $p = \frac{1}{3}q + \frac{55}{3}$. 3.



5. (5, 212.50). **7.** (9, 38). **9.** (15, 5). **11.**



13. Cannot break even at any level of production. **15.** 10 units or 40 units.

17. (a) \$12; (b) \$12.18. **19.** 5840 units; 840 units; 1840 units. **21.** \$4.

23. Total cost always exceeds total revenue—no break-even point.

Exercise 6-2

26. (a) 12,500 units.
    (b) \$1,875,000.
    (c) -\$10,000(loss)

29. (a) 5000.
    (b) 9000.

30. (a) \$4.00.
    (b) \$7.20.

31. (a) 40 min.
    (b) R = \$3,000,000
    C = \$2,950,000
    D = \$50,000.

33. (a) \$0.244.
    (b) \$0.264.

Exercise 7-1

1. 16. 3. 7. 5. 20. 7. 2.82.

9. 0. 11. -1. 13. $-\frac{5}{2}$. 15. 0.

17. 5. 19. 3. 21. $-\frac{5\sqrt{3}}{6}$.

23. 1. 25. 0. 27. $\frac{1}{6}$. 29. $-\frac{1}{5}$.

31. $\frac{7}{2}$. 33. $\frac{11}{9}$. 35. 4. 37. 2x.

Exercise 7-2a

1. (a) 2; (b) 3; (c) Does not exist; (d) $-\infty$; (e) ∞; (f) ∞; (g) ∞; (h) 0; (i) 1; (j) 1; (k) 1.

3. 1. 5. $-\infty$. 7. $-\infty$. 9. ∞. 11. 0. 13. Does not exist.

15. 0. 17. 0. 19. 1. 21. 0. 23. ∞. 25. $-\frac{2}{5}$. 27. $-\infty$.

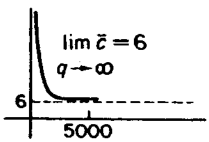
29. $\frac{2}{5}$. 31. $\frac{11}{5}$. 33. $-\frac{1}{2}$. 35. ∞. 37. ∞.

39. Does not exist. 41. $-\infty$. 43. 0. 45. 1.

47. (a) 1; (b) 2; (c) Does not exist; (d) 1; (e) 2.

49. (a) 0; (b) 0; (c) 0; (d) $-\infty$; (e) $-\infty$.

51.



53. 20,000. 55. -1. 57. $2x$.

59. 1, .5, .525, .631, .912, .986, .998; conclude limit is 1.

Exercise 7-2b

1. 0. 3. $13\frac{3}{4}$. 5. $-\frac{11}{3}$.

7. 18. 9. 250. 11. -4.

13. 13. 15. $b^2 + 2b + 1$. 17. 1.

Exercise 7-3a

1. none. 3. none. 5. $x = 3$.

7. none. 9. $x = 3, \frac{3}{2}$.

11. $x = 0, 3, -3$.

Exercise 7-3b

7. Continuous at -2 and 0. **9.** Discontinuous at ± 3.

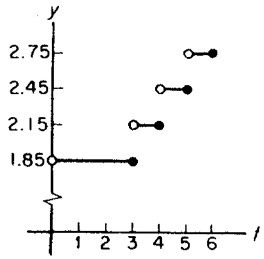
11. Continuous at 2 and 0. **13.** f is a polynomial function.

15. f is a quotient of polynomials and the denominator is never zero.

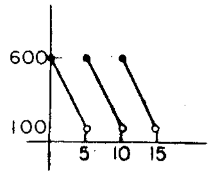
17. None. **19.** $x = 4$. **21.** None. **23.** $x = -5, 3$. **25.** $x = 0, \pm 1$.

27. None. **29.** $x = 2$. **31.** None. **33.** $x = 0, 2$.

35. Discontinuities at $t = 3, 4, 5$.



37.



Exercise 8-1

1. 1. 3. 2. 5. -2. 7. 0.

9. $2x + 4$. 11. $4p + 5$. 13. $-\frac{1}{x^2}$.

15. $\frac{1}{2\sqrt{x+2}}$. 17. -4. 19. 0.

21. $y = x+4$. 23. $y = -3x - 7$.

25. $y = -\frac{1}{3}x + \frac{5}{3}$.

Exercise 8-2

1. $(4x + 1)(6) + (6x + 3)(4) = 48x + 18 = 6(8x + 3)$.

3. $(8 - 7t)(2t) + (t^2 - 2)(-7) = 14 + 16t - 21t^2$.

5. $(3r^2 - 4)(2r - 5) + (r^2 - 5r + 1)(6r) = 12r^3 - 45r^2 - 2r + 20$.

7. $(x^2 + 3x - 2)(4x - 1) + (2x^2 - x - 3)(2x + 3) = 8x^3 + 15x^2 - 20x - 7$.

9. $(8w^2 + 2w - 3)(15w^2) + (5w^3 + 2)(16w + 2) = 200w^4 + 40w^3 - 45w^2 + 32w + 4$.

11. $3[(x^3 - 2x^2 + 5x - 4)(4x^3 - 6x^2 + 7) + (x^4 - 2x^3 + 7x + 1)(3x^2 - 4x + 5)]$
$= 3(7x^6 - 24x^5 + 45x^4 - 28x^3 - 15x^2 + 66x - 23)$.

13. $(x^2-1)(9x^2-6)+(3x^3-6x+5)(2x)-[(x+4)(8x+2)+(4x^2+2x+1)(1)]$
$=15x^4-39x^2-26x-3.$

15. $\frac{3}{2}\left[(p^{1/2}-4)(4)+(4p-5)\left(\frac{1}{2}p^{-1/2}\right)\right]=\frac{3}{4}(12p^{1/2}-5p^{-1/2}-32).$

17. $(2x^{.45}-3)(1.3x^{.3}-7)+(x^{1.3}-7x)(.9x^{-.55})=3.5x^{.75}-20.3x^{.45}-3.9x^{.3}+21.$

19. $18x^2+94x+31.$ 21. 0. 23. $\frac{(x-1)(1)-(x)(1)}{(x-1)^2}=-\frac{1}{(x-1)^2}.$

25. $\frac{(x-1)(1)-(x+2)(1)}{(x-1)^2}=\frac{-3}{(x-1)^2}.$

27. $\frac{(z^2-4)(-2)-(5-2z)(2z)}{(z^2-4)^2}=\frac{2(z-4)(z-1)}{(z^2-4)^2}.$

29. $\frac{(x^2-5x)(16x-2)-(8x^2-2x+1)(2x-5)}{(x^2-5x)^2}=\frac{-38x^2-2x+5}{(x^2-5x)^2}.$

31. $\frac{(2x^2-3x+2)(2x-4)-(x^2-4x+3)(4x-3)}{(2x^2-3x+2)^2}=\frac{5x^2-8x+1}{(2x^2-3x+2)^2}.$

33. $\frac{-100x^{99}}{(x^{100}+1)^2}.$ 35. $\frac{4(v^5+2)}{v^2}.$ 37. $\frac{15x^2-2x+1}{3x^{4/3}}.$

39. $\frac{4}{(x-8)^2}+\frac{2}{(3x+1)^2}.$

41. $\frac{(s-5)(2s-2)-[(s+2)(s-4)](1)}{(s-5)^2}=\frac{s^2-10s+18}{(s-5)^2}.$

43. $\frac{[(x+2)(x-4)](1)-(x-5)(2x-2)}{[(x+2)(x-4)]^2}=\frac{-(x^2-10x+18)}{[(x+2)(x-4)]^2}.$

45. $\frac{[(t^2-1)(t^3+7)](2t+3)-(t^2+3t)(5t^4-3t^2+14t)}{[(t^2-1)(t^3+7)]^2}$
$=\frac{-3t^6-12t^5+t^4+6t^3-21t^2-14t-21}{[(t^2-1)(t^3+7)]^2}.$

