AU J.T. 2(4):138-142 (Apr. 1999)

Analysis of Two New Fuzzy Relational Products for Information Processing

Pratit Santiprabhob and Kitikorn Dowpiset*
Faculty of Science and Technology, Assumption University
Bangkok, Thailand

Abstract

This paper first reviews the four classical fuzzy relational products. Then, two new fuzzy relational products are introduced. The formulas, semantics, and characteristics of these two new fuzzy relational products are described and analyzed. The advantages and limitations of these products in the context of information processing are discussed in comparison to those of the classical ones.

***Keywords**: Fuzzy relation, fuzzy relational product, information processing.*

Introduction

Fuzzy (binary) relations, as discussed in Klir and Folger (1988), have long been used in information processing and data analysis in a variety of fields such as information protection (Santiprabhob and Kohout1992; 1993), information retrieval (Kohout *et al.* 1984; Kohout and Bandler 1985), medical diagnosis (Bandler and Kohout 1981; Kohout and Bandler 1992), etc. During the course of our development of an alternative framework for social network analysis, the authors have developed a new fuzzy relational product called *Similarity product*. This *Similarity product* is employed successfully in the new framework proposed. Further investigation by the authors has discovered yet another interesting fuzzy relational products named *Star product*, which can be used to determine correlation between objects with respect to the strength of their relationships to some third-party objects.

In this paper, after reviewing the four classical fuzzy relational products, the two new products are described and analyzed. These two new products, having different semantics from those of the classical ones, provide innovative means to analyze information represented by fuzzy relations. They yield new results that could not be calculated before.

* *Department of Computer Information Management, St.John University, Bangkok , Thailand.*

Example fuzzy relations

For the sake of our discussion in the following sections, we define two generic fuzzy relations, to which various fuzzy relational products will apply. The two fuzzy relations are named *R* and *S*, and defined in Figs.1 and 2, respectively.

	z_1	z_2	z_3	z_4	z_5	z_6
x_1	0.0	0.5	0.9	0.3	0.0	0.4
x_2	0.0	0.0	0.0	0.0	0.0	0.6
x_3	0.0	0.0	0.0	0.0	0.0	0.0
x_4	0.0	0.5	0.1	0.2	0.0	0.0
x_5	1.0	0.8	0.7	0.7	0.8	0.9

Fig.1. Fuzzy relation *R*

	y_1	y_2	y_3	y_4	y_5
z_1	0.9	0.6	0.0	0.6	0.0
z_2	0.7	0.0	0.0	0.0	0.5
z_3	0.9	0.0	0.0	0.0	0.8
z_4	0.8	0.0	0.0	0.0	0.2
z_5	1.0	0.0	0.0	0.5	0.0
z_6	0.6	0.0	0.0	0.0	0.4

Fig.2. Fuzzy relation *S*.

The above two fuzzy relations could, in practice, represent any kind of relationships between pairs of objects (x's, z's) and (z's, y's) where there is one common set of objects (z's) in both relations. For example, the pairs in the first relation could be (persons, symptoms) and those in the second could be (symptoms, diseases), or the pairs in the first relation could be (persons, concepts) and those in the second could be (concepts, documents), etc.

For the sake of genrality, the discussion will be based on generic relationships among generic objects. The two fuzzy relations above are used to demonstrate characteristics of different fuzzy relational products.

Review of Classical Fuzzy Relational Products

Four classical fuzzy relational products as discussed in Bandler and Kohout (1980), Kohout *et al.* (1984), Kohout and Bandler (1985), and Kohout and Bandler (1990) are *Circle product*, *Triangle Subproduct*, *Triangle Superproduct*, and *Square product*. They are reviewed one by one in this section. Each product is described in terms of its fuzzy membership degree calculation and its semantics. Then, fuzzy relations resulted from an application of each product to the example fuzzy relations R and S are respectively given.

Circle product can be defined in terms of membership degree calculation as

$$\mu_{R\circ S}(x_i, y_k) = \vee_j(\mu_R(x_i, z_j) \wedge \mu_S(z_j, y_k))$$

This gives a degree of relationship for each pair of objects (x, y) according to the degree to which they are related to at least one common object in the set of z's.

Triangle Subproduct can be defined in terms of membership degree calculation as

$$\mu_{R\lhd S}(x_i, y_k) = \wedge_j(\mu_R(x_i, z_j) \to \mu_S(z_j, y_k))$$

This gives a degree of relationship for each pair of objects (x, y) according to the degree to which the set of objects z's that relate to the object x in R is included in the set of object z's that relate to the object y in S.

