AU. J.T.2(2): 65-67 (Oct. 1998)

Circumference of a k-Connected Graph*

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Abstract

A new lower bound for the circumference of a graph has been obtained for this long-standing problem. Let G be a simple k-connected graph with the number of vertices $|G| \geq 3$, minimum degree δ, and C a longest cycle in G. In 1980, Bondy conjectured that, if the degree sum of any k+1 independent vertices in G is at least $|G|+k(k-1)$, then G-C contains no path of length k-1. This conjecture has essentially been settled only for the values of $k \leq 8$. The present study shows that if G-C has a path of length k-1, then the length of C is at least $(1/2)k(\delta - k + 3)$.

Keywords: *k-connected graph, longest cycle, connectivity, vertices, Hamilton cycle, Hall's theorem, circumference.*

Introduction

The Hamilton problem, determining when a graph has a cycle containing all vertices (called a *Hamilton cycle*) has been a fundamental problem in graph theory since 1850. The research on finding sufficient degree conditions for a graph to have a Hamilton cycle originated from the work of Dirac (1952).

The basic terminology and notation can be found in any standard text on graph theory, Bondy & Murty (1976), for instance. Necessary terms and notation will be difined at appropriate places.

In this paper, G denotes a graph with |G| vertices and minimum degree δ.

Theorem A: (Dirac 1952). *If $|G| \leq 2\delta$, then G has a Hamilton cycle.*

A more general problem is finding bounds for the *circumference* (the length of a longest cycle) of a graph, and the origin of estimating the circumference is also a result of Dirac.

Theorem B: (Dirac 1952). *If C is a longest cycle in G and G is 2-connected, then $|C| \geq \min(|G|, 2\delta)$.*

The main problem in this direction should be the following conjecture of Bondy (1980).

Conjecture 1: (Bondy 1980). Let G be a k-connected graph for which every independent set of k+1 vertices has degree sum at least |G|+k(k-1) and let C be a longest cycle in G. Then, G-C contains no path of length k-1.

Related Problem

A related problem is to find the bound for the circumference. The corresponding conjecture for the circumference of a graph can be stated as follows:

Conjecture 2: Let G be a k-connected graph and let C be a longest cycle in G. If G-C contains a path of length k-1, then

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**Paper presented at the International Congress of Mathematicians 1998 Berlin, held at TU Berlin, August 18 – 27, 1998. A product of the joint project with Prof. H. A. Jung, TU Berlin.*

$|C| \geq (\Sigma\ d(v_i))$-$k(k$-$2)$ for some independent set of k+1 vertices.

One will obtain a weaker version of conjecture 2 if one requires the weaker conclusion "$|C| \geq k(\delta - k + 2)$".

Conjecture 3: Let G be a k-connected graph and let C be a longest cycle in G. If G-C contains a path of length k-1, then $|C| \geq k(\delta - k + 2)$.

Even this weak version has been settled only for very few k, namely $k \leq 8$, as can be seen in Jung (1990), Dopheide (1994), and Neddermeier (1994). For some results involving independent vertices, the readers are referred to Fraisse and Jung (1989).

In this paper, the proof of the following Theorem is outlined.

Theorem

Let G be a simple k-connected graph and C a longest cycle in G. Let H be a component of G-C. If H contains a path of length k - 1 then $|C| \geq (1/2)k(\delta - k + 3)$.

Assume, from now on, G is a *k*-connected graph, C is a longest cycle in G, and H is a component of G-C.

Fix an orientation of C. For any two distinct vertices x, y on C, C[x, y] denotes the segment of C which is an (x, y)-path taken in the orientation of C from x to y. Note that:

$$|C[x, y]| \geq 3$$

as C is a longest cycle; so, the subpath C(x, y) obtained by removing x, y from C[x, y] is nonempty. For subgraphs A, B of G, let $N_A(B)$ be the set of vertices of *A* adjacent to vertices of B. If B = { u } then we write $N_A(u)$ for $N_A(B)$ and $d_A(u)$ for $|N_A(u)|$, the number of elements of $N_A(u)$. Write N(u) for $N_G(u)$, and d(u) for $d_G(u)$, the degree of u. For any $x, y \in N_C(H)$ such that $x \neq y$, we call the segment C(x, y) a *basic segment* when $C(x, y) \cap N_C(H) = \varnothing$.