47. $\frac{(x^2-7x+12)(2x-3)-(x^2-3x+2)(2x-7)}{[(x-3)(x-4)]^2}=\frac{-2(2x^2-10x+11)}{[(x-3)(x-4)]^2}.$

49. $3-\frac{2x^3+3x^2-12x+4}{[x(x-1)(x-2)]^2}.$ 51. $-6.$ 53. $y=-\frac{3}{2}x+\frac{15}{2}.$

55. $y=16x+24.$

xercise 8-3

1. $(2u-2)(2x-1)=4x^3-6x^2-2x+2.$ **3.** $\left(-\frac{2}{w^3}\right)(-1)=\frac{2}{(2-x)^3}.$

5. $-2.$ **7.** $0.$ **9.** $56(7x+4)^7.$ **11.** $-56p(3-2p^2)^{13}.$

13. $\frac{40(3x^2-2)(4x^3-8x+2)^9}{3}.$ **15.** $-30(4r-5)(4r^2-10r+3)^{-16}.$

17. $-49x(3x-2)(x^3-x^2+2)^{-8}.$ **19.** $6z(6z-1)(4z^3-z^2+2)^{-4/5}.$

21. $\frac{1}{2}(4x-1)(2x^2-x+3)^{-1/2}.$ **23.** $\frac{6}{5}x(x^2+1)^{-2/5}.$

25. $10\left(\frac{x-7}{x+4}\right)^9\left[\frac{(x+4)(1)-(x-7)(1)}{(x+4)^2}\right]=\frac{110(x-7)^9}{(x+4)^{11}}.$

27. $10\left(\frac{q^3-2q+4}{5q^2+1}\right)^4\left[\frac{5q^4+13q^2-40q-2}{(5q^2+1)^2}\right]$

$=\frac{10(5q^4+13q^2-40q-2)(q^3-2q+4)^4}{(5q^2+1)^6}.$

29. $\frac{5}{2(x+3)^2}\left(\frac{x-2}{x+3}\right)^{-1/2}=\frac{5}{2(x+3)^2}\sqrt{\frac{x+3}{x-2}}.$

31. $(x^2+2x-1)^3(5)+(5x+7)[3(2x+2)(x^2+2x-1)^2]$

$=(x^2+2x-1)^2(35x^2+82x+37).$

33. $8[(4x+3)(6x^2+x+8)]^7[(4x+3)(12x+1)+(6x^2+x+8)(4)]$

$=8(72x^2+44x+35)[(4x+3)(6x^2+x+8)]^7.$

35. $\frac{(w^2+4)[6(2w+3)^2]-(2w+3)^3(2w)}{(w^2+4)^2}=\frac{2(w^2-3w+12)(2w+3)^2}{(w^2+4)^2}.$

37. $6\{(5x^2+2)[2x^3(x^4+5)^{-1/2}]+(x^4+5)^{1/2}(10x)\}$

$=12x(x^4+5)^{-1/2}(10x^4+2x^2+25).$

39. $(4-3x^2)^2[3(2-3x)^2(-3)]+(2-3x)^3[2(4-3x^2)(-6x)]$

$=3(4-3x^2)(2-3x)^2(21x^2-8x-12).$

41. $8+\frac{5}{(t+4)^2}-(8t-7)=15-8t+\frac{5}{(t+4)^2}.$

43. $\frac{(3x-1)^3[40(8x-1)^4]-(8x-1)^5[9(3x-1)^2]}{(3x-1)^6}=\frac{(8x-1)^4(48x-31)}{(3x-1)^4}.$

45. $\frac{(x^2-7)^4[(2x+1)(2)(3x-5)(3)+(3x-5)^2(2)]-(2x+1)(3x-5)^2[4(x^2-7)^3(2x}{(x^2-7)^8}$

47. 0. 49. 0. 51. $y = 4x - 11$. 53. $y = -\frac{1}{6}x + \frac{5}{3}$.

Exercise 8-4

1. $\frac{3}{3x-4}$. 3. $\frac{2}{x}$. 5. $-\frac{2x}{1-x^2}$. 7. $\frac{6p^2+3}{2p^3+3p} = \frac{3(2p^2+1)}{p(2p^2+3)}$.

9. $\frac{4\ln^3(ax)}{x}$. 11. $\frac{2x+4}{x^2+4x+5} = \frac{2(x+2)}{x^2+4x+5}$. 13. $t\left(\frac{1}{t}\right) + (\ln t)(1) = 1 + \ln t$.

15. $\frac{2\log_3 e}{2x-1}$. 17. $\frac{2(x^2+1)}{2x+1} + 2x\ln(2x+1)$. 19. $\frac{4x}{x^2+2} + \frac{3x^2+1}{x^3+x-1}$.

21. $\frac{2}{1-l^2}$. 23. $\frac{x}{1+x^2}$. 25. $\frac{5(x+1)^4+4(x+2)^3+8x^7}{(x+1)^5+(x+2)^4+x^8}$. 27. $\frac{x}{1-x^4}$.

29. $\frac{z\left(\frac{1}{z}\right) - (\ln z)(1)}{z^2} = \frac{1-\ln z}{z^2}$. 31. $\frac{x}{2(x-1)} + \ln\sqrt{x-1}$.

33. $\frac{1}{2x\sqrt{4+\ln x}}$.

Exercise 8-5

1. $2xe^{x^2+1}$. 3. $-5e^{3-5x}$. 5. $(6r+4)e^{3r^2+4r+4} = 2(3r+2)e^{3r^2+4r+4}$.

7. $x(e^x) + e^x(1) = e^x(x+1)$. 9. $2xe^{-x^2}(1-x^2)$. 11. $\frac{e^x - e^{-x}}{2}$.

13. $(6x)4^{3x^2}\ln 4$. 15. $\frac{2e^{2w}(w-1)}{w^3}$. 17. $\frac{e^{1+\sqrt{x}}}{2\sqrt{x}}$.

19. $3x^2 - 3^x\ln 3$. 21. $\frac{2e^x}{(e^x+1)^2}$. 23. $e^{e^x}e^x = e^{e^x+x}$.

25. 1. 27. $e^{x\ln x}(1+\ln x)$. 29. $(\log 2)^x \ln(\log 2)$.

31. $y - e^2 = e^2(x-2)$ or $y = e^2x - e^2$.

Exercise 8-6

1. $-\frac{x}{4y}$. 3. $-\frac{y}{x}$. 5. $\frac{4-y}{x-1}$. 7. $\frac{4y-x^2}{y^2-4x}$. 9. $-\frac{y^{1/4}}{x^{1/4}} = -\left(\frac{y}{x}\right)^{1/4}$.

11. $\frac{5}{12y^3}$. **13.** $-\frac{\sqrt{y}}{\sqrt{x}} = -\sqrt{\frac{y}{x}}$. **15.** $\frac{6y^{2/3}}{3y^{1/6}+2}$. **17.** $\frac{1-6xy^3}{1+9x^2y^2}$.

19. $\frac{xe^y - y}{x(\ln x - xe^y)}$. **21.** $-\frac{e^y}{xe^y+1}$. **23.** $-\frac{3}{5}$. **25.** $y = -\frac{3}{4}x + \frac{5}{4}$.

Exercise 8-7

1. (a) 1; (b) $\frac{1}{x+4}$; (c) 1; (d) $\frac{1}{9} \approx .111$; (e) 11.1%.

3. (a) $6x$; (b) $\frac{2x}{x^2+2}$; (c) 12; (d) $\frac{2}{3} \approx .667$; (e) 66.7%.