Triangle Superproduct can be defined in terms of membership degree calculation as:

$$\mu_{R\rhd S}(x_i, y_k) = \wedge_j(\mu_R(x_i, z_j) \leftarrow \mu_S(z_j, y_k))$$

This gives a degree of relationship for each pair of objects (x, y) according to the degree to which the set of objects z's that relate to the object x in R includes the set of object z's that relate to the object y in S.

Square product can be defined in terms of membership degree calculation as:

$$\mu_{R\Box S}(x_i, y_k) = \wedge_j(\mu_R(x_i, z_j) \leftrightarrow \mu_S(z_j, y_k))$$

where $\leftrightarrow$ is defined as

$$\mu_A(x) \leftrightarrow \mu_B(y) = \mu_A(x) \to \mu_B(y) \wedge \mu_A(x) \leftarrow \mu_B(y)$$

This gives a degree of relationship for each pair of objects (x, y) according to the degree to which the set of objects z's that relate to the object x in R exactly matches the set of object z's that relate to the object y in S.

In the following calculations, the fuzzy implication operator $\to$ used in the above definitions is defined as the *Standard Star*, S^*, operator discussed in Bandler & Kohout (1981)

$$\mu_A(x) \to \mu_B(y) = 1 \text{ if } \mu_A(x) \le \mu_B(y)$$
$$\mu_B(y) \text{ if } \mu_A(x) > \mu_B(y)$$

The resulting fuzzy relations from applications of the *Circle product*, the *Triangle Subproduct*, the *Triangle Superproduct*, and the *Square product* to the example fuzzy relations R and S are shown in Figs.3-6, respectively.

	y_1	y_2	y_3	y_4	y_5
x_1	0.9	0.0	0.0	0.0	0.8
x_2	0.6	0.0	0.0	0.0	0.4
x_3	0.0	0.0	0.0	0.0	0.0
x_4	0.5	0.0	0.0	0.0	0.5
x_5	0.9	0.6	0.0	0.6	0.7

Fig.3. Resulting fuzzy relation $R \circ S$

	y_1	y_2	y_3	y_4	y_5
x_1	1.0	0.0	0.0	0.0	0.2
x_2	1.0	0.0	0.0	0.0	0.4
x_3	1.0	1.0	1.0	1.0	1.0
x_4	1.0	0.0	0.0	0.0	1.0
x_5	0.6	0.0	0.0	0.0	0.0

Fig.4. Resulting fuzzy relation $R \lhd S$

	y_1	y_2	y_3	y_4	y_5
x_1	0.0	0.0	1.0	0.0	1.0
x_2	0.0	0.0	1.0	0.0	0.0
x_3	0.0	0.0	1.0	0.0	0.0
x_4	0.0	0.0	1.0	0.0	0.0
x_5	0.7	1.0	1.0	1.0	0.7

Fig.5. Resulting fuzzy relation $R \triangleright S$

	y_1	y_2	y_3	y_4	y_5
x_1	0.0	0.0	0.0	0.0	0.2
x_2	0.0	0.0	0.0	0.0	0.0
x_3	0.0	0.0	1.0	0.0	0.0
x_4	0.0	0.0	0.0	0.0	0.0
x_5	0.6	0.0	0.0	0.0	0.0

Fig.6. Resulting fuzzy relation $R \square S$

New fuzzy relational products

Two new fuzzy relational products namely *Similarity product* and *Star product* are introduced in this section. Both the formulas for membership degree calculation and the semantics of the two products are given and discussed. Then, the resulting fuzzy relations from applications of both products to the example fuzzy relation *R* and *S* are shown.

Similarity product is defined in terms of membership degree calculation as:

$$\mu_{R \diamond S}(x_i, y_k) = \frac{\sum_j (1 - |\mu_R(x_i, z_j) - \mu_S(z_j, y_k)|)}{\Sigma j}$$

This gives a degree of relationship for each pair of objects (*x*, *y*) according to the degree to which a set of objects *z*'s that relate to the object *x* in *R* is similar to the set of object *z*'s that relate to the object *y* in *S*.

In the definition of the *Similarity product* above, the absolute difference in their relationships to a common object *z* between an object *x* in *R* and an object *y* in *S* is defined as:

$$|\mu_R(x_i, z_j) - \mu_S(z_j, y_k)|$$

Therefore, the degree of similarity between the two objects *x* and *y* with respect to their relationships to a particular object *z* can be defined as

$$1 - |\mu_R(x_i, z_j) - \mu_S(z_j, y_k)|$$

The *Similarity product*, then, averages out the degrees of similarity over all the common objects *z*'s for each pair of objects (*x*, *y*).