Selection of Vertices

Let P be a longest path in H. Assume, P has at least k vertices. Label vertices of P in the same order as that on P: $v_1, v_2, \ldots, v_r$. There is a Depth-First Search tree T containing P, in which v_r is the root. The spanning tree T of H determines a partial ordering $\leq_T$ on the vertex set V(H) of H: $u \leq_T w$ if the unique path $T[u, v_r]$ contains w. A characteristic property of the Depth-First Search tree T is that if vertices x and y in H are joined by an edge, then x and y are *comparable* (with respect to $\leq_T$), that is, either $x \leq_T y$ or $y \leq_T x$ (See Jung 1968). Choose u_i, for $1 \leq i \leq k$, as follows. Put $I_1 :=$ $\{i : d_T(v_i) \geq 3, 1 \leq i \leq k\}$ and, for each $i \in I_1$, let u_i ($\leq_T v_i$) be a vertex in T–P with $d_T(u_i) = 1$. Put $U_1 := \{u_i : i \in I_1\}$ and $K := \{1, 2, \ldots, k\}$. Find the maximum subset I_2 of K-I_1 and $W := \{ w_i : i \in I_2 \}$ such that:

(i) $w_i \in N(v_i)$-$(P \cup U_1)$ for each $i \in I_2$ and

(ii) a and b are incomparable for any two distinct $a, b \in U_1 \cup W$.

For each $i \in I_2$, fix a vertex u_i ($\leq_T w_i$) of degree one in T and put $u_i := v_i$ for each $i \in K$-$(I_1 \cup I_2)$. Let $U := \{ u_i : 1 \leq i \leq k \}$. For disjoint subgraphs A and B of G, let us write e(A;B) for the number of edges joining vertices of A and vertices of B. For $1 \leq j \leq k$, define

$\delta_j' = d_P(u_j)$ if $u_j = v_j$, or else

$\delta_j' = d_H(u_j)+e(\{v_i : v_i = u_i\}; T[u_j, u_j'])$,

where u_j' is the first vertices on $T[u_j, v_r]$ which is in P.

Lemmas

The following lemmas are obtained:

Lemma 1: $\Sigma\ \delta_i' \geq \Sigma\ d_H(u_i)$ where the summations are taken over $1 \leq i \leq k$.

Lemma 2: For distinct $x, y \in N_C(H)$, $|C(x, y)| \geq (1/2)(d_U(x)+d_U(y))$.

By Menger's theorem, there are k disjoint paths connecting C and P as G is k-connected. Let $x_1, x_2, \ldots, x_k$ be end vertices on C of these k paths in the orientation around C. Using Hall's theorem, applying Lemma 2 to each basic segment in the segment $C(x,y)$, and making additional measurements, we obtain Lemma 3. The subscripts of x are taken modulo $k+1$.

Lemma 3: There is a permutation θ of K such that for each $j \in K$, $|C(x_{\theta(j)}, x_{\theta(j)+1})| \geq (1/2)(\delta_j' - k + 1 + e(U; C(x_{\theta(j)}, x_{\theta(j)+1})) + (1/2)d_U(x_{\theta(j)}) + (1/2)d_U(x_{\theta(j)+1}))$.

Estimate for the Circumference

By Lemma 3,

$$|C(x_1,x_2)| + \ldots + |C(x_k,x_1)| \geq (1/2)(\delta_1' + \ldots + \delta_k' - k(k-1) + e(U;C)).$$

Since $\delta_1' + \ldots + \delta_k' \geq d_H(u_1) + \ldots + d_H(u_k)$ by Lemma 1, $e(U;C) = d_C(u_1) + \ldots + d_C(u_k)$, and $d_H(u_i) + d_C(u_i) = d(u_i)$ for each i, one obtains

$$|C(x_1,x_2)| + \ldots + |C(x_k,x_1)| \geq (1/2)(d(u_1) + \ldots + d(u_k) - k(k-1)) \geq (1/2)(k\delta - k(k-1)) = (1/2)k(\delta - k + 1).$$

Consequently,

$$|C| \geq k + (1/2)k(\delta - k + 1) = (1/2)k(\delta - k + 3).$$

Conclusion

The result presented in this paper is the first estimate for the circumference of a k-connected graph in terms of k. The next attempt should be directed to Conjecture 3, and thereafter Conjecture 2. Conjecture 1 will naturally be the target after that.

Acknowledgement

The author would like to express his gratitude to the German Academic Exchange Service (DAAD) for the invitation for a research visit. He is greatly indebted to the Bro . P. Martin Komolmas, President of Assumption University for his kind permission to undertake this joint research work. During that visit, he participated in and presented the result of his work to the International Congress of Mathematicians, Berlin in 1998.

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