5. (a) $-3x^2$; (b) $-\frac{3x^2}{8-x^3}$; (c) -3; (d) $-\frac{3}{7} \approx -.429$; (e) -42.9%.

7. $\frac{dc}{dq} = 10$; 10. **9.** $\frac{dc}{dq} = .6q + 2$; 3.8. **11.** $\frac{dc}{dq} = 2q + 50$; 80, 82, 84.

13. $\frac{dc}{dq} = .02q + 5$; 6, 7. **15.** $\frac{dc}{dq} = .00006q^2 - .02q + 6$; 4.6, 11.

17. $\frac{dr}{dq} = .7$; .7, .7, .7. **19.** $\frac{dr}{dq} = 250 + 90q - 3q^2$; 625, 850, 625.

21. 3.2; 21.3%. **23.** $\frac{dc}{dq} = 6.750 - .000656q$; 3.47.

25. $\frac{dr}{dq} = 25\frac{(q+2)\ln(q+2) - q}{(q+2)\ln^2(q+2)}$. **27.** $\frac{dp}{dq} = -.015e^{-.001q}$; $-.015e^{-.5}$.

31. $\frac{dc}{dq} = 10e^{(q+3)/400}$; $10e^{.25}$, $10e^{.5}$. **33.** 0.

Exercise 8-8

1. $\frac{dC}{dI} = 0.672$. 3. $\frac{1}{3}$; $\frac{2}{3}$.

5. 0.615; 0.385.

Exercise 8-9

1. 20. 3. 13.99.

5. (a) $-\frac{q}{\sqrt{q^2+20}}$; (b) $-\frac{q}{100\sqrt{q^2+20} - q^2 - 20}$;

(c) $100 - \frac{q^2}{\sqrt{q^2+20}} - \sqrt{q^2+20}$.

7. -325. 9. 0.456; 0.544.

11. $\dfrac{dc}{dq} = \dfrac{5q(q^2+6)}{(q^2+3)^{3/2}}$.

Exercise 8-10

1. 24. 3. 0. 5. e^x. 7. $3 + 2\ln x$. 9. $-\dfrac{10}{p^6}$. 11. $-\dfrac{1}{4(1-r)^{3/2}}$.

13. $\dfrac{50}{(5x-6)^3}$. 15. $\dfrac{4}{(x-1)^3}$. 17. $-\left[\dfrac{1}{x^2} + \dfrac{1}{(x+1)^2}\right]$. 19. $e^z(z^2+4z+2)$.

21. $-\dfrac{1}{y^3}$. 23. $-\dfrac{4}{y^3}$. 25. $\dfrac{1}{8x^{3/2}}$. 27. $\dfrac{2(y-1)}{(1+x)^2}$. 29. $\dfrac{2y}{(2-y)^3}$.

31. $300(5x-3)^2$. 33. .6.

Exercise 8-11

1. 0. 3. $28x^3 - 18x^2 + 10x = 2x(14x^2 - 9x + 5)$.

5. $2e^x + e^{x^2}(2x) = 2(e^x + xe^{x^2})$. 7. $\dfrac{2x}{5}$. 9. $\dfrac{1}{r^2+5r}(2r+5) = \dfrac{2r+5}{r(r+5)}$.

11. $(x^2+6x)(3x^2-12x) + (x^3-6x^2+4)(2x+6) = 5x^4 - 108x^2 + 8x + 24$.

13. $100(2x^2+4x)^{99}(4x+4) = 400(x+1)[(2x)(x+2)]^{99}$.

15. $(8+2x)(4)(x^2+1)^3(2x) + (x^2+1)^4(2) = 2(x^2+1)^3(9x^2+32x+1)$.

17. $\dfrac{4}{3}(4x-1)^{-2/3}$. 19. $e^x(2x) + (x^2+2)e^x = e^x(x^2+2x+2)$.

21. $\dfrac{(z^2+1)(2z) - (z^2-1)(2z)}{(z^2+1)^2} = \dfrac{4z}{(z^2+1)^2}$.

23. $\dfrac{e^x\left(\dfrac{1}{x}\right) - (\ln x)(e^x)}{e^{2x}} = \dfrac{1 - x\ln x}{xe^x}$.

25. $e^{x^2+4x+5}(2x+4) = 2(x+2)e^{x^2+4x+5}$.

27. $\dfrac{(2x^2+3)(2x+1) - (x^2+x)(4x)}{(2x^2+3)^2} = \dfrac{-2x^2+6x+3}{(2x^2+3)^2}$. 29. $-\dfrac{y}{x+y}$.

31. $\dfrac{y}{2}\left[\dfrac{1}{x-6} + \dfrac{1}{x+5} - \dfrac{1}{9-x}\right] = \dfrac{y}{2}\left[\dfrac{1}{x-6} + \dfrac{1}{x+5} + \dfrac{1}{x-9}\right]$.

33. $\dfrac{2}{q+1} + \dfrac{3}{q+2}$. 35. $-\dfrac{1}{2}(1-x)^{-3/2}(-1) = \dfrac{1}{2}(1-x)^{-3/2}$.

37. $\dfrac{16 \log_2 e}{8x + 5}$. 39. $y[1 + \ln(x + 1)]$.

41. $\dfrac{\sqrt{x^2 + 5}\,(2x) - (x^2 + 6)(1/2)(x^2 + 5)^{-1/2}(2x)}{x^2 + 5} = \dfrac{x(x^2 + 4)}{(x^2 + 5)^{3/2}}$. 43. $-\dfrac{y}{x}$.

45. $2\left(-\dfrac{3}{8}\right)x^{-11/8} + \left(-\dfrac{3}{8}\right)(2x)^{-11/8}(2) = -\dfrac{3}{4}(1 + 2^{-11/8})x^{-11/8}$.

47. $\dfrac{1 + 2l + 3l^2}{1 + l + l^2 + l^3}$.

49. $\left(\dfrac{3}{5}\right)(x^3 + 6x^2 + 9)^{-2/5}(3x^2 + 12x) = \dfrac{9}{5}x(x + 4)(x^3 + 6x + 9)^{-2/5}$.

51. $2\left(\dfrac{1}{u}\right) + \dfrac{1}{2}\left(\dfrac{1}{1 - u}\right)(-1) = \dfrac{5u - 4}{2u(u - 1)}$.

53. $y\left[\dfrac{3}{2}\left(\dfrac{1}{x^2 + 2}\right)(2x) + \dfrac{4}{9}\left(\dfrac{1}{x^2 + 9}\right)(2x) - \dfrac{4}{11}\left(\dfrac{1}{x^3 + 6x}\right)(3x^2 + 6)\right]$

$= y\left[\dfrac{3x}{x^2 + 2} + \dfrac{8x}{9(x^2 + 9)} - \dfrac{12(x^2 + 2)}{11x(x^2 + 6)}\right]$.

55. 12. 57. $-\dfrac{1}{128}$. 59. $\dfrac{4}{9}$. 61. $-\dfrac{1}{4}$. 63. $y = -4x + 3$.

65. $y = 2x + 2(1 - \ln 2)$ or $y = 2x + 2 - \ln 4$. 67. $7x - 2\sqrt{10}\,y - 9 = 0$.

69. $\frac{5}{7} \approx .714$; 71.4%. 71. $\dfrac{dr}{dq} = 20 - .2q$. 73. .569, .431.

75. $\dfrac{dr}{dq} = 450 - q$. 77. $\dfrac{dc}{dq} = .125 + .00878q$; .7396.

Exercise 9-1

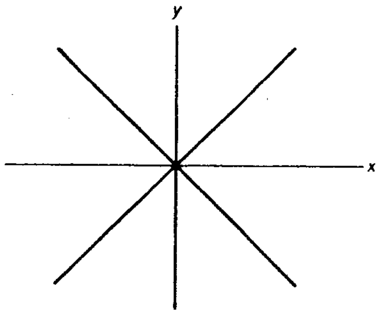
1. (0, 0); sym. to origin. 3. $(\pm 2, 0)$, (0, 8); sym. to y-axis.