Star product is defined in terms of membership degree calculation as:

$$\mu_{R*S}(x_i, y_k) = \frac{\Sigma\,(\mu_R(x_i, z_j)\,\mu_S(z_j, y_k))}{\Sigma j}$$

This gives a degree of relationship for each pair of objects (*x*, *y*) according to the combined strength of their relationships to a common set of objects *z*'s.

In the definition of the *Star product* above, the combined strength of relationships of an object *x* in *R* and of an object *y* in *S* with respect to a common object *z* is defined as

$$(\mu_R(x_i, z_j)\,\mu_S(z_j, y_k))$$

Like the *Similarity product*, the *Star product*, then averages out the combined relationship degrees over all the common objects *z*'s for each pair of objects (*x*, *y*).

The fuzzy relations resulted from applications of the *Similarity product* and the *Star product* to the example fuzzy relations *R* and *S* are shown in Fig.7 and Fig.8, respectively.

	y_1	y_2	y_3	y_4	y_5
x_1	0.5	0.6	0.7	0.5	1.0
x_2	0.3	0.8	0.9	0.7	0.7
x_3	0.2	0.9	1.0	0.8	0.7
x_4	0.4	0.7	0.8	0.6	0.9
x_5	0.8	0.3	0.2	0.4	0.5

Fig.7. Resulting fuzzy relation $R \diamond S$

	y_1	y_2	y_3	y_4	y_5
x_1	0.3	0.0	0.0	0.0	0.2
x_2	0.1	0.0	0.0	0.0	0.0
x_3	0.0	0.0	0.0	0.0	0.0
x_4	0.2	0.0	0.0	0.0	0.1
x_5	0.7	0.1	0.0	0.2	0.2

Fig. 8. Resulting fuzzy relation $R * S$

Analysis of the New Fuzzy Relational Products

Both new fuzzy relational products proposed, namely the *Similarity product* and the *Star product* have their own niches of applications that fill the gap between the *Circle product* and the *Square product* in their own ways. With the use of the *Circle product*, the resulting fuzzy relation contains, for each pair of objects (x, y), a degree to which that object x and that object y relate to at least one same object z. On the other extreme, the *Square product* yields a fuzzy relation that contains, for each pair of objects (x, y), a degree to which both objects relate to exactly the same common set of objects z's.

The *Square product* is in effect an *and*-combination of the *Triangle Subproduct* and the *Triangle Superproduct*. The two *Triangle* products determine the degree of inclusion in two different directions of the relationships between an object x and an object y with respect to their relationships to a common set of objects z's.

The *Similarity product* looks at things from a different perspective. It is not as stringent as the *Square product*. The degree to which a particular object x is related to another object y in the resulting fuzzy relation is determined by the similarity of their overall relationships to a common set of objects z's in the original fuzzy relations.

The *Square product* disregards a relationship between two objects (x, y) right away if there is just one absence of relationship with respect to some common object z in one of the original fuzzy relations while the corresponding relationship presents in the other relation. This is not the case of the *Similarity product*. For example, the resulting relationship degree of (x_4, y_5) from an application of the *Square product* as shown in Figure 6 is 0.0. However, the same relationship degree, which is resulted from an application of the *Similarity product* as shown in Fig. 7, is 0.8. In this case, if we look back at the original relations R and S, we would find that the relationships between x_4 and z's are very similar to those of y_5 and z's. The main discrepancy is on the relationships with respect to z_6. Here, the relationship degree between x_4 and z_6 is 0.0, while the relationship degree between y_5 and z_6 is 0.4.

Therefore, the *Similarity product* would very much be suitable for an application that wants to determine relationships among different objects based on the similarity of their relationships to a set of some third-party objects. One good example of such an application is the social network analysis in which relationships among different people can be determined from the opinions they have.

On the other hand, the *Star product* emphasizes the presence (or absence) of relationship/information. Due to the use of the mathematics multiplication on the relationship degrees from the two original fuzzy relations in the membership degree calculation, any absence of relationship/information (represented by the degree of 0.0) in any of the two original fuzzy relations would result in a combined relationship degree of 0.0. However, the resulting combined degrees are then averaged out over the common objects z's for each pair of objects (x, y). Therefore, the non-zero results still have their effects on the final relationship degree between the objects (x, y) unlike the case of the *Square product*.

