n-Centrum and Steiner n-Center of a Tree

San San Aye and Min Aung
Faculty of Business Administration, Assumption University
Bangkok, Thailand

Abstract

This paper is a contribution to the theory of centrality in graphs which finds its application in facility location problems. For a graph G and an integer n such that 1 £ n £ |G|, Slater (1978), and Chartrand et al. (1989) defined non-negative integer-valued functions r_n and e_n , respectively, on the vertex set V(G). The subgraphs induced by vertices of minimum values are called n-centrum and Steiner n-center, respectively. We prove that the n-centrum of a tree is a subtree of its Steiner n-center.

Keywords: *Theory of centrality, graph G, integer n, non-negative integer-valued functions, vertex set, n-centrum, Steiner n-center, eccentricity.*

Introduction

The centrality concept, originated in the classic work of Jordan, has been generalized in several directions [see Harary and Buckley (1990) as a brief survey]. Naturally, then, it is worthwhile to study the interrelations among these different centrality concepts. This paper can be considered to be a contribution to this direction of studies.

G denotes a simple, connected graph, u a vertex of G, and, n a positive integer not exceeding |G|, where |G| is the number of vertices. We identify the edge joining vertices u and v with uv. To express that G is a subgraph of a graph H, we write $G \subset H$.

Generalizing the standard definitions of center and centroid, P.J. Slater (1978) defined the concept of n-centrum. For a nonempty subset S of V(G), he puts

$$d_S(u) := \sum\{ d(u, v) : v \in S \}$$

where d(u, v) denotes the distance between u and v. The (*Slater*) n-eccentricity of u is then defined to be

$$r_n(u) := \max \{ d_S(u) : S \subset V(G) \text{ and } |S| = n \}.$$

The subgraph $SlC_n(G)$ induced by the set of vertices of minimum n-eccentricity is called the *n-centrum*. Slater (1978) obtained several results on the structure of n-centrum and the union of all n-centra.

Chartrand *et al.* (1989) defined another generalization of the standard center concept. The *Steiner distance* d(S) of a nonempty subset S of V(G) is defined as the minimum number of edges in a connected subgraph of G that contains S. The *(Steiner) n-eccentricity* of u is defined to be

$$e_n(u) := \max \{ d(S) : S \subset V(G), |S| = n \text{ and } u \in S \}$$

[In fact, it is not defined for n = 1 in Chartrand *et al.* (1989).] The subgraph induced by the set of vertices of minimum n-eccentricity is called the *Steiner n-center*; we will denote it by $StC_n(G)$. Oellermann and Tian (1990) obtained results on the location and structure of Steiner n-centers of a tree.

It is proved in Oellermann and Tian (1990) that the Steiner n-center of a tree can be located by repeatedly removing end-vertex sets until the first subtree with less end-vertices than n remains. Unlike Steiner n-center, n-centrum cannot be found by a simple procedure, although it is simply isomorphic to either K_1 or K_2. According to Slater (1978), n-centra can be far apart from each other, and the union of all n-centra of a tree is a subtree which can be large and contain more than a path. We consider the relative positions of n-centrum and Steiner n-center of a tree. We will show that $SlC_n(T) \subset StC_n(T)$ for any tree T and any n |T|.

We will list the observations we need.