5. $(\pm 3, 0)$; sym. to x-axis, y-axis, origin. 7. (−2, 0); sym. to x-axis.

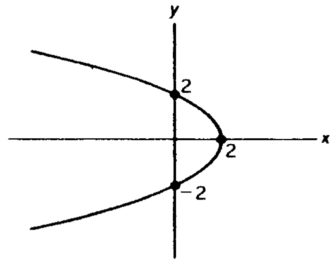
9. Sym. to x-axis. 11. (−21, 0), (0, −7), (0, 3). 13. (0, 0); sym. to origin.

15. $(\pm\sqrt{\ln 5}, 0)$, $(0, \pm\sqrt{\ln 5})$; sym. to x-axis, y-axis, origin.

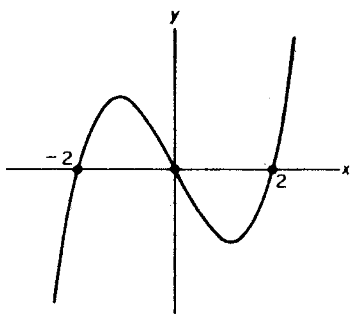
17. (0, 0); sym. to x-axis, y-axis, origin.



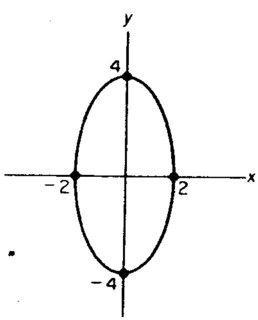
19. (2, 0), (0, ± 2); sym. to x-axis.



21. (± 2, 0), (0, 0); sym. to origin.



23. (± 2, 0), (0, ± 4); sym. to x-axis, y-axis, origin.



Exercise 9-2

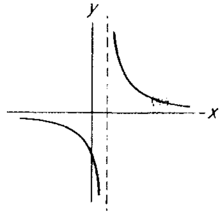
1. $y = 0, x = 0.$ **3.** $y = 4, x = 6.$ **5.** None. **7.** $y = \frac{1}{2}, x = -\frac{3}{2}.$

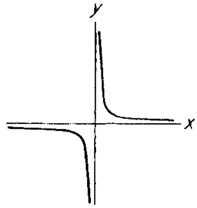
9. $y = 2, x = -3, x = 2.$ **11.** None. **13.** $y = 4.$

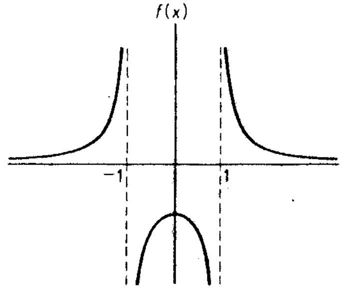
15. $(0, -3)$; $y = 0, x = 1.$

17. Sym. to origin; $y = 0, x = 0.$

19. $(0, -1)$; sym. to y-axis; $y = 0, x = 1, x = -1.$



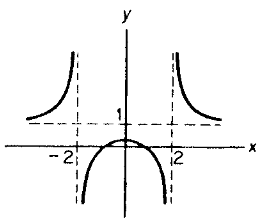


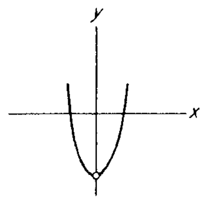


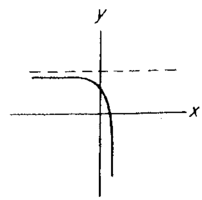
21. $(\pm 1, 0), (0, \frac{1}{4})$; sym. to y-axis; $y = 1, x = 2, x = -2.$

23. $(\pm 3, 0)$; sym. to y-axis.

25. $((\ln 3)/2, 0), (0, 2)$; $y = 3$







Exercise 9-3a

1. Dec. on $(-\infty, 0)$; inc. on $(0, \infty)$; rel. min. when $x = 0.$

3. Inc. on $\left(-\infty, \frac{1}{2}\right)$; dec. on $\left(\frac{1}{2}, \infty\right)$; rel. max. when $x = \frac{1}{2}.$

5. Dec. on $(-\infty, -5)$ and $(1, \infty)$; inc. on $(-5, 1)$; rel. min. when $x = -5$; rel. max when $x = 1.$

7. Dec. on $(-\infty, -1)$ and $(0, 1)$; inc. on $(-1, 0)$ and $(1, \infty)$; rel. max. when $x = 0$; rel min. when $x = \pm 1.$

9. Inc. on $(-\infty, 1)$ and $(3, \infty)$; dec. on $(1, 3)$; rel. max. when $x = 1$; rel. min. when $x = 3.$

11. Inc. on $(-\infty, -1)$ and $(1, \infty)$; dec. on $(-1, 0)$ and $(0, 1)$; rel. max. when $x = -1$; rel. min. when $x = 1$.

13. Dec. on $(-\infty, -4)$ and $(0, \infty)$; inc. on $(-4, 0)$; rel. min. when $x = -4$; rel. max. when $x = 0$.

15. Dec. on $(-\infty, 1)$ and $(1, \infty)$; no rel. max. or min.

17. Dec. on $(0, \infty)$; no rel. max. or min.

19. Dec. on $(-\infty, 0)$ and $(2, \infty)$; inc. on $(0, 1)$ and $(1, 2)$; rel. min. when $x = 0$; rel. max. when $x = 2$.

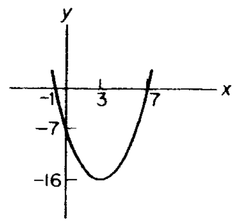
21. Inc. on $(-\infty, -2)$, $(-2, 11/5)$, and $(5, \infty)$; dec. on $(11/5, 5)$; rel. max. when $x = 11/5$; rel. min. when $x = 5$.

23. Dec. on $(-\infty, \infty)$; no rel. max. or min.

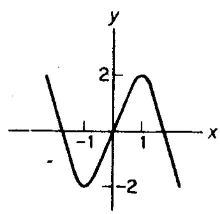
25. Dec. on $(0, 1)$; inc. on $(1, \infty)$; rel. min. when $x = 1$.

27. Dec. on $(-\infty, 0)$; inc. on $(0, \infty)$; rel. min. when $x = 0$.

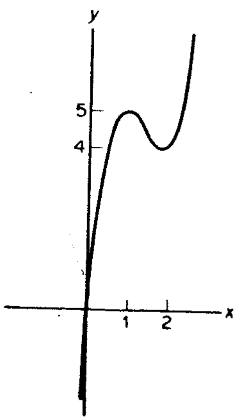
29. Dec. on $(-\infty, 3)$; inc. on $(3, \infty)$; rel. min. when $x = 3$; intercepts: $(7, 0)$, $(-1, 0)$, $(0, -7)$.



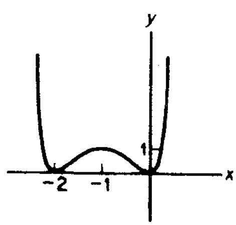
31. Dec. on $(-\infty, -1)$ and $(1, \infty)$; inc. on $(-1, 1)$; rel. min. when $x = -1$; rel. max. when $x = 1$; sym. to origin; intercepts: $(\pm\sqrt{3}, 0)$, $(0, 0)$.



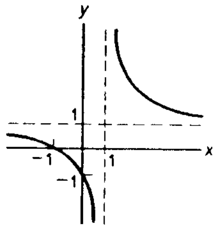
33. Inc. on $(-\infty, 1)$ and $(2, \infty)$; dec. on $(1, 2)$; rel. max. when $x = 1$; rel. min. when $x = 2$; intercept: $(0, 0)$.