With the use of the *Star product*, only the pairs of objects (x's, y's) that have strong correlation with a set of common objects z's will have high resulting relationship degrees, e.g. the resulting relationship degree of 0.7 of (x_5, y_1) in Fig. 8. This is a characteristic of the *Star product* that other fuzzy relational products do not posses.

In the case of the *Circle product*, just one pair of strong relationships to a single common object z can result in a high resulting relationship degree, e.g. the resulting relationship degree of 0.6 of (x_2, y_1) or of (x_5, y_2) in Fig. 3.

In the case of the *Square product*, an absence of relationship/information in both original fuzzy relations results in a very high relationship degree, e.g. the relationship degree of 1.0 of (x_3, y_3) in Fig. 6. Here, an absence of relationship/information in the two original fuzzy relations is considered a perfect match!

The *Similarity product* also yields a perfect match for an absence of relationship/information in the two original fuzzy relations, e.g. the relationship degree of 1.0 of (x_3, y_3) in Fig. 7. In this very same case, the *Star product* yields a degree of 0.0.

Moreover, the *Similarity product* considers the case where, most of the time, the relationship/information is absent in both original fuzzy relations, i.e. no relationship/ information with respect to third-party objects, e.g. the relationship degree of 0.8 of (x_2, y_2) in Fig. 7.

Although the resulting relationship degrees from an application of the *Star product* would, in general, be quite low (because of the use of mathematics multiplication) as shown in Fig. 8, all non-zero degrees do signify some sort of valid correlation with respect to non-zero relationship degrees in the two original fuzzy relations. Furthermore, any high-number relationship degree guarantees that really strong correlation exists between the objects in the pair. Therefore, the *Star product* would be quite suitable for an application where only non-zero relationship degrees (in both original fuzzy relations) do count. One potential application of this *Star product* is in the criminal case examination where strong correlation of evidences and facts should be established.

Conclusion

In this paper two new fuzzy relational products have been introduced, the *Similarity product* and the *Star product* which have unique niches in their applications in information processing. The *Similarity product* can be used to determine the similarity of relationships of objects in focus with respect to some third-party objects. On the other hand, the *Star product* can be used to determine the strength of the correlation of objects in focus with respect to their non-null relationships with some third-party objects. Both fuzzy relational products complement the four classical products. Together they will definitely enhance our ability to manipulate fuzzy relations, so that more novel information processing tasks can be accomplished.

References

Bandler, W.; and Kohout, L.J. 1981. The identification of hierarchies in symptoms and patients through computation of fuzzy relational products. *In* R. D. Parslow, ed. BCS'81: Information Technology for the Eighties, pp.191-194, Heyden & Son, London.

Bandler, W.; and L. J. Kohout, L.J. 1980. Semantics of implication operators and fuzzy relational products. Int. J. Man-Machine Studies 12: 89-116. (Also *In* E. H. Mamdani and B. R. Gaines, eds., Fuzzy Reasoning and Its Applications, pp. 219-246, AP, London, 1981).

Klir, G.L.; and Folger, T.A. 1988. Fuzzy Sets, Uncertainty, and Information. Prentice Hall, New York, NY.

Kohout, L.J.; Keravnou, E.; and Bandler, W. 1984. Automatic documentary information retrieval by means of fuzzy relational products. *In* B. R. Gaines, L. A. Zadeh, and H. J. Zimmermann, eds. Fuzzy Sets in Decision Analysis, pp. 383-404, North-Holland, Amsterdam.

Kohout, L.J.; and Bandler, W. 1985. Relational-product architectures for information processing. Infor. Sci. 37: 25-37.

Kohout, L.J.; and Bandler, W. 1990. Fuzzy relational products in knowledge engineering. Proc. Int. Symp. Fuzzy Approach to Reasoning and Decision-Making, Bechyne, Czechoslovakia. (Also *In* V. Novak, *et al*. Fuzzy Approach to Reasoning and Decision-Making, pp. 51-66, Kluwer Academic Publ., 1992).

Santiprabhob, P.; and Kohout, L.J. 1992. Connected protection structures: A key to secure distributed knowledge based systems. Proc. 5th Florida AI Res. Symp., pp. 134-138, Ft. Lauderdale, FL.

Santiprabhob, P.; and Kohout, L.J. 1993. Distributed dynamic protection of interacting systems. Proc. 5th Int. Fuzzy Systems Ass. World Congr., pp. 493-496, Seoul, Korea.