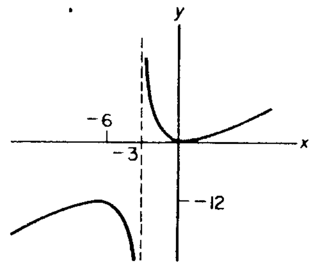
35. Inc. on $(-2, -1)$ and $(0, \infty)$; dec. on $(-\infty, -2)$ and $(-1, 0)$; rel. max. when $x = -1$; rel. min. when $x = -2, 0$; intercepts $(0, 0)$, $(-2, 0)$.



37. Dec. on $(-\infty, 1)$ and $(1, \infty)$; asym. $y = 1$, $x = 1$; intercepts: $(0, -1)$, $(-1, 0)$.



39. Inc. on $(-\infty, -6)$ and $(0, \infty)$; dec. on $(-6, -3)$ and $(-3, 0)$; rel. max. when $x = -6$; rel. min. when $x = 0$; asym. $x = -3$; intercept: $(0, 0)$.



41. Ab. max. when $x = -1$; ab. min. when $x = 1$.

43. Ab. max. when $x = 0$; ab. min. when $x = 2$.

45. Ab. max. when $x = 3$; ab. min. when $x = 1$. **49.** Never. **51.** 40.

Exercise 9-3b

1. Rel. min. when $x = \frac{5}{2}$; abs. min.
3. Rel max. when $x = \frac{1}{4}$; abs. max.
5. Rel. max. when $x = -3$, rel. min. when $x = 3$.
7. Rel. min, when $x = 0$, rel. max. when $x = 2$.
9. Test fails, when $x = 0$ there is rel, min. by first-derive. test.

Exercise 9-4

1. Conc. down $(-\infty, \infty)$.

3. Conc. down $(-\infty, -1)$; conc. up $(-1, \infty)$; inf. pt. when $x = -1$.

5. Conc. up $(-\infty, -1)$, $(1, \infty)$; conc. down $(-1, 1)$; inf. pt. when $x = \pm 1$.

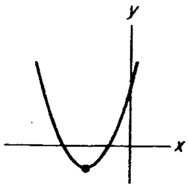
7. Conc. down $(-\infty, 1)$; conc. up $(1, \infty)$.

9. Conc. down $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, \infty\right)$; conc. up $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$; inf. pt. when $x = \pm \frac{1}{\sqrt{3}}$.

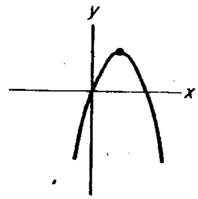
11. Conc. up $(-\infty, \infty)$.

13. Conc. down $(-\infty, -2)$; conc. up $(-2, \infty)$; inf. pt. when $x = -2$.

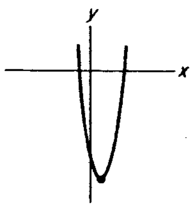
15. Int. $(-3, 0)$, $(-1, 0)$, $(0, 3)$; dec. $(-\infty, -2)$; inc. $(-2, \infty)$; rel. min. when $x = -2$; conc. up $(-\infty, \infty)$.



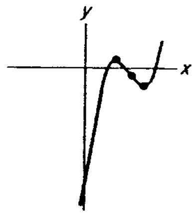
17. Int. $(0, 0)$, $(4, 0)$; inc. $(-\infty, 2)$; dec. $(2, \infty)$; rel. max. when $x = 2$; conc. down $(-\infty, \infty)$.



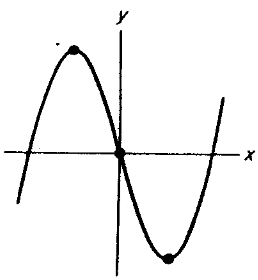
19. Int. $(-3/2, 0)$, $(4, 0)$, $(0, -12)$; dec. $(-\infty, 5/4)$; inc. $(5/4, \infty)$; rel. min. when $x = 5/4$; conc. up $(-\infty, \infty)$.



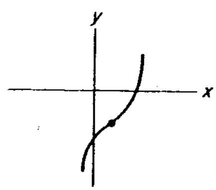
21. Int. $(0, -19)$; inc. $(-\infty, 2)$, $(4, \infty)$; dec. $(2, 4)$; rel. max. when $x = 2$; rel. min. when $x = 4$; conc. down $(-\infty, 3)$; conc. up $(3, \infty)$; inf. pt. when $x = 3$.



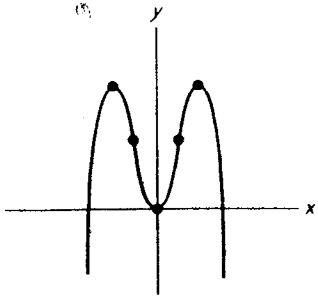
23. Int. $(0, 0)$, $(\pm 3, 0)$; inc. $(-\infty, -\sqrt{3})$, $(\sqrt{3}, \infty)$; dec. $(-\sqrt{3}, \sqrt{3})$; rel. max. when $x = -\sqrt{3}$; rel. min. when $x = \sqrt{3}$; conc. down $(-\infty, 0)$; conc. up $(0, \infty)$; inf. pt. when $x = 0$; sym. to origin.



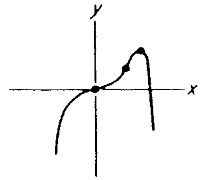
25. Int. $(0, -3)$; inc. $(-\infty, 1)$, $(1, \infty)$; no rel. max. or min.; conc. down $(-\infty, 1)$; conc. up $(1, \infty)$; inf. pt. when $x = 1$.



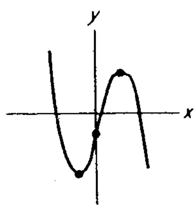
27. Int. (0, 0), $(\pm 2, 0)$; inc. $(-\infty, -\sqrt{2})$, $(0, \sqrt{2})$; dec. $(-\sqrt{2}, 0)$, $(\sqrt{2}, \infty)$; rel. max. when $x = \pm\sqrt{2}$; rel. min. when $x = 0$; conc. down $(-\infty, -\sqrt{2/3})$, $(\sqrt{2/3}, \infty)$; conc. up $(-\sqrt{2/3}, \sqrt{2/3})$; inf. pt. when $x = \pm\sqrt{2/3}$; sym. to y-axis.



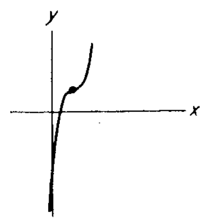
29. Int. (0, 0), (4/3, 0); inc. $(-\infty, 0)$, (0, 1); dec. $(1, \infty)$; rel. max. when $x = 1$; conc. up (0, 2/3); conc. down $(-\infty, 0)$, $(2/3, \infty)$; inf. pt. when $x = 0$, $x = 2/3$.



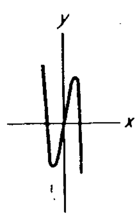
31. Int. (0, −2); dec. $(-\infty, -2)$, $(2, \infty)$; inc. (−2, 2); rel. min. when $x = -2$; rel. max. when $x = 2$; conc. up $(-\infty, 0)$; conc. down $(0, \infty)$; inf. pt. when $x = 0$.



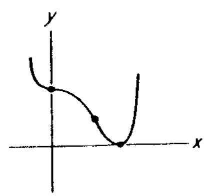
33. Int. (0, −6); inc. $(-\infty, 2)$, $(2, \infty)$; conc. down $(-\infty, 2)$; conc. up $(2, \infty)$; inf. pt. when $x = 2$.



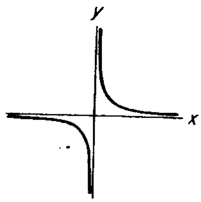
35. Int. (0, 0), $(\pm\sqrt[4]{5}, 0)$; dec. $(-\infty, -1)$, $(1, \infty)$; inc. (−1, 1); rel. min. when $x = -1$; rel. max. when $x = 1$; conc. up $(-\infty, 0)$; conc. down $(0, \infty)$; inf. pt. when $x = 0$; sym. to origin.



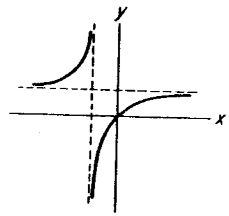
37. Int. (0, 1), (1, 0); dec. $(-\infty, 0)$, (0, 1); inc. $(1, \infty)$; rel. min. when $x = 1$; conc. up $(-\infty, 0)$, $(2/3, \infty)$; conc. down (0, 2/3); inf. pt. when $x = 0$, $x = 2/3$.



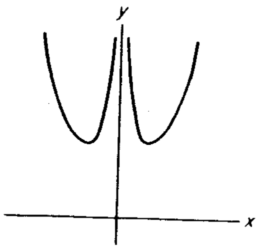
39. Dec. $(-\infty, 0)$, $(0, \infty)$; conc. down $(-\infty, 0)$; conc. up $(0, \infty)$; sym. to origin; asymptotes $x = 0$, $y = 0$.



41. Int. $(0, 0)$; inc. $(-\infty, -1)$, $(-1, \infty)$; conc. up $(-\infty, -1)$; conc. down $(-1, \infty)$; asymptotes $x = -1$, $y = 1$.



43. Dec. $(-\infty, -1)$, $(0, 1)$; inc. $(-1, 0)$, $(1, \infty)$; rel. min. when $x = \pm 1$; conc. up $(-\infty, 0)$, $(0, \infty)$; sym. to y-axis; asymptote $x = 0$.



Exercise 9-5

1. 100. 3. \$15. 5. 525,\$51,\$10,525.

7. \$22. 9. 625,\$4. 11. 750.

13. 20 and 20. 15. 300 ft by 250 ft.

17. 4 ft by 4 ft by 2 ft.

25. 250 per lot(4 lots). 27. 35.

29. 5000. 31. $5 - \sqrt{3}$ tons, $5 - \sqrt{3}$ tons.

33. 500 units. 35. x = 10 ft., h = 7.5 ft.

38. 10,000 fish. 39. 5 runs.

41. x = 150. 44. 4 runs.

45. x = 6; R(x) = 18. 46 p = \$6.

47. x = 2500 units, p = \$75/unit,

Profit = \$52,500.

48. x = 2000 units, p = \$80/unit,

Profit = \$30,000.

49. \$1. 51. 32. 53. 5.

55. x = 20 units, p = \$133.33.

57. 2 million tons, \$156 per ton.

58. (a) \$1.00; (b) \$1.15.

60. 2000 cans.

62. (a) $x = 15.10^5$, p = \$45;

(b) No profit is maximized when price is increased to \$50.

Exercise 9-6

1. $f_x(x, y) = 1$; $f_y(x, y) = -5$. **3.** $f_x(x, y) = 3$; $f_y(x, y) = 0$.

5. $g_x(x, y) = 5x^4y^4 - 12x^3y^3 + 21x^2 - 3y$; $g_y(x, y) = 4x^5y^3 - 9x^4y^2 + 4y - 3x$.

7. $g_p(p, q) = \dfrac{q}{2\sqrt{pq}}$; $g_q(p, q) = \dfrac{p}{2\sqrt{pq}}$.

9. $h_s(s, t) = \dfrac{2s}{t - 3}$; $h_t(s, t) = -\dfrac{s^2 + 4}{(t - 3)^2}$.

11. $u_{q_1}(q_1, q_2) = \dfrac{3}{4q_1}$; $u_{q_2}(q_1, q_2) = \dfrac{1}{4q_2}$.

13. $h_x(x, y) = (x^3 + xy^2 + 3y^3)(x^2 + y^2)^{-3/2}$;

$h_y(x, y) = (3x^3 + x^2y + y^3)(x^2 + y^2)^{-3/2}$.

15. $\dfrac{\partial z}{\partial x} = 5ye^{5xy}$; $\dfrac{\partial z}{\partial y} = 5xe^{5xy}$.

17. $\dfrac{\partial z}{\partial x} = 5\left[\dfrac{2x^2}{x^2 + y} + \ln(x^2 + y)\right]$; $\dfrac{\partial z}{\partial y} = \dfrac{5x}{x^2 + y}$.

19. $f_r(r, s) = \sqrt{r + 2s}\,(3r^2 - 2s) + \dfrac{r^3 - 2rs + s^2}{2\sqrt{r + 2s}}$;

$f_s(r, s) = 2(s - r)\sqrt{r + 2s} + \dfrac{r^3 - 2rs + s^2}{\sqrt{r + 2s}}$.

21. $f_r(r, s) = -e^{3-r} \ln(7-s)$; $f_s(r, s) = \dfrac{-e^{3-r}}{7-s}$.

23. $g_x(x, y, z) = 6xy + 2y^2z$; $g_y(x, y, z) = 3x^2 + 4xyz$; $g_z(x, y, z) = 2xy^2 + 9z^2$.

25. $g_r(r, s, t) = 2re^{s+t}$; $g_s(r, s, t) = (7s^3 + 21s^2 + r^2)e^{s+t}$;
$g_t(r, s, t) = e^{s+t}(r^2 + 7s^3)$.

27. 50. 29. $\frac{1}{3}$. 31. 0.

Exercise 9-7

1. $(-2,1)$. 3. $(26,11)$.
5. $(1,-3),(-1,-3)$.
7. $(\sqrt{5},1)$; $(\sqrt{5},-1)$; $(-\sqrt{5},1)$; $(-\sqrt{5},-1)$.
9. $(\frac{1}{3},\frac{4}{3})$. 11. $(0,0)$ min.
13. $(-1,-4)$ max. 15. $(0,-1)$ min.
17. $(-1,2)$ max; $(1,2)$ neither max nor min.
19. $(\frac{1}{4},+2)$ min; $(\frac{1}{4},-2)$ neither max nor min.
21. $(\frac{1}{2},\frac{1}{6},\frac{1}{2})$. 23. (14 in. × 14 in. × 28 in.).
25. $x = 120$, $y = 80$.

Exercise 10-1

2. $-50x + C$. 4. $\frac{x^2}{8} + C$.
6. $10x - 3x^2 + C$. 8. $\frac{x^5}{5} + C$.
10. $15x^{1/3} + C$. 12. $x + C$.
14. $\frac{ax^3}{3} + \frac{bx^2}{2} + Cx + C$. 16. $\frac{(x^2-5)^4}{4} + C$.
18. $2\sqrt{2x^2-5} + C$. 20. $\frac{1}{40}(4x^4 - 16x)^{5/2} + C$.
22. $(\frac{x^2}{6} - 20)^6/2 + C$. 24. $\frac{e^{5x}}{5} + C$.

26. $(\frac{1}{4})e^{2x^2-4x} + C.$ 28. $\frac{1}{3}\ln(x^3 - 3x) + C.$

30. $(\frac{1}{m}) \ln (mx + b) + C.$

31. $\int 6\, dx = 6x + K.$

33. $\int x^{-4}\, dx = \frac{x^{-3}}{-3} + K = -\frac{1}{3}x^{-3} + K.$

35. $\int (4x^3 + 3x^2 - 4)\, dx = x^4 + x^3 - 4x + K.$

37. $\int (3\sqrt{x} + x)\, dx = \frac{3x^{3/2}}{3/2} + \frac{x^2}{2} + K$

$= 2x^{3/2} + \frac{1}{2}x^2 + K.$

39. $\int (4x^{3/2} + x^{1/2})\, dx = \frac{4x^{5/2}}{5/2} + \frac{x^{3/2}}{3/2} + K$

$= \frac{8}{5}x^{5/2} + \frac{2}{3}x^{3/2} + K.$

41. $\int \frac{3x^5 + 1}{x^2}\, dx = \int (3x^3 + x^{-2})\, dx = \frac{3}{4}x^4 - x^{-1} + K.$

43. $\int \frac{x^3 - 8}{x - 2}\, dx = \int (x^2 + 2x + 4)\, dx$

$= \frac{1}{3}x^3 + x^2 + 4x + K.$

46. $R(x) = \int (50x - x^2)\, dx = \frac{50}{2}x^2 - \frac{1}{3}x^3 + K$

Since $R(0) = 0, K = 0$, and

$R(x) = 25x^2 - \frac{1}{3}x^3$

Maximum revenue occurs at

$R'(x) = 50x - x^2 = 0$

$50 - x = 0$

$x = 50.$

48. $C(x) = \int (14x - 280)\, dx$

$$= \frac{14}{2} x^2 - 280x + K$$

Since the fixed cost is 4300, $K = C(0)$

$= 4300$

Hence $C(x) = 7x^2 - 280x + 4300$

The minimum cost occurs for

$C'(x) = 14x - 280 = 0$

$x = 20$

$C(20) = 1500$

50. (a) $\int (3x - 1)^3\, dx = \frac{1}{3}\int (3x - 1)^3 3\, dx$

$$= \frac{1}{3}\frac{(3x - 1)^4}{4} + K = \frac{1}{12}(3x - 1)^4 + K$$

(c) $\int (x^2 + 2)^6 4x\, dx = 2\int (x^2 + 2)^6 2x\, dx$

$$= 2\frac{(x^2 + 2)^7}{7} + K = \frac{2}{7}(x^2 + 2)^7 + K$$

(e) $\int \frac{dx}{(3 + 5x)^2} = \frac{1}{5}\int (3 + 5x)^{-2} 5\, dx = -\frac{1}{5}(3 + 5x)^{-1} + K$

(g) $\int \frac{(x^{1/3} - 1)^6}{x^{2/3}}\, dx = 3\int (x^{1/3} - 1)^6 \frac{1}{3} x^{-2/3}\, dx = \frac{3}{7}(x^{1/3} - 1)^7 + K$

(i) $\int \frac{(x + 1)\, dx}{(x^2 + 2x + 3)^{1/4}} = \frac{1}{2}\int (x^2 + 2x + 3)^{-1/4}(2x + 2)\, dx$

$$= \frac{1}{2}\frac{(x^2 + 2x + 3)^{3/4}}{3/4} + K$$

$$= \frac{2}{3}(x^2 + 2x + 3)^{3/4} + K.$$

51. $f(x) = \int \frac{dx}{(x + 1)^{1/2}} = \frac{(x + 1)^{1/2}}{1/2} + K = 2(x + 1)^{1/2} + K$

$f(0) = 2(0 + 1)^{1/2} + K = 1$

$2 + K = 1$

$K = -1$

$f(x) = 2(x + 1)^{1/2} - 1.$

53. $C(x) = \int x\sqrt{3x^2+1}\,dx = \frac{1}{6}\int (3x^2+1)^{1/2}6x\,dx$

$$= \frac{1}{6}\frac{(3x^2+1)^{3/2}}{3/2} + K$$

$$= \frac{1}{9}(3x^2+1)^{3/2} + K$$

$$C(0) = \frac{1}{9}(1)^{3/2} + K = 20$$

$$\frac{1}{9} + K = 20$$

$$K = \frac{179}{9}$$

$$C(x) = \frac{1}{9}(3x^2+1)^{3/2} + \frac{179}{9}.$$

Exercise 10-2

1. 12. **3.** $\frac{9}{2}$. **5.** $\frac{100}{3}$. **7.** -24. **9.** $\frac{14}{3}$. **11.** $\frac{7}{3}$. **13.** $\frac{15}{2}$.

15. $-\frac{7}{6}$. **17.** 0. **19.** $\frac{5}{3}$. **21.** $\frac{32}{3}$. **23.** $-\frac{1}{6}$. **25.** 4 ln 8.

27. $\frac{1}{3}(e^8 - 1)$. **29.** $\frac{3}{4}$. **31.** $\frac{38}{9}$. **33.** $\frac{15}{28}$. **35.** $\frac{1}{2}\ln 3$.

37. $e + \frac{1}{2e^2} - \frac{3}{2}$. **39.** $\frac{3}{2} - \frac{1}{e} + \frac{1}{2e^2}$. **41.** $\frac{e^3}{2}(e^{12} - 1)$.

Exercise 10-3

In Problems **1–33**, *answers are assumed to be expressed in square units.*

1. 8. **3.** $\frac{19}{2}$. **5.** 8. **7.** $\frac{19}{3}$. **9.** 9. **11.** $\frac{50}{3}$. **13.** 36.

15. 8. **17.** $\frac{32}{3}$. **19.** 1. **21.** 18. **23.** $\frac{26}{3}$. **25.** $\frac{3}{2}\sqrt[3]{2}$. **27.** $e^2 - 1$.

29. $\frac{3}{2} + 2\ln 2 = \frac{3}{2} + \ln 4$. **31.** 68. **33.** 2.

Exercise 10-4a

In Problems **1–21**, *the answers are assumed to be expressed in square units.*

1. $\frac{4}{3}$. **3.** $\frac{16}{3}$. **5.** $8\sqrt{6}$. **7.** 40. **9.** $\frac{125}{6}$. **11.** $\frac{32}{81}$.

13. $\frac{125}{12}$. **15.** $\frac{9}{2}$. **17.** $\frac{44}{3}$. **19.** $\frac{4}{3}(5\sqrt{5} - 2\sqrt{2})$. **21.** $\frac{1}{2}$.

Exercise 10-4b

2. 7. 4. 50. 6. 20.34. 8. $10\frac{2}{3}$.

10. -38. 12. $e^3 - 1$. 14. ln(2).

16. $\frac{8a}{3} + 2b + 2c$. 18. $\frac{875}{3}$. 20. 42.

22. 1248. 24. 54.

25. $\frac{2}{5}(4\sqrt{2}) + \frac{1}{2}(4)$. 26. $\frac{1}{5} + \frac{3}{8}$.

27. $(-x^{-2} - x^4) + C$. 33. $28\frac{1}{8} - 12\log_4 e$.

34. $\frac{1}{2}e^{2x} + x + x^{-1} - \log_e x + C$.

35. $\frac{1}{8}e^{4x} + C$. 36. $\frac{1}{\log_e 2}(2^b - 2^a)$.

37. $e^{x+2} + \frac{x^3}{3} + C$. 38. $\frac{3}{2}e^{(x^2+4)} + C$.

39. $\frac{2}{9}(16\sqrt{2} - 1)$.

40. $\frac{1}{b(d+1)}(a + bx)^{d+1} + C$.

46. (a) $a = 2$ or $a = -4$; (b) $b = \frac{20}{13}$;

(c) there are many pairs of values that will work. For example, if we set $a = 0$, then $b = \sqrt{8}$.

49. (a) $c = 0$ and $f(x) = \frac{3}{2}x^2 + 4x^3$;

(b) $f(x) = \frac{7}{2}x^2 + \frac{x^5}{5}$ and both constants are Zero;

(c) $\frac{x^2}{2} + \frac{x^3}{6} + \frac{x^6}{30}$, and both constants are Zero.

50. $6x + C$. 52. $\frac{x}{3} + C$. 54. $ax + bx + C$.

56. $x + C$. 58. $\frac{x^2}{2} + C$. 60. $-x^{-1} + C$.

62. $-\frac{1}{2} x^{-2} + C$. 64. $\frac{2x^{3/2}}{3} + C$.

66. $\frac{8}{3}$. 68. 5. 70. $7499\frac{1}{4}$.

72. $\frac{3x^2}{3^2} + 4x + C$. 74. $\frac{x^4}{2} + \frac{x^3}{3} + C$.

76. $\frac{x^3}{3} + x^2 + x + C$. 78. 15.

80. $e^{3x} + C$. 82. $\frac{e^{2x}}{4} + C$. 84. 1.

86. $x \ln(x) - x + C$.

88. $x \log_a (3x) - x \log_a e + C$.

90. 14.0260. 92. $\frac{2}{7} (x - 1)^{7/2} + C$.

94. $\frac{(4x^3 - 2)^{10}}{120} + C$. 96. $\frac{e^{x^3}}{3} + C$.

98. $\frac{1}{5} \ln(5x - 6) + C$. 100. 2.4423.

102. $\frac{1}{2} (x^2 - 6) \ln (x^2 - 6) - \frac{1}{2} (x^2 - 6) + C$.

104. 0.6932.

Exercise 11-1

1. (376.17).P dollars.
2. $\int_1^{10} 6000\pi \, dr \approx 169{,}646$.
3. $\int_0^{15} A(t)\, dt = \int_0^{15} 20.3e^{0.09t}\, dt \approx 644.5$bbl.

Exercise 11-2

1. $20 - 5x = 4x + 8;\ 9x = 12;\ x^* = \frac{4}{3};$

$$p^* = D(x^*) = D\left(\frac{4}{3}\right) = \frac{40}{3}$$

$$CS = \int_0^{4/3} (20 - 5x)\,dx - \frac{40}{3}\cdot\frac{4}{3}$$

$$= \left(20x - \frac{5}{2}x^2\right)\Bigg|_0^{4/3} - \frac{160}{9}$$

$$= \frac{200}{9} = \frac{160}{9} = \frac{40}{9} = \$4.44$$

$$PS = \frac{40}{3}\cdot\frac{4}{3} - \int_0^{4/3} (4x + 8)\,dx$$

$$= \frac{160}{9} - (2x^2 + 8x)\Bigg|_0^{4/3}$$

$$= \frac{160}{9} - \frac{128}{9} = \frac{32}{9} = \$3.56.$$

4. $\int_4^8 \left(25 + 2t - \frac{1}{4}t^2\right) dt = 110\frac{2}{3}.$

6. (767.74)P dollars. 8. \$1.37.

10. $\int_0^5 400\pi r e^{-r}\,dr.$

11. $10[e^5 - 1] \approx 1474.$ 13. \$6890.

Exercise 11-3

1. (a) \$1900; (b) \$875.

3. (b) 9 days; (c) \$53,550; (d) \$24,300.

4. (b) p = 5, q = 75; (c) \$166.67.

6. approximately 4 days (4.02).

8. (a) 4.234 billion tons/year;

(b) 29.79 billion tons.

10. 319.455 billion barrels.

11. (a) 137,200 units/year; (b) 1,294,833.2.

12. (a) \$248/year; (b) \$432.

13. $p = 100 - \sqrt{2q}$. 15. \$1900.

17. $CS = 166\frac{2}{3}$, $PS = 53\frac{1}{3}$.

Exercise 11-4

1. (a) $A = \int_3^6 (2x - 5)\,dx = (x^2 - 5x)\Big|_3^6$

$= 6 - (-6) = 12$

(c) $A = -\int_2^{5/2} f(x)\,dx + \int_{5/2}^4 f(x)\,dx$

$= -(x^2 - 5x)\Big|_2^{5/2} + (x^2 - 5x)\Big|_{5/2}^4$

$= \frac{1}{4} + 2\frac{1}{4} = 2\frac{1}{2}$.

2. Points of intersection $x^3 = x^2$

$x^3 - x^2 = x^2(x - 1) = 0$

Thus the curves interesect at (0,0) and (1,1).

For $0 \leqslant x \leqslant 1$, $f(x) \geqslant g(x) \geqslant 0$.

$A = \int_0^1 (x^2 - x^3)\,dx = \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1$

$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

5. Since $f(x) \geqslant 0$ for $x \geqslant 0$,

$$A = \int_0^1 x\, e^{3x^2} dx = \frac{1}{6}\int_0^1 e^{3x^2}\, 6x dx$$

$$= \frac{1}{6}\left(e^{3x^2}\right)\Big|_0^1.$$

$$= \frac{1}{6}(20.086 - 1) = \frac{1}{6}(19.086) = 3.181.$$

7. Points of intersection:

$$x^2 = \sqrt{x}$$

$$x^4 = x$$

$$x(x^3 - 1) = 0$$

$$x = 0 \text{ or } x = 1$$

Thus the curves intersect at (0,0) and (1,1)

For $0 \leqslant x \leqslant 1$, $g(x) \geqslant f(x) \geqslant 0$.

$$A = \int_0^1 (x^{\frac{1}{2}} - x^2)\, dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

11-5

1. $$\int_{35}^{60} 1000x^{-0.5}\, dx = 2000x^{\frac{1}{2}}\Big|_{35}^{60}$$

$$= 2000(7.746 - 5.916) = 3660 \text{ hours}.$$

3. The direct labor required approaches a constant value irrespective of the number of units produced.

Exercise 11-6

1. (a) He should choose Investment B since its value exceeds both B's cost and the value of A.

   (b) namely Investment A.

2. (a) To get total sales of 10 (in thousands), a must be $\geqslant 3$. Thus $a < 3$ will make them abort.

   (b) $5\frac{1}{4}$.

6. (a) $S(A) = 4 \log_e A + 5$.

   (b) Profit as a function of $A = f(A) = p[4 \log_e A + 5] - A$

   (c) $\frac{df}{dA} = \frac{4p}{A} - 1 = 0$ when $A = 4p$.

9. $20 - 5x = 4x + 8$

$$9x = 12$$

$$x^* = \frac{4}{3}$$

$$p^* = D(x^*) = D\left(\frac{4}{3}\right) = \frac{40}{3}.$$

$$CS = \int_0^{4/3} (20 - 5x)\, dx - \frac{40}{3} \cdot \frac{4}{3}$$

$$= \left(20x - \frac{5}{2} x^2\right)\Bigg|_0^{4/3} - \frac{160}{9}$$

$$= \frac{200}{9} - \frac{160}{9} = \frac{40}{9} = \$4.44$$

$$PS = \frac{40}{3} \cdot \frac{4}{3} - \int_0^{4/3} (4x + 8)\, dx$$

$$= \frac{160}{9} - (2x^2 + 8x) \Big|_0^{4/3}$$

$$= \frac{160}{9} - \frac{128}{9} = \frac{32}{9}$$

$$= \$3.56$$

12. (a) \$37,599; (b) \$4924.

13. (a) 5481; (b) \$535. 15. 4,000,000.

16. $R'(t) = C'(t)$

$$19 - t^{\frac{1}{2}} = 3 + 3t^{\frac{1}{2}}$$

$$4t^{\frac{1}{2}} = 16$$

$$t^{\frac{1}{2}} = 4$$

$$t = 16$$

The operation should continue for 16 years.

$$P(t) = \int_0^{16} [R'(t) - C'(t)]dt$$

$$= \int_0^{16} [(19 - t^{\frac{1}{2}}) - (3 + 3t^{\frac{1}{2}})]dt$$

$$= \int_0^{16} (16 - 4t^{\frac{1}{2}})dt = \left(16t - \frac{4t^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_0^{16}$$

$$= 256 - \frac{512}{3} \approx 85.33 \text{ millions of dollars.}$$